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ON WEAK* CONTINUOUS CHARACTERS AND WEAK SPECTRA

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Abstract

In this note we discuss the weak spectra of operators via weak*-continuous characters on a given singly generated dual algebra of operators on Hilbert space. In particular, the results unify some known examples, and is shown that for a certain class of such algebras, the set of such characters is empty.

1. Introduction. This is based on the joint work with B. Chevreau, E. Ko, and C. Pearcy ([7]) and was talked at the 2006 RIMS conference: Recent developments in theory of linear operators and its applications, which was held at Kyoto University on October 11-13 in 2006.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$ we write, as usual, $\sigma(T)$, $\sigma_p(T)$, and $\sigma_e(T)$ for the spectrum, point spectrum, and essential spectrum of T, respectively, $r(T)$ for the spectral radius of T, and $W(T)$ for the numerical range of T. We denote the kernel and range of T, as usual, by $\ker(T)$ and $\operatorname{ran}(T)$. We also denote by $\mathbb{D}$ the open unit disc $\{\zeta : |\zeta| < 1\}$ in the complex plane $\mathbb{C}$, set $\mathbb{T} := \partial\mathbb{D}$, and write $H^\infty(\mathbb{D})$, as usual, for the Banach algebra of bounded holomorphic functions on $\mathbb{D}$. If $K \neq \emptyset$ is a compact set in $\mathbb{C}$ then the (closed) convex hull of K will be denoted by $\operatorname{conh}(K)$ and the outer boundary of K (i.e., $\partial(\mathbb{C}\backslash K)$), by $\partial^\infty K$. The unbounded component of $\mathbb{C}\backslash K$ will be written as $\operatorname{unbd}(\mathbb{C}\backslash K)$. A subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ that contains $1_{\mathcal{H}}$ and is closed in the weak* topology on $\mathcal{L}(\mathcal{H})$ is called a *dual algebra*, and the dual algebra generated by a single operator T in $\mathcal{L}(\mathcal{H})$ is denoted by $\mathcal{A}_T$. It follows from general principles (cf., e.g., [3]) that if $\mathcal{A}$ is a dual algebra, then $\mathcal{A}$ can be identified with the dual space of the quotient space $Q_{\mathcal{A}} = \mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}$, where ${}^{\perp}\mathcal{A}$ is the preannihilator of $\mathcal{A}$ in $\mathcal{C}_1(\mathcal{H})$, under the pairing $\langle T, [L]\rangle = \operatorname{tr}(TL)$, $T \in \mathcal{A}$, $[L] \in Q_{\mathcal{A}}$, where, of course, $[L]$ is the coset in $Q_{\mathcal{A}}$ containing the operator $L \in \mathcal{C}_1(\mathcal{H})$. In particular, if x and y are nonzero vectors in $\mathcal{H}$, then the rank-one operator $x \otimes y$, defined by $(x \otimes y)(u) = (u, y)x$, $u \in \mathcal{H}$, belongs to $\mathcal{C}_1(\mathcal{H})$, so $[x \otimes y]$ denotes the image of $x \otimes y$ in the quotient space $Q_{\mathcal{A}}$. For brevity we write Q_T for the predual $Q_{\mathcal{A}_T}$. Recall that a *weak***-continuous character* on

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a dual algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is by definition, a multiplicative linear functional $\varphi \in \mathcal{A}^*$ that is weak* continuous and satisfies $\varphi(1_{\mathcal{H}}) = 1$.

The purpose of this note is to study the properties of the collection of weak* continuous characters on a dual algebra $\mathcal{A}_T$. This enables us to generalize some results of Cassier (Theorem 2.1 e), g) and i) below), who has made a penetrating study of this topic [4], [5], [6] in his study of uniform dual algebras, and to unify what, until now, have seemed to be some disparate examples.

2. Some known results. If we denote the maximal ideal space of such a dual algebra by $\mathrm{Max}(\mathcal{A}_T)$, then obviously the set $\mathcal{C}_w(\mathcal{A}_T)$ of weak* continuous characters on $\mathcal{A}_T$ can be identified with $\mathrm{Max}(\mathcal{A}_T) \cap Q_T$, which we call the *weak* character space* of $\mathcal{A}_T$. As noted above, $\mathcal{C}_w(\mathcal{A}_T)$ may be empty, but it is easy to see that in any case, $\mathcal{C}_w(\mathcal{A}_T)$ is a weakly closed subset of Q_T. Note that if $\varphi \in \mathcal{C}_w(\mathcal{A}_T)$, then φ is completely determined by its value $\varphi(T) = \lambda_\varphi$. Thus there is a 1-1 mapping $\varphi \to \varphi(T) = \lambda_\varphi$ of $\mathcal{C}_w(\mathcal{A}_T)$ onto the set $\sigma^*(T) := \{\lambda_\varphi \in \mathbb{C} : \varphi \in \mathcal{C}_w(\mathcal{A}_T)\}$, which was introduced by Cassier [4] and called the *weak spectrum* of T. Clearly turns $\sigma^*(T)$ into a complete metric space, even though, as is seen below, $\sigma^*(T)$ need not be closed as a subset of $\mathbb{C}$. There are various relations known between $\sigma^*(T)$ and $\sigma(T)$, and to review some of these, we need a bit more notation. We write $\mathcal{L}(\mathcal{A}_T)$ $[\mathcal{L}(Q_T)]$ for the algebra of all bounded linear operators on the Banach space $\mathcal{A}_T$ $[Q_T]$, $M_T[m_T]$ for the operator in $\mathcal{L}(\mathcal{A}_T)[\mathcal{L}(Q_T)]$ defined by $M_T(A) = AT$ $(= TA)$, $A \in \mathcal{A}_T$, $[m_T([L]) = [LT]$ $(= [TL])$, $[L] \in Q_T]$, and $\sigma_{\mathcal{A}_T}(T) = \{\varphi(T) : \varphi \in \mathrm{Max}(\mathcal{A}_T)\} \supset \sigma(T)$ for the spectrum of T as an element of the unital Banach algebra $\mathcal{A}_T$. Recall from general principles that $\partial\sigma_{\mathcal{A}_T}(T) \subset \partial\sigma(T)$, and thus that $\sigma_{\mathcal{A}_T}(T)$ consists of $\sigma(T)$ together with some of its holes (i.e., bounded components of $\mathbb{C}\backslash\sigma(T)$). Note also that for every $0 \neq \lambda \in \mathbb{C}$, $\mathcal{A}_T = \mathcal{A}_{T-\lambda} = \mathcal{A}_{\lambda T}$, so $\mathcal{C}_w(\mathcal{A}_T) = \mathcal{C}_w(\mathcal{A}_{T-\lambda}) = \mathcal{C}_w(\mathcal{A}_{\lambda T})$. In other words, $\mathcal{C}_w(\mathcal{A}_T)$ does not depend on which particular generator for $\mathcal{A}_T$ is singled out, but $\sigma^*(T)$ is related to $\sigma^*(T-\lambda)$ and $\sigma^*(\lambda T)$ as in d) below.

Parts a)-d) of the following theorem are essentially elementary and parts e)-i) were proved by Cassier in the articles cited above.

Theorem 2.1. *For every operator T in $\mathcal{L}(\mathcal{H})$, the following are valid:*

a) $\sigma_p(T) \cup (\sigma_p(T^*))^* \subset \sigma^*(T)$,

b) $\lambda \in \sigma^*(T) \iff \{(T - \lambda 1_{\mathcal{H}})\mathcal{A}_T\}^{-w^*} (= \{(M_T - \lambda 1_{\mathcal{A}_T})\mathcal{A}_T\}^{-w^*}) \neq \mathcal{A}_T$
$\iff \ker(m_T - \lambda 1_{Q_T}) \neq 0$,

c) *for every invertible S in $\mathcal{L}(\mathcal{H})$, $\sigma^*(T) = \sigma^*(STS^{-1})$,*

d) *for every $0 \neq \lambda \in \mathbb{C}$, $\sigma^*(T - \lambda) = \sigma^*(T) - \lambda$ and $\sigma^*(\lambda T) = \lambda\sigma^*(T)$,*

e) $\sigma^*(T) \cap \{\zeta \in \mathbb{C} : |\zeta| = \|T\|\} \subset \sigma_p(T)$,

f) *$\partial\sigma^*(T) \subset \sigma(T)$, which implies that $\sigma^*(T)$ is a subset of the union of $\sigma(T)$ with its holes (i.e., the polynomial hull of $\sigma(T)$).*

g) *if $\lambda \in \mathbb{C}\backslash\sigma^*(T)$, then either $T - \lambda$ is not a semiFredholm operator or $\lambda \notin \sigma_{\mathcal{A}_T}(T)$,*

h) *if $\mathcal{J}$ is a simple closed Jordan curve in $\mathbb{C}$ and $Int(\mathcal{J})$ denotes the interior domain of $\mathcal{J}$ given by the Jordan curve theorem, then $\mathcal{J} \subset \sigma^*(T)^\circ \implies Int(\mathcal{J}) \subset \sigma^*(T)^\circ$, and*

i) *if $\mathcal{A}_T$ is a uniform dual algebra (i.e., the Gelfand map of $\mathcal{A}_T$ into the space $C(Max(\mathcal{A}_T))$ of continuous functions on $Max(\mathcal{A}_T)$ is an isometry), then $\sigma(T) \cup \sigma^*(T) = \sigma_{\mathcal{A}_T}(T)$.*

3. Some new results. Recall that a subspace $\mathcal{M} \subset \mathcal{H}$ is called a *semi-invariant subspace* for $T \in \mathcal{L}(\mathcal{H})$ if there exist invariant subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ for T with $\mathcal{N}_2 \subset \mathcal{N}_1$ such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2$. Relative to the decomposition $\mathcal{H} = \mathcal{N}_2 \oplus \mathcal{M} \oplus \mathcal{N}_1^{\perp}$, $T_{\mathcal{M}} \in \mathcal{L}(\mathcal{M})$ is defined by $T_{\mathcal{M}}x = P_{\mathcal{M}}Tx$, $x \in \mathcal{M}$,(with $P_{\mathcal{M}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$). The map $A \to A_{\mathcal{M}}$ defined on $\mathcal{A}_T$ is clearly a weak* continuous algebra homomorphism of $\mathcal{A}_T$ into $\mathcal{A}_{T_{\mathcal{M}}}$. Thus we obtain, by composing the appropriate maps, the following.

Proposition 3.1. *If $T \in \mathcal{L}(\mathcal{H})$ and $T_{\mathcal{M}}$ is the compression of T to a semi-invariant subspace $\mathcal{M}$, then $\sigma^*(T_{\mathcal{M}}) \subset \sigma^*(T)$.*

Corollary 3.2. *If $T_1 \oplus T_2 \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, then $\sigma^*(T_1) \cup \sigma^*(T_2) \subset \sigma^*(T_1 \oplus T_2)$, but equality need not hold. However, if $\mathcal{A}_{T_1 \oplus T_2} = \mathcal{A}_{T_1} \oplus \mathcal{A}_{T_2}$ which happens (at least) whenever $\sigma(T_2) \subset$* unbd$(\mathbb{C}\backslash\sigma(T_1))$ *(or, equivalently, $\sigma(T_1) \subset$* unbd$(\mathbb{C}\backslash\sigma(T_2))$*), then $\sigma^*(T_1 \oplus T_2) = \sigma^*(T_1) \cup \sigma^*(T_2)$.*

The following corollary of Proposition 3.1 has been known for some time.

Corollary 3.3. *For every absolutely continuous contraction $T \in \mathcal{L}(\mathcal{H})$ such that the Sz.-Nagy-Foiaş functional calculus $H^\infty(\mathbb{D}) \to \mathcal{A}_T$ is an isometry, $\sigma^*(T) = \mathbb{D}$.*

The following also seems to be new.

Theorem 3.4. *Suppose $T \in \mathcal{L}(\mathcal{H})$. Then for every $\lambda \in \mathbb{C}\backslash\sigma_l(T)$ (the complement of the left spectrum of T), the following are equivalent:*

a) $\lambda \in \sigma^*(T)$,

b) $m_T - \lambda (= m_T - \lambda 1_{Q_T})$ *is a Fredholm operator in $\mathcal{L}(Q_T)$ with index* 1,

c) $M_T - \lambda (= M_T - \lambda 1_{\mathcal{A}_T})$ *is a Fredholm operator in $\mathcal{L}(\mathcal{A}_T)$ with index* -1.

The following corollary generalizes Theorem 2.1 g) and i).

Corollary 3.5. *For every T in $\mathcal{L}(\mathcal{H})$, $\sigma_{\mathcal{A}_T}(T) = \sigma_l(T) \cup \sigma^*(T)$.*

The following contains another new idea.

Theorem 3.6. *Suppose $T \in \mathcal{L}(\mathcal{H})$, $\lambda_0 \in \mathbb{C}\backslash(\sigma_p(T) \cup \sigma_p(T^*)^*)$, and there exist a number $K > 0$ and a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ lying in* unbd$(\mathbb{C}\backslash\sigma(T))$ *such that $\lambda_n \to \lambda_0$ and $\|(T-\lambda_n)^{-1}\| \le K/|\lambda_n - \lambda_0|$, $n \in \mathbb{N}$. Then $\lambda_0 \notin \sigma^*(T)$.*

The following is a consequence of Theorem 3.6 and generalizes Theorem 2.1 e).

Theorem 3.7. *For every T in $\mathcal{L}(\mathcal{H})$, the set $\partial(W(T))\backslash(\sigma_p(T) \cup \sigma_p(T^*)^*)$ does not intersect $\sigma^*(T)$.*

Corollary 3.8. *Suppose T is a quasinilpotent quasiaffinity in $\mathcal{L}(\mathcal{H})$ and some closed half-plane H determined by a line through the origin contains the numerical range $W(T)$ of T. Then $C_w(\mathcal{A}_T) = \sigma^*(T) = \emptyset$.*

Proposition 3.9. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is an absolutely continuous contraction, $\varphi \in C_w(\mathcal{A}_T)$,* and *$\lambda_\varphi = \varphi(T)$. Then $\lambda_\varphi \in \mathbb{D}$ and for every $f \in H^\infty(\mathbb{D})$, $\varphi(f(T)) = f(\lambda_\varphi)$. (Here $f(T)$ is given by the Sz.-Nagy-Foiaş functional calculus.)*

Corollary 3.10. *Let T be any C_0-contraction in $\mathcal{L}(\mathcal{H})$ such that the minimal function m of T does not vanish on $\mathbb{D}$ (for definitions and examples, see* [1] *or* [2]*). Then $C_w(\mathcal{A}_T) = \sigma^*(T) = \emptyset$.*

For completeness, we include here the following known result.

Proposition 3.11 (Cassier). *Suppose N is a normal operator in $\mathcal{L}(\mathcal{H})$ without point spectrum such that $N^* \in \mathcal{A}_N$ (which happens, of course, if $\sigma(N)^\circ = \emptyset$ and $\sigma(N)$ doesn't separate the plane). Then $\mathcal{C}_w(\mathcal{A}_N) = \sigma^*(N) = \emptyset$.*

Finally, we close this note with some open problems. The one most pertinent to the invariant subspace problem is the following.

Problem 3.12. If $T \in \mathcal{L}(\mathcal{H})$ and $\sigma(T)$ contains a nonempty open set, must $\mathcal{C}_w(\mathcal{A}_T)$ be nonvoid?

Problem 3.13. If T is a completely nonunitary contraction in $\mathcal{L}(\mathcal{H})$ and $\sigma(T)$ contains a nonempty open set, must every $A \in \mathcal{A}_T$ such that $A \neq \lambda 1_{\mathcal{H}}$ satisfy $\sigma(A)^\circ \neq \emptyset$? (In this connection, using a transfinite induction argument, one sees that it is enough to show that every $A \in \mathcal{A}_T$ which is a weak* limit of a sequence $\{p_n(T)\}_{n\in\mathbb{N}}$ of polynomials has this property.)

Problem 3.14. For an operator T in $\mathcal{L}(\mathcal{H})$ with connected spectrum and with $\sigma_p(T) \cup \sigma_p(T^*)^* = \emptyset$, is $\sigma^*(T)$ always either an open set or a closed set? Can $\sigma^*(T)$ of such a T be a circle? (With respect to the first question, we note that without the hypothesis that $\sigma(T)$ is connected, the answer is obviously no by Corollary 3.2, Corollary 3.3.)

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