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作用素の q-正定値性と q-正規拡大について
(On q-positive definiteness and q-normal extensions of an operator)

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調和振動子に現れる生成作用素 $S=\frac{1}{\sqrt{2}}\left(x-\frac{\mathrm{d}}{\mathrm{dx}}\right)$ は Segal-Bargmann 空間上の独立変数を掛ける積作用素にユニタリー同値になる (V. Bargmann [1961])。よって、S はより広いヒルベルト空間上の正規作用素に拡大される。近年、量子群の理論に関連して、多様な q-調和振動子が調べられている。その中のひとつに、$S^*S-qSS^*=1$ がある。一方、作用素に関しても 1 次元量子平面の生成元や Wess グループにより導入された q-ハイゼンベルグ代数の生成元に現れる要素や関係式等から、通常の正規作用素等に対応して q-正規作用素等の変形作用素が導入され調べられている。この講演では上記 q-調和振動子に現れる q-生成作用素の q-正規拡大に関連して、作用素に「q-正定値性」の概念を導入し、作用素の q- 正規拡大に関して得られたいくつかの結果を報告する。

1 q-formally normal operators

Let q be a real positive number such that $q\neq 1$. For operators S and T in $\mathcal{H}$, the relation $S\subseteq T$ means that $\mathcal{D}(S)\subseteq\mathcal{D}(T)$ and $S\eta=T\eta$ for all $\eta\in\mathcal{D}(S)$.

Let T be a closed densely defined operator in $\mathcal{H}$. If T satisfies

$$TT^*=qT^*T,$$

then T is called a *deformed normal operator with deformation parameter* q and we will simply say T is **q-normal**. The q-normality of a closed densely defined operator T is equivalent to the conditions:

$$\mathcal{D}(T)=\mathcal{D}(T^*)\quad\text{and}\quad\|T^*\eta\|=\sqrt{q}\|T\eta\|\quad(\eta\in\mathcal{D}(T)).$$

A densely defined operator T is called a **q-hyponormal** if it satisfies

$$\mathcal{D}(T)\subseteq\mathcal{D}(T^*)\quad\text{and}\quad\|T^*\eta\|\leq\sqrt{q}\,\|T\eta\|$$

for all $\eta \in \mathcal{D}(T)$. If a q-hyponormal operator satisfies

$$\|T^*\eta\| = \sqrt{q}\,\|T\eta\|$$

for all $\eta \in \mathcal{D}(T)$, then T is called **q-formally normal**. Let T be a closed densely defined operator in $\mathcal{H}$ with polar decomposition $T = U|T|$. If T satisfies the equality $U|T| = \sqrt{q}\,|T|U$, then T is called a **q-quasinormal operator**.

$$\begin{array}{ccccc} & \nearrow & q\text{-formally normal} & \searrow & \\ q\text{-normal} & & & & q\text{-hyponormal} \\ & \searrow & q\text{-quasinormal} & \nearrow & \end{array}$$

Let T be a q-hyponormal operator in $\mathcal{H}$. Then there exists uniquely a contraction K_T such that

$$T^* \supseteq \sqrt{q} K_T T \quad \text{and} \quad \ker K_T \supseteq \ker T^* .$$

If T is q-formally normal, then K_T is a partial isometry with the initial space $\overline{\mathcal{R}}(T)$ and the final space $\overline{\mathcal{R}}(T^*|_{\mathcal{D}(T)})$.

命題 1 *A non-trivial q-formally normal operator is unbounded.*

命題 2 *If a closed q-formally normal operator T is q-quasinormal, then T is q-normal.*

命題 3 *Let T be a q-formally normal operator in a Hilbert space. If the domain $\mathcal{D}(T)$ is a core for T^*, then the closure $\tilde{T}$ is q-normal.*

定理 4 *Let T be a q-formally normal operator in a Hilbert space $\mathcal{H}$. If K_T is unitary, T is injective and the inverse T^{-1} is also q-formally normal.*

系 5 *The spectrum of a q-formally normal operator in $\mathcal{H}$ must contain zero if $\mathcal{R}(T^*|_{\mathcal{D}(T)})$ is dense in $\mathcal{H}$.*

Proof. Assume T has a bounded inverse T^{-1}. Then, T^{-1} is also q-formally normal. This is a contradiction since T^{-1} must be unbounded. □

Every q-quasinormal operator has a q-normal extension in a possibly larger Hilbert space.

命題 6 *If a densely defined operator T has a q-formally normal extension in a possibly larger Hilbert space, then T is q-hyponormal.*

2 q-positive definiteness

Suppose a densely defined operator S in $\mathcal{H}$ has invariant domain, namely; $S\mathcal{D}(S) \subseteq \mathcal{D}(S)$. Then, **$q$-positive definite condition** means that

$$\sum_{i,j=0}^{n} q^{ij} \langle S^i f_j \,, S^j f_i \rangle \geqq 0 \qquad \textbf{(q-PD)}$$

for $f_0, ..., f_n \in \mathcal{D}(S)$, $n \in \mathbb{N}_0 = \{0, 1, ...\}$.

例 7 (q-oscillator) *Suppose S is closable in $\mathcal{H}$. If $\mathcal{D}(S)$ is invariant for S and S^* and such that*

$$S^*S - qSS^* = 1 \quad \text{on} \quad \mathcal{D}(S),$$

then S satisfies (q^{-1}-PD).

命題 8 *Suppose S has invariant domain. If S has a q-formally normal extension N in a possibly larger Hilbert space such that*

$$N\mathcal{D}(N) \subseteq \mathcal{D}(N) \quad \text{and} \quad N^*\mathcal{D}(N) \subseteq \mathcal{D}(N), \qquad (\star)$$

then it satisfies the q-positive definite condition (q-PD).

Proof. For $f_0, ..., f_n \in \mathcal{D}(S)$, $n \in \mathbb{N}_0$,

$$\begin{aligned}\sum_{i,j=0}^{n} q^{ij}\langle S^i f_j \,, S^j f_i \rangle &= \sum_{i,j=0}^{n} q^{ij}\langle N^i f_j \,, N^j f_i \rangle \\ &= \sum_{i,j=0}^{n}\langle N^{*^j} f_j \,, N^{*^i} f_i \rangle \\ &= \|\sum_{i=0}^{n} N^{*^i} f_i\|^2\end{aligned}$$

□

例 9 *Let $\mathcal{H}$ be a separable Hilbert space and $\{e_n\}_{n\in\mathcal{N}_0}$ be an orthonormal basis of $\mathcal{H}$. Define an operator S in $\mathcal{H}$ by*

$$\mathcal{D}(S) = \textit{linear span of } \{e_n : n \in \mathcal{N}_0\}$$

and

$$Se_n = \left(\frac{1}{\sqrt{q}}\right)^n e_{n+1}$$

for all $n \in \mathcal{N}_0$.

定理 10 *Suppose S in $\mathcal{H}$ has invariant domain. If S satisfies the q-positive definite condition* (q-PD), then S has a q-formally normal extension N in a possibly larger Hilbert space such that the condition $(\star)$ holds.

Proof. Define $K : (\mathbb{N}_0 \times \mathcal{D}(S)) \times (\mathbb{N}_0 \times \mathcal{D}(S)) \to \mathbb{C}$ by

$$K((m,x),(n,y)) = q^{mn}\langle S^n x \,, S^m y \rangle$$

for $x, y \in \mathcal{D}(S)$ and $m, n \in \mathbb{N}_0$.

- K is a positive definite kernel on $\mathbb{N}_0 \times \mathcal{D}(S)$. The corresponding R.K.H.S is denoted by $\mathcal{K}$.

- Define

  $$K_{(m,x)}(n,y) = K((m,x),(n,y)) \quad \text{and put}$$

  $$\mathcal{D} \equiv \text{linear span of}\{K_{(m,x)} : m \in \mathbb{N}_0, x \in \mathcal{D}(S)\}$$

  Then, $\mathcal{D}$ is dense in $\mathcal{K}$ and $\langle K_{(m,x)} \,, K_{(n,y)} \rangle_{\mathcal{K}} = K_{(m,x)}(n,y)$.

- Define an operator N on $\mathcal{D} \equiv \mathcal{D}(N)$ by

$$N\,K_{(m,x)} = q^m K_{(m,Sx)} \qquad m \in \mathbb{N}_0,\ x \in \mathcal{D}(S).$$

Then,

1. $\mathcal{D} \subsetneqq \mathcal{D}(N^*)$ and $N^* K_{(n,y)} = K_{(n+1,y)}$
2. $\langle\, N\,K_{(m,x)}\,, N\,K_{(n,y)}\,\rangle_{\mathcal{K}} = q^{mn+m+n} \langle\, S^{n+1}x\,, S^{m+1}y\,\rangle_{\mathcal{H}}$.
3. $\langle\, N^*\,K_{(m,x)}\,, N^*\,K_{(n,y)}\,\rangle_{\mathcal{K}} = q^{(m+1)(n+1)} \langle\, S^{n+1}x\,, S^{m+1}y\,\rangle_{\mathcal{H}}$.

Finally, Define an isometry $V : \mathcal{D}(S) \ni x \to K_{(0,x)} \in \mathcal{K}$ (and it is extended to $\mathcal{H}$). By $VS \subseteq NV$, N extend S. □

注意 11

$$\mathcal{D}(N) = \textit{linear span of} \left\{ N^{*^n} f : f \in \mathcal{D}(S) \right\}.$$

定理 12 *Suppose S has invariant domain. If S satisfies the q-positive definite condition* (q-PD), then S satisfies the (q'-PD) for every $q' > q$.

Proof. This follows from the fact (by Man-Duen Choi): "If $q > 1$, then the matrix $(q^{ij})_{i,j=0}^{n}$ is positive semi-definite for any $n \in \mathbb{N}_0$".

系 13 *Suppose S has invariant domain. If $q > 1$, then for S to satisfy the positive definite condition* (1-PD), implies S to satisfy the (q-PD).

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