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Kyoto University
ON RICCATI INEQUALITY
(RICCATI 不等式について)

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1. Introduction.
As stated in our preceding discussion, the algebraic Riccati equation

\[ X^*B^{-1}X - T^*X - X^*T = C \]  \(B, C\); positive definite matrices)

has solutions given by \(X = W + BT\) for some solution \(W\) of

\[ W^*B^{-1}W = C + T^*BT \]

because

\[ X^*B^{-1}X - T^*X - X^*T = W^*B^{-1}W - T^*BT. \]

Namely the operator equation

(1) \[ X^*B^{-1}X = C \]

is essential, so we call it the Riccati equation.

Related to this, we recall Ando’s definition of operator geometric mean [2]: For positive operators \(B, C\) on a Hilbert space,

(2) \[ B \# C = \max\{X \geq 0; \begin{pmatrix} B & X \\ X & C \end{pmatrix} \geq 0\}. \]

is called the geometric mean of \(B\) and \(C\). If \(B\) is invertible, it is expressed by

(3) \[ B \# C = B^{\frac{1}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{\frac{1}{4}}B^{\frac{1}{2}}. \]

It is known that \(X_0 = B \# C\) is the unique positive solution of the Riccati equation \(X^*B^{-1}X = C\), see [1,4,6,11].

Very recently, Izumino and Nakamura [5] gave some interesting consideration to weakly positive operators introduced by Wigner [12]. An operator \(T\) is weakly positive if \(T = SCS^{-1}\) for some \(S, C > 0\), where \(X > 0\) means it is positive and invertible. It is equivalent to be of form \(T = AB\) for some \(A, B > 0\). (Take \(A = S^2\) and \(B = S^{-1}CS^{-1}\).) They pointed out that the square root \(T^{\frac{1}{2}}\) of a weakly positive operator \(T = SCS^{-1} = AB\) can be defined by \(T^{\frac{1}{2}} = SC^{\frac{1}{4}}S^{-1} = A^{\frac{1}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{4}}A^{-\frac{1}{4}}\), and

\[ A^{-1} \# B = A^{-1}(AB)^{\frac{1}{2}}. \]

As an easy consequence, \(A^{-1} \# B\) is a (unique) positive solution of Riccati equation \(XAX = B\) for given \(A, B > 0\).

Inspired by Ando’s work (2) and Izumino-Nakamura’s consideration, we would like to introduce a Riccati inequality by the positivity of an operator matrix, i.e.,

\[ \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \text{ for given } B, A \geq 0 \]
2. Riccati inequality.
In this section, we investigate solutions of Riccati inequality, in which we characterize them by factorization.

The following lemma is well-known, but important.

**Lemma 1.** Let $A$ be a positive operator. Then
\[
\begin{pmatrix}
   1 & X \\
   X^* & A
\end{pmatrix} \geq 0 \text{ if and only if } A \geq X^*X.
\]

**Proof.** Since
\[
\begin{pmatrix}
   1 & 0 \\
   0 & A - X^*X
\end{pmatrix} = \begin{pmatrix}
   1 & 0 & X \\
   -X^* & 1 & A \\
   0 & 1
\end{pmatrix},
\]
it follows that
\[
\begin{pmatrix}
   1 & X \\
   X^* & A
\end{pmatrix} \geq 0 \text{ if and only if } A \geq X^*X.
\]

We here note the existence of the maximum of the geometric mean (2), as an application of an idea in Lemma 1: We may assume that $B$ is invertible. Then
\[
\begin{pmatrix}
   B & X \\
   X^* & C
\end{pmatrix} \geq 0 \text{ if and only if } C \geq X^*B^{-1}X
\]
because
\[
\begin{pmatrix}
   1 & 0 \\
   -X^*B^{-1} & 1
\end{pmatrix} \begin{pmatrix}
   B & X \\
   X^* & C
\end{pmatrix} \begin{pmatrix}
   1 & -B^{-1}X \\
   0 & 1
\end{pmatrix} = \begin{pmatrix}
   B & 0 \\
   0 & C - X^*B^{-1}X
\end{pmatrix}.
\]
Hence, if $C \geq XB^{-1}X$, then
\[
B^{-\frac{1}{2}}CB^{-\frac{1}{2}} \geq (B^{-\frac{1}{2}}XB^{-\frac{1}{2}})^2.
\]
By Lowner-Heinz inequality, we have
\[
(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^\frac{1}{2} \geq B^{-\frac{1}{2}}XB^{-\frac{1}{2}},
\]
so that
\[
B \# C = B^{\frac{1}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}} \geq X.
\]
Consequently, the maximum
\[
\max\{X \geq 0; \begin{pmatrix}
   B & X \\
   X & C
\end{pmatrix} \geq 0\} = \max\{X \geq 0; C \geq XB^{-1}X\}
\]
is given by $B \# C$.

**Lemma 2.** Let $A$ and $B$ be positive operators. Then
\[
\begin{pmatrix}
   B & W \\
   W^* & A
\end{pmatrix} \geq 0 \text{ implies } \text{ran} W \subseteq \text{ran} B^{\frac{1}{2}}.
\]
and so $X = B^{-\frac{1}{2}}W$ is well-defined as a mapping.
Proof. Let \( S = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \) be the square root of \( R = \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \). Then
\[
R = S^2 = R = \begin{pmatrix} a^2 + bb^* & ab + bd \\ b^*a + db^* & b^*b + d^2 \end{pmatrix},
\]
that is,
\[
B = a^2 + bb^* \quad \text{and} \quad W = ab + bd.
\]
Therefore \( \text{ran} \ B^{\frac{1}{2}} \) contains both \( \text{ran} \ a \) and \( \text{ran} \ b \), so that it contains \( \text{ran} \ a + \text{ran} \ b \). Moreover \( \text{ran} \ W \) is contained in \( \text{ran} \ a + \text{ran} \ b \) by \( W = ab + bd \).

Under the preparation of Lemmas 1 and 2, Riccati inequality can be solved as follows:

**Theorem 3.** Let \( A \) and \( B \) be positive operators on \( K \) and \( H \) respectively, and \( W \) be an operator from \( K \) to \( H \). Then \( \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \) if and only if
\[
W = B^{\frac{1}{2}}X \quad \text{for some operator} \quad X \quad \text{from} \quad K \quad \text{to} \quad H \quad \text{and} \quad A \geq X^*X.
\]

**Proof.** Suppose that \( \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \). Since \( \text{ran} \ W \subseteq \text{ran} \ B^{\frac{1}{2}} \) by Lemma 2, Douglas' majorization theorem [3] says that \( W = B^{\frac{1}{2}}X \) for some operator \( X \). Moreover we restrict \( X \) by \( P_B X = X \), where \( P_B \) is the range projection of \( B \). Noting that \( y \in \text{ran} \ B \) if and only if \( y = B^{\frac{1}{2}}x \) for some \( x \in \text{ran} \ B^{\frac{1}{2}} \), the assumption implies that
\[
\begin{pmatrix} P_B & X \\ X^* & A \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \geq 0
\]
for all \( y \in \text{ran} \ B \) and \( z \in K \). This means that \( \begin{pmatrix} P_B & X \\ X^* & A \end{pmatrix} \geq 0 \), and so
\[
\begin{pmatrix} P_B & 0 \\ 0 & A - X^*X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A - X^*X \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \geq 0,
\]
that is, \( A \geq X^*X \), as required. The converse is easily checked.

The following factorization theorem [2; Theorem 1.1] is led by Theorem 3 and Douglas' factorization theorem [3].

**Theorem 4.** Let \( A \) and \( B \) be positive operators. Then
\[
\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \quad \text{if and only if} \quad W = B^{\frac{1}{2}}VA^{\frac{1}{2}} \quad \text{for some contraction} \quad V.
\]

**Proof.** Suppose that \( \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \). Then it follows from Theorem 3 that \( W = B^{\frac{1}{2}}X \) for some bounded \( X \) satisfying \( A \geq X^*X \). Hence we can find a contraction \( V \) with \( X = VA^{\frac{1}{2}} \) by [3], so that \( W = B^{\frac{1}{2}}VA^{\frac{1}{2}} \) is shown.

The converse is proved by Lemma 1 as follows:
\[
\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} = \begin{pmatrix} B & B^{\frac{1}{2}}VA^{\frac{1}{2}} \\ A^{\frac{1}{2}}V^*B^{\frac{1}{2}} & A \end{pmatrix} = \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & V \\ V^* & 1 \end{pmatrix} \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} \geq 0.
\]
3. An exact expression of the harmonic mean. Recall that the harmonic mean is defined by

\[ B \frac{1}{\mathbf{C}} = \max\{X \geq 0; \begin{pmatrix} 2B & 0 \\ 0 & 2C \end{pmatrix} \preceq \begin{pmatrix} X & X \\ X & X \end{pmatrix} \}. \]

To obtain the exact expression of the harmonic mean, we need the following lemma which is more explicit than Theorem 3.

**Lemma 5.** If \( \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \succeq 0 \), then \( X = B^{-\frac{1}{2}}W \) is bounded and \( A \succeq X^*X \).

**Proof.** For a fixed vector \( x \), we put \( x_1 = B^{-\frac{1}{2}}Wx \). Since \( B^\frac{1}{2}x_1 = Wx \), we may assume \( x_1 \in (\ker B^\frac{1}{2})^\perp \). So it follows that

\[
\|B^{-\frac{1}{2}}Wx\| = \sup\{(Wx, v); \|v\| = 1\} = \sup\{|(B^{-\frac{1}{2}}Wx, B^\frac{1}{2}u)|; \|B^\frac{1}{2}u\| = 1\} = \sup\{|(Wx, u); (Bu, u) = 1\}.
\]

On the other hand, since

\[
\left( \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \begin{pmatrix} u \\ tx \end{pmatrix}, \begin{pmatrix} u \\ tx \end{pmatrix} \right) = \|t\|^2(Ax, x) + 2\text{Re }t(Wx, u) + (Bu, u) \geq 0
\]

for all scalars \( t \), we have

\[
|(Wx, u)|^2 \leq (Ax, x)(Bu, u).
\]

Hence it follows that

\[
\|B^{-\frac{1}{2}}Wx\|^2 = \sup\{|(Wx, u)|; (Bu, u) = 1\} \leq (Ax, x),
\]

which implies that \( X = B^{-\frac{1}{2}}W \) is bounded and \( A \succeq X^*X \).

**Theorem 6.** Let \( B, C \) be positive operators. Then

\[ B \frac{1}{\mathbf{C}} = 2(B - [(B + C)^{-\frac{1}{2}}B]^*[B + C]^{-\frac{1}{2}}B). \]

In particular, if \( B + C \) is invertible, then

\[ B \frac{1}{\mathbf{C}} = 2(B - B(B + C)^{-1}B) = 2B(B + C)^{-1}C. \]

**Proof.** First of all, the inequality \( \begin{pmatrix} 2B & 0 \\ 0 & 2C \end{pmatrix} \succeq \begin{pmatrix} X & X \\ X & X \end{pmatrix} \) is equivalent to

\[
\begin{pmatrix} 2(B + C) & -2B \\ -2B & 2B - X \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2B - X & -X \\ -X & 2C - X \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \succeq 0.
\]

Then it follows from Lemma 5 that \( D = [2(B + C)]^{-\frac{1}{2}}(-2B) \) is bounded and \( D^*D \leq 2B - X \). Therefore we have the explicit expression of \( B \frac{1}{\mathbf{C}} \) even if both \( B \) and \( C \) are non-invertible:

\[ B \frac{1}{\mathbf{C}} = \max\{X \geq 0; D^*D \leq 2B - X\} = 2B - D^*D. \]

In particular, if \( B + C \) is invertible, then

\[ B \frac{1}{\mathbf{C}} = 2B - D^*D = 2(B - B(B + C)^{-1}B) = 2B(B + C)^{-1}C. \]
4. Pedersen-Takesaki theorem. Finally we review a work of Pedersen and Takesaki [8] from the viewpoint of Riccati inequality; we add another equivalent condition to their theorem:

**Theorem 7.** Let $B$ and $C$ be positive operators and $B$ be nonsingular. Then the following statements are mutually equivalent:

1. Riccati equation $XBX = C$ has a positive solution.
2. $(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} \leq kB$ for some $k > 0$.
3. There exists the minimum of $\{X \geq 0; C \leq XBX\}$.

(3') There exists the minimum of $\{X \geq 0; \frac{1}{C^{\frac{1}{2}}} XBX \geq 0\}$.

**Proof.** We first note that (3) and (3') are equivalent by Lemma 1.

Now we suppose (1), i.e., $X_{0}BX_{0} = C$ for some $X_{0} \geq 0$. If $X \geq 0$ satisfies $C \leq XBX$, then

$$(B^{\frac{1}{2}}X_{0}B^{\frac{1}{2}})^{2} = B^{\frac{1}{2}}CB^{\frac{1}{2}} \leq (B^{\frac{1}{2}}XB^{\frac{1}{2}})^{2}$$

and so

$$B^{\frac{1}{2}}X_{0}B^{\frac{1}{2}} \leq B^{\frac{1}{2}}XB^{\frac{1}{2}}.$$ 

Since $B$ is nonsingular, we have $X_{0} \leq X$, namely (3) is proved.

Next we suppose (3). Since $C \leq XBX$ for some $X$, we have

$$B^{\frac{1}{2}}CB^{\frac{1}{2}} \leq (B^{\frac{1}{2}}XB^{\frac{1}{2}})^{2}$$

and so

$$(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{2} \leq B^{\frac{1}{2}}X^{2}B^{\frac{1}{2}} \leq ||X||B,$$

which shows (2).

The implication (2) $\rightarrow$ (1) has been shown by Pedersen-Takesaki [8] and Nakamoto [7], but we sketch it for convenience. By Douglas' majorization theorem [3], we have

$$(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} = ZB^{\frac{1}{2}}$$

for some $Z$, so that

$$(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} = B^{\frac{1}{2}}Z^{*}ZB^{\frac{1}{2}} \quad \text{and} \quad B^{\frac{1}{2}}CB^{\frac{1}{2}} = B^{\frac{1}{2}}(Z^{*}ZB^{\frac{1}{2}}Z)B^{\frac{1}{2}}.$$ 

Since $B$ is nonsingular, $Z^{*}Z$ is a solution of $XBX = C$.

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