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Kyoto University
ON RICCATI INEQUALITY  
(RICCATI 不等式について)

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1. Introduction.
As stated in our preceding discussion, the algebraic Riccati equation
\[ X^*B^{-1}X - T^*X - X^*T = C \quad (B,C; \text{positive definite matrices}) \]
has solutions given by \( X = W + BT \) for some solution \( W \) of
\[ W^*B^{-1}W = C + T^*BT \]
because
\[ X^*B^{-1}X - T^*X - X^*T = W^*B^{-1}W - T^*BT. \]
Namely the operator equation
(1) \[ X^*B^{-1}X = C \]
is essential, so we call it the Riccati equation.

Related to this, we recall Ando's definition of operator geometric mean [2]: For positive operators \( B, C \) on a Hilbert space,
(2) \[ B \# C = \max\{X \geq 0; \begin{pmatrix} B & X \\ X & C \end{pmatrix} \geq 0\} \]
is called the geometric mean of \( B \) and \( C \). If \( B \) is invertible, it is expressed by
(3) \[ B \# C = B^{\frac{1}{2}}(B^{-\frac{1}{4}}CB^{-\frac{1}{4}})^{\frac{1}{2}}B^{\frac{1}{2}}. \]
It is known that \( X_0 = B \# C \) is the unique positive solution of the Riccati equation \( X^*B^{-1}X = C \), see [1,4,6,11].

Very recently, Izumino and Nakamura [5] gave some interesting consideration to weakly positive operators introduced by Wigner [12]. An operator \( T \) is weakly positive if \( T = SCS^{-1} \) for some \( S, C > 0 \), where \( X > 0 \) means it is positive and invertible. It is equivalent to be of form \( T = AB \) for some \( A, B > 0 \). (Take \( A = S^2 \) and \( B = S^{-1}CS^{-1} \).) They pointed out that the square root \( T^{\frac{1}{2}} \) of a weakly positive operator \( T = SCS^{-1} = AB \) can be defined by \( T^{\frac{1}{2}} = SC^{\frac{1}{2}}S^{-1} = A^{\frac{1}{2}}(A^{\frac{1}{4}}BA^{\frac{1}{4}})^{\frac{1}{2}}A^{-\frac{1}{4}}, \) and
\[ A^{-1} \# B = A^{-1}(AB)^{\frac{1}{2}}. \]
As an easy consequence, \( A^{-1} \# B \) is a (unique) positive solution of Riccati equation \( XAX = B \) for given \( A, B > 0 \).

Inspired by Ando's work (2) and Izumino-Nakamura's consideration, we would like to introduce a Riccati inequality by the positivity of an operator matrix, i.e.,
\[ \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \quad \text{for given} \quad B, A \geq 0 \]
2. Riccati inequality.
In this section, we investigate solutions of Riccati inequality, in which we characterize them by factorization.

The following lemma is well-known, but important.

**Lemma 1.** Let $A$ be a positive operator. Then
\[
\begin{pmatrix}
1 & X \\
X^* & A
\end{pmatrix} \geq 0 \text{ if and only if } A \geq X^*X.
\]

**Proof.** Since
\[
\begin{pmatrix}
1 & 0 \\
0 & A - X^*X
\end{pmatrix} = \begin{pmatrix} 1 & X \\ -X^* & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ X^* & A \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix},
\]
it follows that
\[
\begin{pmatrix}
1 & X \\
X^* & A
\end{pmatrix} \geq 0 \text{ if and only if } A \geq X^*X.
\]

We here note the existence of the maximum of the geometric mean (2), as an application of an idea in Lemma 1: We may assume that $B$ is invertible. Then
\[
\begin{pmatrix}
B & X \\
X^* & C
\end{pmatrix} \geq 0 \text{ if and only if } C \geq X^*B^{-1}X
\]
because
\[
\begin{pmatrix}
1 & 0 \\
-X^*B^{-1} & 1
\end{pmatrix} \begin{pmatrix} B & X \\ X^* & C \end{pmatrix} \begin{pmatrix} 1 & -B^{-1}X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & C - X^*B^{-1}X \end{pmatrix}.
\]
Hence, if $C \geq XB^{-1}X$, then
\[
B^{-\frac{1}{2}}CB^{-\frac{1}{2}} \geq (B^{-\frac{1}{2}}XB^{-\frac{1}{2}})^2.
\]
By Lowner-Heinz inequality, we have
\[
(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{\frac{1}{2}} \geq B^{-\frac{1}{2}}XB^{-\frac{1}{2}},
\]
so that
\[
B \# C = B^{\frac{1}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}} \geq X.
\]
Consequently, the maximum
\[
\max\{X \geq 0; \begin{pmatrix} B & X \\ X & C \end{pmatrix} \geq 0\} = \max\{X \geq 0; C \geq XB^{-1}X\}
\]
is given by $B \# C$.

**Lemma 2.** Let $A$ and $B$ be positive operators. Then
\[
\begin{pmatrix}
B & W \\
W^* & A
\end{pmatrix} \geq 0 \text{ implies } \text{ran} W \subseteq \text{ran} B^{\frac{1}{2}}.
\]
and so $X = B^{-\frac{1}{2}}W$ is well-defined as a mapping.
Proof. Let $S = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$ be the square root of $R = \begin{pmatrix} B & W \\ W^* & A \end{pmatrix}$. Then

$$R = S^2 = R = \begin{pmatrix} a^2 + bb^* & ab + bd \\ b^*a + db^* & b^*b + d^2 \end{pmatrix},$$

that is,

$$B = a^2 + bb^*$$

and $W = ab + bd$.

Therefore ran $B^{\frac{1}{2}}$ contains both ran $a$ and ran $b$, so that it contains ran $a+ \text{ran } b$. Moreover ran $W$ is contained in ran $a+ \text{ran } b$ by $W = ab + bd$.

Under the preparation of Lemmas 1 and 2, Riccati inequality can be solved as follows:

**Theorem 3.** Let $A$ and $B$ be positive operators on $K$ and $H$ respectively, and $W$ be an operator from $K$ to $H$. Then \( \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \) if and only if $W = B^{\frac{1}{2}}X$ for some operator $X$ from $K$ to $H$ and $A \geq X^*X$.

Proof. Suppose that \( \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \). Since ran$W \subseteq \text{ran } B^{\frac{1}{2}}$ by Lemma 2, Douglas' majorization theorem [3] says that $W = B^{\frac{1}{2}}X$ for some operator $X$. Moreover we restrict $X$ by $P_B X = X$, where $P_B$ is the range projection of $B$. Noting that $y \in \text{ran } B$ if and only if $y = B^{\frac{1}{2}}x$ for some $x \in \text{ran } B^{\frac{1}{2}}$, the assumption implies that

$$\begin{pmatrix} P_B \\ X^* \\ A \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \geq 0$$

for all $y \in \text{ran } B$ and $z \in K$. This means that \( \begin{pmatrix} P_B \\ X^* \\ A \end{pmatrix} \geq 0 \), and so

$$\begin{pmatrix} P_B & 0 \\ 0 & A - X^*X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -X^* & A \end{pmatrix} \begin{pmatrix} P_B & X \\ X^* & A \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} \geq 0,$$

that is, $A \geq X^*X$, as required. The converse is easily checked.

The following factorization theorem [2; Theorem 1.1] is led by Theorem 3 and Douglas' factorization theorem [3].

**Theorem 4.** Let $A$ and $B$ be positive operators. Then \( \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \) if and only if $W = B^{\frac{1}{2}}VA^{\frac{1}{2}}$ for some contraction $V$.

Proof. Suppose that \( \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \). Then it follows from Theorem 3 that $W = B^{\frac{1}{2}}X$ for some bounded $X$ satisfying $A \geq X^*X$. Hence we can find a contraction $V$ with $X = VA^{\frac{1}{2}}$ by [3], so that $W = B^{\frac{1}{2}}VA^{\frac{1}{2}}$ is shown.

The converse is proved by Lemma 1 as follows:

$$\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} = \begin{pmatrix} B & B^{\frac{1}{2}}VA^{\frac{1}{2}} \\ A^{\frac{1}{2}}V^*B^{\frac{1}{2}} & A \end{pmatrix} = \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & V \\ V^* & 1 \end{pmatrix} \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} \geq 0.$$
3. An exact expression of the harmonic mean. Recall that the harmonic mean is defined by

\[ B \| C = \max \{ X \geq 0; \left( \begin{array}{cc} 2B & 0 \\ 0 & 2C \end{array} \right) \succeq \left( \begin{array}{cc} X & X \\ X & X \end{array} \right) \}. \]

To obtain the exact expression of the harmonic mean, we need the following lemma which is more explicit than Theorem 3.

**Lemma 5.** If \( \left( \begin{array}{cc} B & W \\ W^* & A \end{array} \right) \succeq 0 \), then \( X = B^{-\frac{1}{2}}W \) is bounded and \( A \geq X^*X \).

**Proof.** For a fixed vector \( x \), we put \( x_1 = B^{-\frac{1}{2}}Wx \). Since \( B^{\frac{1}{2}}x_1 = Wx \), we may assume \( x_1 \in \text{ker}B^{\frac{1}{2}} \). So it follows that

\[
\|B^{-\frac{1}{2}}Wx\| = \sup\{\|Wx,v\|; \|v\| = 1\} = \sup\{\|(B^{-\frac{1}{2}}Wx,B^{\frac{1}{2}}u)\|; \|B^{\frac{1}{2}}u\| = 1\} = \sup\{\|(Wx,u)\|; (Bu,u) = 1\}.
\]

On the other hand, since

\[
\left( \begin{array}{cc} B & W \\ W^* & A \end{array} \right) \left( \begin{array}{c} u \\ tx \end{array} \right), \left( \begin{array}{c} u \\ tx \end{array} \right) = |t|^2(Ax,x) + 2\text{Re}(Wx,u) + (Bu,u) \geq 0
\]

for all scalars \( t \), we have

\[
|(Wx,u)|^2 \leq (Ax,x)(Bu,u).
\]

Hence it follows that

\[
\|B^{-\frac{1}{2}}Wx\|^2 = \sup\{\|(Wx,u)\|; (Bu,u) = 1\} \leq (Ax,x),
\]

which implies that \( X = B^{-\frac{1}{2}}W \) is bounded and \( A \geq X^*X \).

**Theorem 6.** Let \( B, C \) be positive operators. Then

\[ B \| C = 2(B - [(B + C)^{-\frac{1}{2}}B]^*[B + C)^{-\frac{1}{2}}B]). \]

In particular, if \( B + C \) is invertible, then

\[ B \| C = 2(B - B(B + C)^{-1}B) = 2B(B + C)^{-1}C. \]

**Proof.** First of all, the inequality \( \left( \begin{array}{cc} 2B & 0 \\ 0 & 2C \end{array} \right) \succeq \left( \begin{array}{cc} X & X \\ X & X \end{array} \right) \) is equivalent to

\[
\left( \begin{array}{cc} 2(B+C) & -2B \\ -2B & 2B - X \end{array} \right) \succeq \left( \begin{array}{cc} -1 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} 2B - X & -X \\ -X & 2C - X \end{array} \right) \left( \begin{array}{cc} -1 & 1 \\ 1 & 0 \end{array} \right) \succeq 0.
\]

Then it follows from Lemma 5 that \( D = [2(B + C)]^{-\frac{1}{2}}(-2B) \) is bounded and \( D^*D \leq 2B - X \). Therefore we have the explicit expression of \( B \| C \) even if both \( B \) and \( C \) are non-invertible:

\[ B \| C = \max\{X \geq 0; D^*D \leq 2B - X\} = 2B - D^*D. \]

In particular, if \( B + C \) is invertible, then

\[ B \| C = 2B - D^*D = 2(B - B(B + C)^{-1}B) = 2B(B + C)^{-1}C. \]
4. Pedersen-Takesaki theorem. Finally we review a work of Pedersen and Takesaki [8] from the viewpoint of Riccati inequality; we add another equivalent condition to their theorem:

**Theorem 7.** Let $B$ and $C$ be positive operators and $B$ be nonsingular. Then the following statements are mutually equivalent:

1. Riccati equation $XBX = C$ has a positive solution.
2. $(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} \leq kB$ for some $k > 0$.
3. There exists the minimum of $\{X \geq 0; C \leq XBX\}$.

(3') There exists the minimum of $\{X \geq 0; \left(\frac{1}{C^{\frac{1}{2}}XBX} \right) \geq 0\}$.

**Proof.** We first note that (3) and (3') are equivalent by Lemma 1.

Now we suppose (1), i.e., $X_0BX_0 = C$ for some $X_0 \geq 0$. If $X \geq 0$ satisfies $C \leq XBX$, then

\[(B^{\frac{1}{2}}X_0B^{\frac{1}{2}})^2 = B^{\frac{1}{2}}CB^{\frac{1}{2}} \leq (B^{\frac{1}{2}}XBX^{\frac{1}{2}})^2\]

and so

\[B^{\frac{1}{2}}X_0B^{\frac{1}{2}} \leq B^{\frac{1}{2}}XBX^{\frac{1}{2}}.\]

Since $B$ is nonsingular, we have $X_0 \leq X$, namely (3) is proved.

Next we suppose (3). Since $C \leq XBX$ for some $X$, we have

\[B^{\frac{1}{2}}CB^{\frac{1}{2}} \leq (B^{\frac{1}{2}}XBX^{\frac{1}{2}})^2\]

and so

\[(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} \leq B^{\frac{1}{2}}XBX^{\frac{1}{2}} \leq \|X\|B,\]

which shows (2).

The implication (2) $\rightarrow$ (1) has been shown by Pedersen-Takesaki [8] and Nakamoto [7], but we sketch it for convenience. By Douglas' majorization theorem [3], we have

\[(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} = ZB^{\frac{1}{2}}\]

for some $Z$, so that

\[(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} = B^{\frac{1}{2}}Z^{*}ZB^{\frac{1}{2}} \quad \text{and} \quad B^{\frac{1}{2}}CB^{\frac{1}{2}} = B^{\frac{1}{2}}(Z^{*}ZBZ^{*}Z)B^{\frac{1}{2}}.\]

Since $B$ is nonsingular, $Z^{*}Z$ is a solution of $XBX = C$.

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REFERENCES


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