数理解析研究所講究録
1536 巻 2007 年 172-184

Scattering problem for nonlinear Klein-Gordon equations

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1 Introduction

In this note, we will survey a series of recent joint works with P.I. Naumkin, [11], [12], [13], [14] for the nonlinear Klein-Gordon equation with a power nonlinearity

$$u_{tt}-\Delta u+u=\mu|u|^{p-1}u,\ \ (t,x)\in\mathbf{R}\times\mathbf{R}^{n} \tag{1}$$

where $p\geq 3,\mu\in\mathbf{R}$ for $n=1,p>1+\frac{2}{n}$, $\mu\in\mathbf{C}$ for $n=1$ and $p>1+\frac{4}{n+2}$, $\mu\in\mathbf{C}$ for $n\geq 2$. When $\mu<0$, and $1+\frac{4}{n}<p<p^{*}(n)$, where $p^{*}(n)=\infty$ for $n=1,2$, $p^{*}(n)=\frac{n+2}{n-2}$ for $n\geq 3$, the completeness of the scattering operator for the nonlinear Klein-Gordon equation (1) in the energy space was established in papers [1], [2], [9], [26], [27] by using the Morawetz type estimates and the energy conservation law for $n\geq 3$. This result was extended in [24] to lower space dimensions $n=1,2$. The condition $\mu<0$ can be removed (see [29]) in the case of small initial data. If we are interested in the global in time existence of solutions to the Cauchy problem for the nonlinear Klein-Gordon equation, it was shown in [29] for the case of $p_{0}(n)<p\leq 1+\frac{4}{n}$ by $\mathbf{L}^{p}-\mathbf{L}^{q}$ time decay estimate of the fundamental solution obtained in [20], where $p_{0}(n)$ is a positive root of $\frac{n}{2}\frac{p-1}{p+1}p>1$. When $n=3,p=2$, $\frac{n}{2}\frac{p-1}{p+1}p=1$, then the $\mathbf{L}^{p}-\mathbf{L}^{q}$ time decay estimates from [20] can not be applied to the Cauchy problem (1) even if the nonlinearity is smooth. In the case of the Cauchy problem, the lower order p was treated in papers [16], [28] and the global existence of small solutions to the quadratic nonlinear Klein-Gordon equations in three space dimensions was studied by the method of the vector fields and the method of normal forms, respectively. In [6], the vector fields method was refined and applied for the case of $p>1+\frac{2}{n}$, $n=1,2,3$, $\mu\in\mathbf{C}$ under the condition that the initial data have a compact support (see also [19]). The Cauchy problem (1) for $n=1$ with a cubic nonlinearity ($p=3$) was studied by [7], where the sharp $\mathbf{L}^{\infty}$ - time decay estimates and nonexistence of the inverse wave operator were obtained. The asymptotic profile of small solutions to (1) for $p=3$ and $n=1$ was found in [3] for the case of regular data having a compact support. These methods do not work for the power nonlinearity of (1) since the nonlinearity is not regular enough. The quadratic nonlinear Klein-Gordon equation for two space dimensions was considered in papers [25] and [4], where the global existence of small solutions was proved (see [25]) by using the method of normal forms [28] and the time decay estimates of linear evolution group [5], and the sharp asymptotic behavior of small solutions was found (see [4]) by virtue of the vector fields method [16]. Note that the critical nonlinearity $|u|\,u$ was out of the scope of these works since it is not smooth. Furthermore, the Cauchy problem (1) with cubic nonlinearities depending on $u, u_t, u_x, u_{xx}, u_{tx}$ was studied in the one dimensional case by [23], [22], [15], where the existence of small solutions in the neighborhood of the free solutions was proved , when the nonlinearity has some special structure, and if the initial data are small, regular and decay rapidly at infinity. Thus we can see that the cubic nonlinearity is not necessarily critical in the one dimensional case (the critical nonlinearity $|u|^2 u$ was excluded there). If the data are small, regular and have a compact support, then sufficient conditions on the cubic nonlinearities which admit global existence and asymptotic behavior of small solutions were given in [3]. This result was generalized in [31] to the cubic nonlinearities including dissipative terms, such as $-|u_t|^2 u_t$. The nonlinearity $|u|^2 u$ was included in these papers, so that the asymptotic profile differs from the free one. Cubic nonlinearity $|u|^2 u$ was considered also in paper [30], where the sufficient conditions on the complex initial data were found which yield global existence and the uniform time decay of order $t^{-\frac{1}{2}}$. However the asymptotic behavior of small solutions was not given for the case of complex data. Recently the asymptotic behavior of solutions to the nonlinear Klein-Gordon equation with cubic nonlinearity was considered in papers [17], [18] by applying the hyperbolic polar coordinates of [16], which implicitly assume a compact support for the solutions (as in papers [3], [4], [6], [7], [16], [30], [31]), therefore these methods are not acceptable for the final state problem. Whereas many works are devoted to the Cauchy problem (1), there are few results on the final state problem.

We put

$$w \equiv \frac{1}{2}\left(\mathbf{a}u + i\mathbf{b}\langle i\nabla\rangle^{-1} u_t\right), \; w^0 \equiv \frac{1}{2}\left(\mathbf{a}u_0 + i\mathbf{b}\langle i\nabla\rangle^{-1} u_1\right),$$

$$\mathcal{L} = E\partial_t + iA\langle i\nabla\rangle$$

and

$$\mathcal{N}(w) = \frac{i\mu}{2}\mathbf{b}\langle i\nabla\rangle^{-1}|(\mathbf{a}\cdot w)|^{p-1}(\mathbf{a}\cdot w),$$

where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the nonlinear Klein-Gordon equation (1) can be rewritten as a system of equations

$$\left\{ \begin{array}{l} \mathcal{L}w = \mathcal{N}(w), \;\; (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\ w(0,x) = w^0(x), \; x \in \mathbf{R}^n. \end{array} \right. \tag{2}$$

The direct Fourier transform $\hat{\phi}(\xi)$ of the function $\phi(x)$ is defined by

$$\mathcal{F}\phi = \hat{\phi} = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-i(x\cdot\xi)}\phi(x)\,dx,$$

then the inverse Fourier transformation is given by

$$\mathcal{F}^{-1}\phi = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i(x\cdot\xi)}\phi(\xi)\,d\xi.$$

Denote the usual Lebesgue space $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbf{R}^n} |\phi(x)|^p\,dx\right)^{\frac{1}{p}}$ if $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \text{ess.sup}_{x\in\mathbf{R}^n} |\phi(x)|$ if $p = \infty$. Weighted Sobolev space is

$$\mathbf{H}_p^{m,k} = \left\{\phi \in \mathbf{S}' : \|\phi\|_{\mathbf{H}_p^{m,k}} \equiv \left\|\langle x\rangle^k \langle i\partial\rangle^m \phi\right\|_{\mathbf{L}^p} < \infty\right\},$$

where $m, k \in \mathbf{R}$, $1 \le p \le \infty$, $\langle x\rangle = \sqrt{1+|x|^2}$. We also write $\mathbf{H}^{m,k} = \mathbf{H}_2^{m,k}$. The usual Sobolev space is $\mathbf{H}^m = \mathbf{H}_2^{m,0}$, so the index 0 we usually omit if it does not cause a confusion. Different positive constants we denote by the same letter C.

We introduce the free evolution group

$$U(t) = \begin{pmatrix} e^{-i\langle i\nabla\rangle t} & 0 \\ 0 & e^{i\langle i\nabla\rangle t} \end{pmatrix}.$$

The operator

$$\mathcal{J} = \langle i\nabla\rangle U(t)\,xU(-t) = E\langle i\nabla\rangle x + iAt\nabla$$

is useful for obtaining the large time decay estimates of solutions. Since $[x, \langle i\nabla\rangle^\alpha] = \alpha\langle i\nabla\rangle^{\alpha-2}\nabla$ it is easy to check that $\mathcal{J}$ commutes with $\mathcal{L}$, i.e. $[\mathcal{L},\mathcal{J}] = 0$, however it is difficult to calculate the action of $\mathcal{J}$ on the nonlinearity $\mathcal{N}$. Therefore we use the first order differential operator

$$\mathcal{P} = E(t\nabla + x\partial_t)$$

which is closely related to $\mathcal{J}$ by $\mathcal{P} = \mathcal{L}x - iA\mathcal{J}$, and it almost commutes with $\mathcal{L}$, i.e. $[\mathcal{L},\mathcal{P}] = iA\langle i\nabla\rangle^{-1}\nabla\mathcal{L}$ (see [16]).

2 Results for the super critical case

In this section we state the existence of the inverse wave operator $\mathcal{W}_+^{-1} : \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^2 \to \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^2$ for super critical case and $n = 1,2$ which were shown in [13].

Theorem 2.1. *Let the initial data $w^0 \in \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^2$ have a small norm $\|w^0\|_{\mathbf{H}^{1+\frac{n}{2},1}}$. Then there exist unique solution $U(-t)\,w \in \mathbf{C}\left([0,\infty); \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^2\right)$ of the Cauchy problem (2) such that*

$$\|w(t)\|_{\mathbf{L}^\infty} \le C(1+t)^{-\frac{n}{2}\left(1-\frac{2}{q}\right)}$$

for all $t \geq 0$, *where* $q = \infty$ *for* $n = 1$ *and* $2 \leq q < \infty$ *for* $n = 2$. *Furthermore there exists a unique final state* $w^{+} \in \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2}$ *such that*

$$\left\|U(-t)\,w(t) - w^{+}\right\|_{\mathbf{H}^{1+\frac{n}{2},1}} \leq C(1+t)^{-\gamma} \tag{3}$$

for all $t \geq 0$, *where* $\gamma = \frac{n}{2}\left(1 - \frac{1}{q}\right)(p-1) - 1 > 0$

We next consider the final state problem for the nonlinear Klein-Gordon equation

$$\begin{cases} \mathcal{L}w = \mathcal{N}(w), \\ \|w(t) - U(t)\,w^{+}\|_{\mathbf{L}^2} \to 0 \text{ as } t \to \infty \end{cases} \tag{4}$$

with a final state $w^{+} \in \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2}$, $n = 1, 2$.

Theorem 2.2. *Let the final state* $w^{+} \in \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2}$. *Then there exist a time* $T \geq 0$ *and a unique solution* $U(-t)\,w \in \mathbf{C}\left([T,\infty);\left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2}\right)$ *of the final state problem for the nonlinear Klein-Gordon equation (4) such that*

$$\|w(t)\|_{\mathbf{L}^{\infty}} \leq C(1+t)^{-\frac{n}{2}\left(1-\frac{2}{q}\right)}$$

for all $t \geq T$, *where* $q = \infty$ *for* $n = 1$ *and* $2 < q < \infty$ *for* $n = 2$. *Furthermore the asymptotics*

$$\left\|U(-t)\,w(t) - w^{+}\right\|_{\mathbf{H}^{1+\frac{n}{2},1}} \leq Ct^{-\gamma}$$

is valid for all $t \geq T$, *where* $\gamma = \frac{n}{2}\left(1 - \frac{1}{q}\right)(p-1) - 1 > 0$

Remark 2.1. *By Theorem 2.2, we can define the wave operator* $\mathcal{W}_{+}$ *which maps any final state* $w^{+} \in \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2}$ *to the solution* $U(-t)\,w \in \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2}$ *if* $t \geq T$. *If we choose a sufficiently small norm* $\|w^{+}\|_{\mathbf{H}^{1+\frac{n}{2},1}}$, *we can take* $T = 0$. *Namely, the wave operator*

$$\mathcal{W}_{+} : w^{+} \in \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2} \to w^{0} \in \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2}$$

is well defined in the neighborhood of the origin in the $\left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2}$ *space. Furthermore if* w^{0} *is also sufficiently small in the norm of* $\left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2}$, *then by applying Theorem 2.1 for the negative time, we can define the inverse wave operator*

$$\mathcal{W}_{-}^{-1} : w^{0} \in \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2} \to w^{-} \in \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2}.$$

This means that the scattering operator

$$\mathcal{S}_{+} = \mathcal{W}_{-}^{-1}\mathcal{W}_{+} : w^{+} \in \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2} \to w^{-} \in \left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2}$$

is well defined in the neighborhood of the origin in the $\left(\mathbf{H}^{1+\frac{n}{2},1}\right)^{2}$ *space.*

In [14], we extended the above results for higher space dimensions. However we can not consider the neighborhood of the critical value $p = 1 + 2/n$ unfortunately.

Theorem 2.3. *Let $1 + \frac{4}{n+2} < p < 1 + \frac{4}{n}$ and $n \geq 3$. Suppose that the initial data $w^0 \in \left(\mathbf{H}^{\beta,1}\right)^2$, $\beta = \max\left(\frac{3}{2}, 1 + \frac{2}{n}\right)$ have a small norm $\|w^0\|_{\mathbf{H}^{\beta,1}}$. Then there exists a unique solution $U(-t)\, w \in \mathbf{C}\left([0,\infty); \left(\mathbf{H}^{\beta,1}\right)^2\right)$ to the Cauchy problem (2) such that*

$$\|w(t)\|_{\mathbf{L}^q} \leq C(1+t)^{-\frac{n}{2}\left(1-\frac{2}{q}\right)}$$

for all $t \geq 0$, where $2 \leq q < \frac{2n}{n-2}$. Furthermore there exists a unique final state $w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2$ such that

$$\left\|U(-t)\, w(t) - w^+\right\|_{\mathbf{H}^{\beta,1}} \leq C(1+t)^{-\gamma} \tag{5}$$

for all $t \geq 0$, where $\gamma = \frac{n}{2}(p-1)\left(1 - \frac{1}{q}\right) - 1 > 0$.

Theorem 2.4. *Let $1 + \frac{4}{n+2} < p < 1 + \frac{4}{n}$ and $n \geq 3$. Suppose that the final state $w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2$, $\beta = \max\left(\frac{3}{2}, 1 + \frac{2}{n}\right)$. Then there exists a time $T \geq 0$ and a unique solution $U(-t)\, w \in \mathbf{C}\left([T,\infty); \left(\mathbf{H}^{\beta,1}\right)^2\right)$ of the final state problem (4) such that*

$$\|w(t)\|_{\mathbf{L}^q} \leq C(1+t)^{-\frac{n}{2}\left(1-\frac{2}{q}\right)}$$

for all $t \geq T$, where $2 \leq q < \frac{2n}{n-2}$. Furthermore the asymptotics

$$\left\|U(-t)\, w(t) - w^+\right\|_{\mathbf{H}^{\beta,1}} \leq C t^{-\gamma}$$

is valid for all $t \geq T$, where $\gamma = \frac{n}{2}(p-1)\left(1 - \frac{1}{q}\right) - 1 > 0$.

Remark 2.2. *By Theorem 2.4, we can define the wave operator $\mathcal{W}_+$ which maps any final state $w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2$ to the solution $U(-t)\, w \in \left(\mathbf{H}^{\beta,1}\right)^2$ if $t \geq T$. If we choose a sufficiently small norm $\|w^+\|_{\mathbf{H}^{\beta,1}}$, we can take $T = 0$. Namely, the wave operator*

$$\mathcal{W}_+ : w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2 \to w^0 \in \left(\mathbf{H}^{\beta,1}\right)^2$$

is well-defined in the neighborhood of the origin in the $\left(\mathbf{H}^{\beta,1}\right)^2$ space. Furthermore since the initial data w^0 are also sufficiently small in the norm of $\left(\mathbf{H}^{\beta,1}\right)^2$, by applying Theorem 2.3 for the negative time we can define the inverse wave operator

$$\mathcal{W}_-^{-1} : w^0 \in \left(\mathbf{H}^{\beta,1}\right)^2 \to w^- \in \left(\mathbf{H}^{\beta,1}\right)^2.$$

This means that the scattering operator

$$\mathcal{S}_+ = \mathcal{W}_-^{-1}\mathcal{W}_+ : w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2 \to w^- \in \left(\mathbf{H}^{\beta,1}\right)^2$$

is well-defined in the neighborhood of the origin in the $\left(\mathbf{H}^{\beta,1}\right)^2$ space. Our results stated above include the quadrtic nonlinear Klein-Gordon equation. Therefore in view of the scattering problem, our results are extensions of the previous works by Klainerman [16] and Shatah [28]

3 Results for the critical case

We consider the final state problem to the nonlinear Klein-Gordon equation

$$(6)\qquad \begin{cases} u_{tt} + u - u_{xx} = \mu u^3, \ \ (t,x) \in \mathbf{R}^+ \times \mathbf{R}, \\ \|u(t) - F_S(t)\|_{\mathbf{L}^2} \to 0 \text{ as } t \to \infty, \end{cases}$$

where $\mu \in \mathbf{R}$. The function $F_S(t)$ we call a final state, defined by the final data u_+. If the final state $F_S(t)$ can be taken in the form $F_S(t) = 2\mathrm{Re}U(t)u_+$, where $U(t) = \mathcal{F}^{-1}e^{-it\langle\xi\rangle}\mathcal{F}$ is the free evolution group, (note that the definition of $U(t)$ is different given in the previous section) $\langle x\rangle = \sqrt{1+x^2}$, and the final problem (6) has a nontrivial solution, then we say that there exists a usual wave operator for the final state problem (6). However the cubic nonlinearity is critical in one space dimension and it is impossible to find a solution in the neighborhood of the free final state $F_S(t) = 2\mathrm{Re}U(t)u_+$ (see [7], Theorem 1 and [8], [21] for higher space dimensions). So in order to find a solution of (6), we need to modify the time dependence of the final state $F_S(t)$ as follows

$$F_S(t) = 2\mathrm{Re}U(t)w_+(t),$$

where

$$\widehat{w}_+(t,\xi) = \widehat{u}_+(\xi)\, e^{\frac{3}{2}i\mu\langle\xi\rangle^2|\widehat{u}_+(\xi)|^2\log t}$$

is defined with a given final data u_+ satisfying suitable conditions stated below. Let u be a solution of the nonlinear Klein-Gordon equation (6), then defining new dependent variables $\widetilde{u} = \frac{1}{2}\left(1 + i\langle i\partial_x\rangle^{-1}\partial_t\right)u$ and $\widetilde{v} = \frac{1}{2}\left(1 - i\langle i\partial_x\rangle^{-1}\partial_t\right)u$ we get a system of equations

$$(7)\qquad \begin{cases} \partial_t\widetilde{u} + i\langle i\partial_x\rangle\widetilde{u} = \frac{\mu i}{2}\langle i\partial_x\rangle^{-1}|\widetilde{u}+\widetilde{v}|^2(\widetilde{u}+\widetilde{v}), \\ \partial_t\widetilde{v} - i\langle i\partial_x\rangle\widetilde{v} = -\frac{\mu i}{2}\langle i\partial_x\rangle^{-1}|\widetilde{u}+\widetilde{v}|^2(\widetilde{u}+\widetilde{v}). \end{cases}$$

In the case of the real-valued function u we have $\widetilde{v}(t) = \overline{\widetilde{u}(t)}$, therefore (7) can be written as

$$(8)\qquad \partial_t\widetilde{u} + i\langle i\partial_x\rangle\widetilde{u} = -\frac{\mu}{2i}\langle i\partial_x\rangle^{-1}\left(\widetilde{u} + \overline{\widetilde{u}}\right)^3.$$

Our main result for the final state problem is

Theorem 3.1. *Let the final data u_+ be a real-valued function and $\widehat{u_+} \in \mathbf{H}_p^{1,3-\frac{3}{2p}}$, $2 < p \le \infty$, with a small norm $\|\widehat{u_+}\|_{\mathbf{H}_p^{1,3-\frac{3}{2p}}}$. Then there exists a unique solution $\widetilde{u} \in \mathbf{C}([1,\infty);\mathbf{H}^1)$ of equation (8). Moreover the asymptotics is true*

$$\left\|\widetilde{u}(t) - U(t)\mathcal{F}^{-1}\widehat{u}_+(\xi)\, e^{\frac{3i\mu}{2}\langle\xi\rangle^2|\widehat{u}_+(\xi)|^2\log t}\right\|_{\mathbf{H}^1} \le Ct^{-b},$$

where $b \in \left(\frac{1}{4}, \frac{1}{2} - \frac{1}{2p}\right)$, $2 < p \le \infty$.

Corollary 3.1. *Let the final data u_+ be a real-valued function and $\widehat{u_+} \in \mathbf{H}_p^{1,3-\frac{3}{2p}}$, $2 < p \leq \infty$, with a small norm $\|\widehat{u_+}\|_{\mathbf{H}_p^{1,3-\frac{3}{2p}}}$. Then there exists a unique solution $u \in \mathbf{C}\left([1,\infty);\mathbf{H}^1\right)$ of the final state problem to the nonlinear Klein-Gordon equation (6) such that*

$$\left\| u(t) - 2ReU(t)\mathcal{F}^{-1}\widehat{u}_+(\xi)\, e^{\frac{3i\mu}{2}\langle\xi\rangle^2|\widehat{u}_+(\xi)|^2\log t}\right\|_{\mathbf{H}^1} \leq Ct^{-b}$$

where $b \in \left(\frac{1}{4}, \frac{1}{2} - \frac{1}{2p}\right)$, $2 < p \leq \infty$.

We now explain our strategy of the proof used in [11]. It was shown in the first theorem of [11] that

$$U(t)\phi = (it)^{-\frac{1}{2}}\theta(\chi)\langle i\chi\rangle^{-\frac{3}{2}} e^{-it\langle i\chi\rangle}\,\widehat{\phi}(\zeta) + O\left(t^{-b}\right)$$

uniformly with respect to $x \in \mathbf{R}$, where $\chi = \frac{x}{t}$, $\zeta = \frac{\chi}{\langle i\chi\rangle}$, $\frac{1}{4} < b < 1$ and $\theta(\chi) = 1$ for $|\chi| < 1$; $\theta(\chi) = 0$ for $|\chi| \geq 1$. Since the cubic nonlinearity is critical we need a modification of the final state. Assuming that the solutions have the same time decay rate as that of the linear equation we find that the leading term in the large time asymptotic behavior of the nonlinearity is

$$\frac{3}{2}i\mu\,|U(t)u_+|^2\,U(t)u_+ \simeq U(t)\mathcal{F}^{-1}\left(\frac{3i}{2t}\mu\langle\xi\rangle^2|\widehat{u}_+(\xi)|^2\,\widehat{u}_+(\xi)\right),$$

where the notation $a \simeq b$ means that the difference $a - b$ is a remainder. Therefore we need to choose a phase function to remove this term taking the modified final state

$$\widehat{w}_+(t,\xi) = \widehat{u}_+(\xi)\, e^{\frac{3}{2}i\mu\langle\xi\rangle^2|\widehat{u}_+(\xi)|^2\log t}$$

as stated above. Define the function $\widehat{w}(t,\xi)$ as a solution of the ordinary differential equation

$$\begin{aligned}\frac{d}{dt}\widehat{w}(t,\xi) &= \frac{3i\mu}{2t}\langle\xi\rangle^2|\widehat{w}(t,\xi)|^2\,\widehat{w}(t,\xi)\\ &\quad + it^{-1}E^{-1}\langle\xi\rangle^{-1}\sum_{j=2}^{4}\lambda_j\sqrt{\frac{i}{\omega_j}}\mathcal{D}_{\omega_j}E^{\omega_j}\langle\xi\rangle^3\,\widehat{w}^{\alpha_j}\overline{\widehat{w}}^{3-\alpha_j}\end{aligned}$$

where $\mathcal{D}_\omega$ is the dilation operator defined by $\mathcal{D}_\omega\phi = (i\omega)^{-1/2}\phi\left(\frac{x}{\omega}\right)$, $E = e^{-it\langle\xi\rangle}$, $\lambda_2 = -\frac{i\mu}{2}$, $\lambda_3 = \frac{3i\mu}{2}$, $\lambda_4 = -\frac{\mu}{2}$, $\omega_j = 2\alpha_j - 3$, $\alpha_2 = 3$, $\alpha_3 = 1$, $\alpha_4 = 0$, with a final state $\widehat{w}(t,\xi) \to \widehat{w}_+(t,\xi)$ as $t \to \infty$. We get from (8)

$$\frac{d}{dt}\left(e^{it\langle\xi\rangle}\widehat{\widetilde{u}}\right) = \frac{i\mu}{2}\langle\xi\rangle^{-1}e^{it\langle\xi\rangle}\mathcal{F}\left(\widetilde{u} + \overline{\widetilde{u}}\right)^3.$$

Hence we find for the difference

$$\begin{aligned}&\frac{d}{dt}\left(e^{it\langle\xi\rangle}\widehat{\widetilde{u}} - \widehat{w}(t,\xi)\right) = \frac{i\mu}{2}\langle\xi\rangle^{-1}e^{it\langle\xi\rangle}\mathcal{F}\Bigg(\left(\widetilde{u} + \overline{\widetilde{u}}\right)^3\\ &-3t^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle}\langle\xi\rangle^3|\widehat{w}(t,\xi)|^2\,\widehat{w}(t,\xi)\\ &-it^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle}\sum_{j=2}^{4}\lambda_j\sqrt{\frac{i}{\omega_j}}\mathcal{D}_{\omega_j}E^{\omega_j}\langle\xi\rangle^3\,\widehat{w}^{\alpha_j}\overline{\widehat{w}}^{3-\alpha_j}\Bigg).\end{aligned}\tag{9}$$

Considering the leading terms of the large time asymptotic behavior for the nonlinearities

$$|\widetilde{u}|^2 \widetilde{u} \simeq t^{-1} U(t) \mathcal{F}^{-1} \langle \xi \rangle^3 |\widehat{w}|^2 \widehat{w},$$

$$|\widetilde{u}|^2 \overline{\widetilde{u}} \simeq \left| U(t) \mathcal{F}^{-1} \widehat{w} \right|^2 \overline{U(t) \mathcal{F}^{-1} \widehat{w}},$$

$$\widetilde{u}^3 \simeq \left(U(t) \mathcal{F}^{-1} \widehat{w} \right)^3, \ \overline{\widetilde{u}}^3 \simeq \overline{\left(U(t) \mathcal{F}^{-1} \widehat{w} \right)}^3$$

we can see that the right-hand side of (9) is a remainder, indeed through the ordinary differential equation of $\widehat{w}$ we will show the last three terms $|\widetilde{u}|^2 \overline{\widetilde{u}}, \widetilde{u}^3, \overline{\widetilde{u}}^3$ are considered as non resonance terms and have a better time decay, so we have the desired result.

We next consider the initial value problem for the cubic nonlinear Klein-Gordon equation

$$\begin{cases} u_{tt} + u - u_{xx} = \mu u^3, \ (t,x) \in \mathbf{R} \times \mathbf{R}, \\ u(0) = u_0, u_t(0) = u_1, x \in \mathbf{R} \end{cases} \tag{10}$$

where $\mu \in \mathbf{R}$ and the data u_0, u_1 are real valued. Define a new dependent variable $u = \frac{1}{2} \left(v + i \langle i\partial_x \rangle^{-1} v_t \right)$ and initial data $u_0 = \frac{1}{2} \left(v_0 + i \langle i\partial_x \rangle^{-1} v_1 \right)$ with $\langle x \rangle = \sqrt{1 + |x|^2}$.In the case of the real-valued function v the nonlinear Klein-Gordon equation (10) can be rewritten as

$$\begin{cases} \mathcal{L}u = \mathcal{N}(u), \ (t,x) \in \mathbf{R} \times \mathbf{R}, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \end{cases} \tag{11}$$

where $\mathcal{L} = \partial_t + i \langle i\partial_x \rangle$ and

$$\mathcal{N}(u) = 4i\mu \langle i\partial_x \rangle^{-1} (\mathrm{Re}u)^3 .$$

Then a solution of (10) is $v = 2\mathrm{Re}u$.

Main result is

Theorem 3.2. *Let $u_0 \in \mathbf{H}^{4,1}$ and the norm $\|u_0\|_{\mathbf{H}^{4,1}}$ be sufficiently small. Then there exists a unique global solution u of (11) such that*

$$u(t) \in \mathbf{C}\left([0,\infty); \mathbf{H}^{4,1}\right)$$

and

$$\|u(t)\|_{\mathbf{H}^1_\infty} \le C(1+t)^{-\frac{1}{2}} .$$

Furthermore there exists a unique final state $\widehat{W}_+ \in \mathbf{H}^{0,1}_\infty \cap \mathbf{H}^{0,1}$ such that

$$\left\| u(t) - U(t) \mathcal{F}^{-1} \widehat{W}_+ \exp\left(\frac{3i\mu}{2} \langle \xi \rangle^2 \left| \widehat{W}_+ \right|^2 \log t \right) \right\|_{\mathbf{H}^{1,0}} \le C\varepsilon^3 t^{\gamma - \frac{1}{4}}$$

and

$$\left\| \mathcal{F} U(-t) u(t) - \widehat{W}_+ \exp\left(\frac{3i\mu}{2} \langle \xi \rangle^2 \left| \widehat{W}_+ \right|^2 \log t \right) \right\|_{\mathbf{H}^{0,1}_\infty} \le C\varepsilon^3 t^{\gamma - \frac{1}{4}},$$

where $\gamma \in \left(0, \frac{1}{4}\right)$.

From the result we find that there exists the inverse modified wave operator $(MW_+)^{-1}$ such that

$$(MW_+)^{-1} : u_0 \in \mathbf{H}^{4,1} \to W_+ \in \mathbf{H}^{1,0}.$$

The result yields the result for the Cauchy problem (10).

Corollary 3.2. *Let $v_0 \in \mathbf{H}^{4,1}, v_1 \in \mathbf{H}^{3,1}$ be real valued functions and the norm $\|v_0\|_{\mathbf{H}^{4,1}} + \|v_1\|_{\mathbf{H}^{3,1}}$ be sufficiently small. Then there exists a unique global solution v of (10) such that*

$$v(t) \in \mathbf{C}\left([0,\infty);\mathbf{H}^{4,1}\right) \cap \mathbf{C}^1\left([0,\infty);\mathbf{H}^{3,1}\right)$$

and

$$\|v(t)\|_{\mathbf{H}^1_\infty} \le C(1+t)^{-\frac{1}{2}}.$$

Furthermore there exists a unique final state $\widehat{W}_+ \in \mathbf{H}^{0,1}_\infty \cap \mathbf{H}^{0,1}$ such that

$$\left\| v(t) - 2ReU(t)\mathcal{F}^{-1}\widehat{W}_+ \exp\left(\frac{3i\mu}{2}\langle\xi\rangle^2\left|\widehat{W}_+\right|^2 \log t\right)\right\|_{\mathbf{H}^{1,0}} \le C\varepsilon^3 t^{\gamma-\frac{1}{4}}$$

and

$$\left\| \mathcal{F}U(-t)v(t) - 2Re\widehat{W}_+ \exp\left(\frac{3i\mu}{2}\langle\xi\rangle^2\left|\widehat{W}_+\right|^2 \log t\right)\right\|_{\mathbf{H}^{0,1}_\infty} \le C\varepsilon^3 t^{\gamma-\frac{1}{4}},$$

where $\gamma \in \left(0,\frac{1}{4}\right)$.

Remark 3.1. *By [13] we have*

$$\begin{aligned} &U(t)\mathcal{F}^{-1}\widehat{W}_+ \exp\left(\frac{3i\mu}{2}\langle\xi\rangle^2\left|\widehat{W}_+\right|^2 \log t\right) \\ = \;& (it)^{-1/2}\theta\left(\frac{x}{t}\right)\left(1-\frac{x^2}{t^2}\right)^{-\frac{3}{4}} e^{-i(t^2-x^2)^{\frac{1}{2}}} \\ & \times \widehat{W}_+\left(\frac{x}{(t^2-x^2)^{\frac{1}{2}}}\right)\exp\left(\frac{3i\mu}{2}\frac{t^2}{(t^2-x^2)}\left|\widehat{W}_+\left(\frac{x}{(t^2-x^2)^{\frac{1}{2}}}\right)\right|^2 \log t\right) \\ & + O\left(\left\|\widehat{W}_+ \exp\left(\frac{3i\mu}{2}\langle\xi\rangle^2\left|\widehat{W}_+\right|^2 \log t\right)\right\|_{\mathbf{H}^{1,\frac{9}{4}}} t^{-\frac{3}{4}+\frac{1}{r}}\right) \end{aligned}$$

where the function $\theta(x) = 1$ for $|x| < 1$ and $\theta(x) = 0$ for $|x| \ge 1$. Therefore regularity of $\widehat{W}_+$ is needed to obtain a sharp asymptotics of solutions to (11) in $\mathbf{L}^\infty$ sense. That is the reason why we can not give a sharp asymptotic formula of the solution $u(t)$ in $\mathbf{L}^\infty$ sense.

We use the operator

$$\mathcal{J} = \langle i\partial_x\rangle U(t)xU(-t) = \mathcal{F}^{-1}\langle\xi\rangle e^{-i\langle\xi\rangle t} i\partial_\xi e^{i\langle\xi\rangle t}\mathcal{F} = \langle i\partial_x\rangle x + it\partial_x,$$

which plays the same role as the operator $x + it\partial_x = \mathcal{U}(t)\, x\mathcal{U}(-t)$ which is an important tool to obtain the time decay estimates of the solutions to nonlinear Schrödinger equations, where $\mathcal{U}(t)$ is the free Schrödinger evolution operator defined by $\mathcal{U}(t) = \mathcal{F}^{-1}e^{-\frac{i}{2}|\xi|^2 t}\mathcal{F}$, see . The free Klein-Gordon evolution group for the Klein-Gordon equation $U(t)$ is written as $U(t) = e^{-i\langle i\nabla\rangle t}$. We have the commutation relation $[\mathcal{L}, \mathcal{J}] = \mathcal{L}\mathcal{J} - \mathcal{J}\mathcal{L} = 0$, since $[x, \langle i\partial_x\rangle^{\alpha}] = \alpha\langle i\partial_x\rangle^{\alpha-2}\partial_x$. However it is difficult to calculate the action of $\mathcal{J}$ on the nonlinearity $\mathcal{N}$. Therefore we use the first order differential operator

$$\mathcal{P} = t\partial_x + x\partial_t$$

which is closely related to $\mathcal{J}$ by the identity $\mathcal{P} = \mathcal{L}x - i\mathcal{J}$ and acts well on the nonlinearity. Moreover, it almost commutes with $\mathcal{L}$ since $[\mathcal{L}, \mathcal{P}] = -i\langle i\partial_x\rangle^{-1}\partial_x\mathcal{L}$. We note here that the operator $\mathcal{J}$ was used in paper [11], [12] to construct the scattering operator for nonlinear Klein-Gordon equations with a super critical nonlinearity. We briefly explain our strategy of the proof in this paper. Nonlinear Klein-Gordon equation (10) is considered as the relativistic version of nonlinear Schrödinger equation

$$\begin{cases} iv_t + \frac{1}{2}v_{xx} = \mu|v|^2 v, & (t,x) \in \mathbf{R}\times\mathbf{R}, \\ v(0) = v_0, x \in \mathbf{R}. \end{cases} \tag{12}$$

Therefore the method used in the study of (12) is used to study nonlinear Klein-Gordon equations. The inverse modified wave operator was constructed in [10], where the main ideas are to translate the equation into another equation by multiplying the both sides of (12) by $\mathcal{F}\mathcal{U}(t)$ and to make use of the factorization technique such that

$$\begin{aligned} \mathcal{U}(t)\phi &= \frac{1}{\sqrt{2\pi i t}}\int e^{\frac{ix^2}{2t} - \frac{ixy}{t} + \frac{iy^2}{2t}}\phi(y)\,dy \\ &= e^{\frac{ix^2}{2t}}\frac{1}{\sqrt{it}}\mathcal{F}\left(e^{\frac{iy^2}{2t}}\phi\right)\left(\frac{x}{t}\right) = \mathcal{M}\mathcal{D}\mathcal{F}\mathcal{M}\phi, \end{aligned}$$

where $\mathcal{M} = e^{\frac{ix^2}{2t}}, \mathcal{D}\phi(y) = \frac{1}{\sqrt{it}}\phi\left(\frac{x}{t}\right)$. By using these ideas, we have from (12)

$$\begin{aligned} i\left(\mathcal{F}\mathcal{U}(-t)\,v\right)_t &= \mu\mathcal{F}\overline{\mathcal{M}}\mathcal{F}^{-1}\mathcal{D}^{-1}\overline{\mathcal{M}}\,|v|^2 v \\ &= \mu t^{-1}\mathcal{F}\overline{\mathcal{M}}\mathcal{F}^{-1}\left|\mathcal{F}\overline{\mathcal{M}}\mathcal{U}(-t)\,v\right|^2\mathcal{F}\overline{\mathcal{M}}\mathcal{U}(-t)\,v \\ &= \mu t^{-1}\left|\mathcal{F}\mathcal{U}(-t)\,v\right|^2\mathcal{F}\mathcal{U}(-t)\,v + R. \end{aligned}$$

It was shown in [10] that the nonlinear term is decomposed into the remainder term R and the resonance term $\mu t^{-1}|\mathcal{F}\mathcal{U}(-t)\,v|^2\mathcal{F}\mathcal{U}(-t)\,v$. Resonance term is canceled by replacing $\mathcal{F}\mathcal{U}(-t)\,v$ by $(\mathcal{F}\mathcal{U}(-t)\,v)\exp\left(\int_1^t \mu\tau^{-1}|\mathcal{F}\mathcal{U}(-\tau)\,v|^2 d\tau\right)$. Therefore a-priori estimate of $\|\mathcal{F}\mathcal{U}(-t)\,v\|_{\mathbf{L}^\infty}$ follows. In this paper we use the same idea used in the nonlinear Schrödinger equation. Therefore we multiply both sides of (11) by $\mathcal{F}U(-t)$ and put $\varphi = \langle i\partial_x\rangle\mathcal{F}U(-t)\,v$ to get

$$\begin{aligned} \varphi_t &= \mathcal{F}U(-t)\langle i\partial_x\rangle\mathcal{N}(v) \\ &= i\frac{3\mu}{2}t^{-1}|\varphi|^2\varphi + f(\varphi,\overline{\varphi}) + O\left(t^{-\frac{5}{4}}\|\phi\|^3_{\mathbf{H}^{4,1}}\right), \end{aligned} \tag{13}$$

where $f(\varphi, \overline{\varphi})$ are cubic nonlinearities. The above ordinary differential equation shows that the nonlinearity can be decomposed into resonance term $i\frac{3\mu}{2}t^{-1}|\varphi|^2\varphi$ and nonresonance terms $f(\varphi, \overline{\varphi})$ and remainder term $O\left(t^{-\frac{5}{4}}\|\phi\|^3_{\mathbf{H}^{4,1}}\right)$. It is shown that nonresonance terms have better time decay through integration by parts. Furthermore we can remove the first term of the right hand side of (13) by multiplying both sides of (13) by $\exp\left(-\int_1^t i\frac{3\mu}{2}|\varphi|^2\tau^{-1}d\tau\right)$.

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