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Riccati方程式をめぐって
(Around Riccati equations)

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There are some operator-theoretical results related to Riccati equations:

Pedersen-Takesaki's theorem [4]. Let H be a nonsingular positive operator and K a positive one on a Hilbert space. Then there exists a solution $X \geq 0$ for $XHX = K$ if and only if there exists a positive number a with

$$(H^{1/2}KH^{1/2})^{1/2} \leq aH.$$

Anderson-Trapp's theorem [1]. There exists a unique positive solution $A\#B$ for a Riccati equation $XB^{-1}X = A$ where A and B are positive operators and $A\#B$ is a geometric operator mean ([2]):

$$A\#B = B^{1/2}(B^{-1/2}AB^{1/2})^{1/2}B^{1/2}.$$

Trapp's theorem [5]. If $BA = A^*B$, then there exists a solution

$$X = (A^*BA + C)\#B + BA$$

for a Riccati equation $XB^{-1}X - A^*X - XA = C$ where B and C are positive operators.

But the last two results are not enough evaluated. One of the reasons is that these are considered as rather restricted cases for Riccati equations. So, considering their idea, we discuss geometric operator means as solutions of Riccati equations on the more general situation, which is slightly different from the view of the engineering.

First we observe Riccati equation derived from the engineering situation (where all the results might hold in an infinite dimensional case): For $x(t)$ a *state vector* and a *control vector* $u(t)$, consider a *linear system dynamical equation*

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

where A is square. Then, a **linear-quadratic problem** (or **optimal regulator problem**) is to find $u(t)$ which minimizes the *cost functional*

$$J(u) = \frac{1}{2}\int_0^T x(t)^*Qx(t) + u(t)^*Ru(t)\, dt$$

where Q and R are positive-definite matrices. To solve this, consider the Lagrange equation

$$F(t) = x(t)^*Qx(t) + u(t)^*Ru(t) + \lambda(t)^*\left(Ax(t) + Bu(t) - \dot{x}(t)\right)$$

and the Euler differential one (which is a condition to solve the above):

$$\frac{\partial F}{\partial x}(t) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)(t) = 0 \quad \text{and} \quad \frac{\partial F}{\partial u}(t) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right)(t) = 0$$

with a *controllable condition* $X(T) = 0$. Putting $u(t) = -R^{-1}B^*\lambda(t)$, we have

$$\dot{x}(t) = Ax(t) - BR^{-1}B^*\lambda(t), \qquad (x(0) = x_0).$$

Moreover putting $\lambda(t) = X(t)x(t)$, we have

$$\dot{\lambda}(t) = \dot{X}(t)x(t) + X(t)\dot{x}(t)$$

and hence

$$\begin{aligned}
\dot{X}(t)x(t) &= \dot{\lambda}(t) - X(t)\dot{x}(t) \\
&= -A^*\lambda(t) - Qx(t) - X(t)\left(Ax(t) - BR^{-1}B^*\lambda(t)\right) \\
&= \left[-A^*X(t) - Q - X(t)A + X(t)BR^{-1}B^*X(t)\right]x(t).
\end{aligned}$$

Thus we have the **Riccati differential equation**:

$$(0) \qquad \dot{X}(t) = -X(t)A - A^*X(t) + X(t)BR^{-1}B^*X(t) - Q \quad (X(T) = 0).$$

As we see later, this equation has a unique positive-definite (resp. negative-definite) solution $X(t)$ by the Lipschitz condition for all $t < T$ (resp. $t > T$). In fact, if a solution $X(t)$ exists, then the positivity of the solution follows from the above settings:

$$\begin{aligned}
x(t)^*X(t)x(t) &= -\int_t^T \left(\frac{d}{dt}x(t)^*X(t)x(t)\right)dt \\
&= -\int_t^T \dot{x}(t)^*X(t)x(t) + x(t)^*X(t)\dot{x}(t) + x(t)^*\dot{X}(t)x(t)\, dt \\
&= \int_t^T x(t)^*X(t)^*BR^{-1}B^*X(t)x(t) + x(t)^*Qx(t)dt > 0.
\end{aligned}$$

Moreover, under the controllability and the observability, the limit $X \equiv \lim_{t\to\infty} X(t)$ exists and the following **algebraic Riccati equation** is obtained:

$$Q = -XA - A^*X + XBR^{-1}B^*X.$$

Under these conditions, the Hamiltonian

$$H = \begin{pmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{pmatrix}$$

is stable, that is, $\mathrm{Re}\sigma(H) < 0$ which assures the existence of the limit $\lim_{t\to\infty} e^{tH}$. Then putting $S = \begin{pmatrix} 1 & -1 \\ X & X \end{pmatrix}$, we have $S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & X^{-1} \\ -1 & X^{-1} \end{pmatrix}$ and

$$S^{-1}HS = \begin{pmatrix} A - BR^{-1}B^*X & -A - X^{-1}A^*X \\ O & BR^{-1}B^*X - X^{-1}A^*X \end{pmatrix},$$

which shows the diagonal element $A - BR^{-1}B^*X$ is also stable. Thereby the uniqueness of the solutions are shown: Let X_k be the positive-definite solutions. Then

$$\begin{aligned} O &= (X_1 - X_2)(A - BR^{-1}B^*X_1) + (A - BR^{-1}B^*X_2)^*(X_1 - X_2) \\ &= (X_1 - X_2)(A - BR^{-1}B^*X_1) + (A^* - X_2^*BR^{-1}B^*)(X_1 - X_2) \end{aligned}$$

Thus the stability of the above terms shows $X_1 = X_2$ is a unique solutions O for Lyapunov equation:

$$X_1 - X_2 = -\int_0^\infty e^{t(A^*-X_2^*BR^{-1}B^*)}Oe^{t(A-BR^{-1}B^*X_1)}dt = O.$$

Thus the theory of Riccati equations, in particular, the algebraic one

$$C = XB^{-1}X - A^*X - XA, \tag{1}$$

has been established, but the following equation with the adjoint cannot be obtained by the above settings:

$$C = X^*B^{-1}X - A^*X - X^*A, \tag{2}$$

which is a mathematically natural equation and has a common positive-definite solution for (1). (Another discussion for (2) was shown in [3].) In the below we observe that general solutions for this equation is easily obtained.

Lemma 1. *For a positive invertible operator B and a positive one A on a Hilbert space, the solutions of the equation $X^*B^{-1}X = A$ are expressed by*

$$X = B^{1/2}WA^{1/2}$$

for some partial isometry W with the initial space $\ker A^\perp$.

Proof. Suppose the equation holds and X is a solution. Then

$$\begin{aligned} &\begin{pmatrix} 1 & O \\ -X^*B^{-1} & 1 \end{pmatrix}\begin{pmatrix} B & X \\ X^* & A \end{pmatrix}\begin{pmatrix} 1 & -B^{-1}X \\ O & 1 \end{pmatrix} \\ &= \begin{pmatrix} B & X \\ O & -X^*B^{-1}X + A \end{pmatrix}\begin{pmatrix} 1 & -B^{-1}X \\ O & 1 \end{pmatrix} = \begin{pmatrix} B & O \\ O & O \end{pmatrix}, \end{aligned}$$

and hence we have

$$\begin{pmatrix} B & X \\ X^* & A \end{pmatrix} \geq 0.$$

Then there exists a contraction W with $X = B^{1/2}WA^{1/2}$ ([2, Theo.1.1]). Since

$$A = X^*B^{-1}X = A^{1/2}W^*WA^{1/2},$$

we have W is a partial isometry with the initial space $\ker A^{\perp}$. The converse is clear. □

Moreover we see that the equation in the above lemma is essential for (2):

Theorem 2. *For a positive invertible operator B and a positive one C on a Hilbert space, the solutions of the equation*

$$C = X^*B^{-1}X - A^*X - X^*A \tag{2}$$

are expressed by

$$X = B^{1/2}W(A^*BA + C)^{1/2} + BA$$

for some partial isometry W with the initial space $\ker(A^*BA + C)^{\perp}$.

Proof. Putting $X = Y + BA$, we have

$$\begin{aligned} C &= X^*B^{-1}X - A^*X - X^*A \\ &= (Y^* + A^*B)B^{-1}(Y + BA) - A^*(Y + BA) - (Y^* + A^*B)A \\ &= Y^*B^{-1}Y + A^*BB^{-1}BA - A^*BA - A^*BA \\ &= Y^*B^{-1}Y - A^*BA, \end{aligned}$$

that is,

$$A^*BA + C = Y^*B^{-1}Y.$$

Then Lemma 1 shows the required result. □

Remark. It is clear that $X = A\#B$ is a unique positive solution of $X^*B^{-1}X = A$ and $X = (A^*BA + C)\#B + BA$ is always a solution of (2), which might be a natural view of Trapp's theorem.

To observe easily how geometric means relate the solutions of (2), we assume the invertibility for related positive operators.

Lemma 3. *For positive invertible operators A and B on a Hilbert space, the solutions of the equation $X^*B^{-1}X = A$ are expressed by*

$$X = U(A\#(U^*BU))$$

for some unitary U.

Proof. Suppose that X is a solution of $X^*B^{-1}X = A$. Lemma 1 shows that X is invertible. Thereby there exists a unitary U with $X = U|X|$. Then

$$A = |X|U^*B^{-1}U|X| = |X|(U^*BU)^{-1}|X|,$$

so that we have $|X| = A\#(U^*BU)$, that is, $X = U|X| = U(A\#(U^*BU))$. Conversely suppose $X = U(A\#(U^*BU))$ for some unitary U. Then the transformer equality shows

$$\begin{aligned} X^*B^{-1}X &= (A\#(U^*BU))(U^*BU)^{-1}(A\#(U^*BU)) \\ &= (U^*BU)^{1/2}((U^*BU)^{-1/2}A(U^*BU)^{-1/2}\#1)^2(U^*BU)^{1/2} \\ &= (U^*BU)^{1/2}(U^*BU)^{-1/2}A(U^*BU)^{-1/2}(U^*BU)^{1/2} = A. \end{aligned}$$

Thus X is a solution. □

Since $A^*BA + C$ is invertible whenever C is invertible, we have a general case:

Theorem 4. *For positive invertible operators B and C on a Hilbert space, the solutions of the equation*

$$C = X^*B^{-1}X - A^*X - X^*A \tag{2'}$$

are expressed by

$$X = U\left((A^*BA + C)\#U^*BU\right) + BA$$

for some unitary U.

REMARK. In the above case, the partial isometry W in Theorem 2 is a unitary.

In the case of matrices, since we can take a unitary U with a polar decomposition $X = U|X|$, we have a simplified results:

Corollary. *For a positive-definite matrix B and a positive-semidefinite one C, the solutions of the equation*

$$C = X^*B^{-1}X - A^*X - X^*A \tag{2}$$

are expressed by

$$X = U\left((A^*BA + C)\#U^*BU\right) + BA$$

for some unitary U.

EXAMPLE 1. The solution of (2) is not unique and U or W is nontrivial even if $BA = A^*B$

and $X = X^*$. In fact, put $A = B = I$ and $C = cI$ $(c > 0)$. Then $X = (1 \pm \sqrt{1+c})I$ is a solution of the equation $X^2 - 2X = cI$. But, for

$$W = U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we also have solutions:

$$X = I \pm \sqrt{1+c}\, U = I \pm \sqrt{1+c}\, W = \begin{pmatrix} 1 & \pm\sqrt{1+c} \\ \pm\sqrt{1+c} & 1 \end{pmatrix}.$$

EXAMPLE 2. U or W is nontrivial even if X is positive invertible. In fact, put

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix}, \quad B = I.$$

Then considering $4 - A = U|A - 4| = U\sqrt{A^*A + C}$, we have

$$X = U\left((A^*A + C)\#U^*U\right) + A = U\sqrt{A^*A + C} + A = 4 > 0$$

where

$$U = \frac{1}{\sqrt{37}} \begin{pmatrix} 6 & -1 \\ 1 & 6 \end{pmatrix}.$$

Finally we observe Riccati differential equations with the adjoint:

$$\dot{X}(t) = X(t)^*B^{-1}X(t) - A^*X(t) - X(t)^*A - C$$

under an initial condition $X(T) = O$. Unfortunately it is hard to express solutions using geometric means. But it can also be naturally solved. Here $\dot{X}(t)$ is self-adjoint, then so is $Y(t) = X(t) - X(0)$. So we have

$$\begin{aligned} \dot{X}(t) &= \dot{Y}(t) \\ &= Y(t)B^{-1}Y(t) - Y(t)(A - B^{-1}X(0)) - (A - B^{-1}X(0))^*Y(t) \\ &\qquad - (C + X(0)^*A + A^*X(0) - X(0)^*B^{-1}X(0)). \end{aligned} \tag{3}$$

As we have seen before, if a solution $Y(t)$ exists, then $Y(t) \geq O$ (resp. $Y(t) \leq O$) for all $t < T$ (resp. $t > T$). Thus we reduce it to the usual Riccati differential equation which can be solved in a well-known way. Here we sketch the solution for the case $Y(0) = O$:

Theorem 5. *For a positive invertible operator B and a positive operator C on a Hilbert space, let*

$$\dot{X}(t) = X(t)^*B^{-1}X(t) - A^*X(t) - X(t)^*A - C \tag{4}$$

be a Riccati differentiable equation. Assume

$$Q \equiv C + X(0)^*A + A^*X(0) - X(0)^*B^{-1}X(0)$$

is a positive invertible operator. If

$$H = \begin{pmatrix} A - B^{-1}X(0) & -B^{-1} \\ -Q & -(A^* - X(0)^*B^{-1}) \end{pmatrix}$$

is stable and the $(1,1)$*-element* $e^{tH}_{(1,1)}$ *is invertible for each* $t > 0$*, then a solution of (3) is given by*

$$X(t) = e^{tH}_{(2,1)}(e^{tH}_{(1,1)})^{-1} + X(0).$$

Proof. Putting a selfadjoint operator $Y(t) = X(t) - X(0)$, we note that H is the Hamiltonian for Y and $Y(0) = O$. Consider a differential equation

$$\dot{V}(t) = HV(t) \tag{5}$$

where $V(t) = \begin{pmatrix} V_1(t) \\ V_2(t) \end{pmatrix}$ and $V(0) = \begin{pmatrix} I \\ O \end{pmatrix}$ which is given by $Y(0) = O$. Then we have

$$V(t) = e^{tH}\begin{pmatrix} I \\ O \end{pmatrix}$$

and hence

$$V_1(t) = e^{tH}_{(1,1)} \quad \text{and} \quad V_2(t) = e^{tH}_{(2,1)}.$$

On the other hand, (5) implies

$$\dot{V}_1(t) = [A - B^{-1}X(0)]V_1(t) - B^{-1}V_2(t) \text{ and } \dot{V}_2(t) = -QV_1(t) - (A^* - X(0)^*B^{-1})V_2(t).$$

Putting

$$Y(t) = V_2(t)V_1(t)^{-1}, \quad \text{that is,} \quad X(t) = V_2(t)V_1(t)^{-1} + X(0),$$

we can get

$$\begin{aligned} \dot{X}(t) &= \dot{V}_2(t)V_1(t)^{-1} - V_2(t)V_1(t)^{-1}\dot{V}_1(t)V_1(t)^{-1} \\ &= -Q - (A - B^{-1}X(0))^*Y(t) - Y(t)(A - B^{-1}X(0)) + Y(t)B^{-1}Y(t) \\ &= X(t)^*B^{-1}X(t) - A^*X(t) - X(t)^*A - C \end{aligned}$$

by (3). Thus $X(t)$ is a solution of (4). □

References

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