数理解析研究所講究録
1536 巻 2007 年 58-72

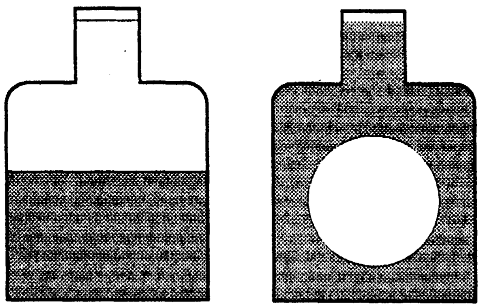
L_p-L_q maximal regularity of the interface problem for the Stokes system in a bounded domain

Senjo SHIMIZU

Faculty of Engineering, Shizuoka University, JAPAN
E-mail: tssshim@ipc.shizuoka.ac.jp

1 Introduction and Results

We consider a time dependent problem for the Navier-Stokes equations with free interface in a bounded domain $\Omega\subset\mathbb{R}^n$ ($n\geqq 2$). Physically, we consider the following model. There is a bottle which contains a half of water with stopper. Suppose we were in a space shuttle with this bottle, then an air bubble floats in water under zero gravity as the right-hand side of the picture.



We formulate such a model mathematically. Let $\Omega_t^+\subset\Omega$ be occupied by the fluid of viscosity $\mu^+>0$ which is given only on the initial time $t=0$, while for $t>0$ it is to be determined. Γ_t denotes the boundary of Ω_t^+ and $\partial\Omega$ denotes the boundary of Ω. Ω_0^+ is strictly contained in Ω, namely $\overline{\Omega_0^+}\subset\Omega$ and the distance between Γ_0 and $\partial\Omega$ is positive. Put $\Omega_t^-=\Omega\setminus(\Omega_t^+\cup\Gamma_t)$. Ω_t^- is occupied by the fluid of viscosity $\mu^->0$. ν_t is the unit outward normal to Γ_t of Ω_t^+ and ν_Ω is the unit outward normal to $\partial\Omega$. We write $\Gamma=\Gamma_0$ and $\Omega^\pm=\Omega_0^\pm$. We assume that Γ is a $C^{2,1}$ hypersurface while $\partial\Omega$ is a $C^{1,1}$ one.

The velocity vector field $v^\pm(x,t)=(v_1^\pm,\dots,v_n^\pm)^*$, where M^* denotes the transpose of M, and the pressure $\theta^\pm(x,t)$ for $x\in\Omega_t^\pm$ satisfy the free boundary problem for the Navier-Stokes equations:

$$
\begin{aligned}
&\partial_t v^\pm+(v^\pm\cdot\nabla)v^\pm-\operatorname{Div}S^\pm(v^\pm,\theta^\pm)=f^\pm && \text{in } \Omega_t^\pm,\ t>0\\
&\operatorname{div}v^\pm=0 && \text{in } \Omega_t^\pm,\ t>0\\
&[S^+(v^+,\theta^+)-S^-(v^-,\theta^-)]\nu_t|_{\Gamma_t}=0,\quad v^+|_{\Gamma_t}=v^-|_{\Gamma_t} && t>0\\
&v^-|_{\partial\Omega}=0,\qquad v^\pm|_{t=0}=v_0^\pm, &&
\end{aligned}\tag{1.1}
$$

where $S^\pm(v^\pm, \theta^\pm)$ are stress tensors defined by

$$S^\pm(v^\pm, \theta^\pm) = \mu^\pm D(v^\pm) - \theta^\pm I, \quad \{D(v^\pm)\}_{jk} = \partial v_j^\pm/\partial x_k + \partial v_k^\pm/\partial x_j,$$

I is the $n \times n$ identity matrix, and $f^\pm = f^\pm(x,t)$ is given external force vector defined on $\Omega^\pm$. In the model the effect of surface tension on Γ_t is excluded. We assume that the kinematic boundary condition:

$$\Gamma_t = \{x = x(\xi, t) \mid \xi \in \Gamma_0\},$$

where $x(\xi, t)$ is the solution of the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi \quad \text{for} \quad \xi \in \Gamma_0.$$

This expresses the fact that the free interface Γ_t consists for all $t > 0$ of the same particles, which do not leave it and are not incident on it from inside of $\Omega_t^\pm$. Here and hereafter, given functions $w^\pm$ defined on $\Omega_t^\pm$ or $\Omega^\pm$, we put

$$w = \begin{cases} w^+ & x \in \Omega_t^+ \text{ (or } \Omega^+), \quad t > 0 \\ w^- & x \in \Omega_t^- \text{ (or } \Omega^-), \quad t > 0. \end{cases}$$

Moreover given function w defined on Ω, $w^\pm$ denote the restriction of w to $\Omega_t^\pm$ or $\Omega^\pm$.

If a velocity vector field $u^\pm(\xi, t)$ is known as a function of the Lagrange coordinates ξ, then this connection can be written in the form:

$$x = \xi + \int_0^\infty u^\pm(\xi, \tau)\, d\tau \equiv X_{u^\pm}(\xi, t).$$

Passing to the Lagrange coordinates in (1.1) and setting $\theta^\pm(X_{u^\pm}(\xi,t),t) = \pi^\pm(\xi,t)$, we obtain the initial boundary value problem on the fixed interface Γ:

$$\begin{aligned}
&u_t^\pm - \operatorname{Div}[S^\pm(u^\pm, \pi^\pm) + U^\pm(u^\pm, \pi^\pm)] = f^\pm(X_{u^\pm}(\xi,t),t) && \text{in } \Omega^\pm,\ t > 0 \\
&\operatorname{div} u^\pm + E^\pm(u^\pm) = \operatorname{div}[u^\pm + \tilde{E}^\pm(u^\pm)] = 0 && \text{in } \Omega^\pm,\ t > 0 \\
&[S^+(u^+, \pi^+) + U^+(u^+, \pi^+)]\nu|_\Gamma = [S^-(u^-, \pi^-) + U^-(u^-, \pi^-)]\nu|_\Gamma && t > 0 \\
&u^+|_\Gamma = u^-|_\Gamma, \qquad u^-|_{\partial\Omega} = 0 && t > 0 \\
&u^\pm|_{t=0} = u_0^\pm(\xi) && \text{in } \Omega^\pm, \qquad (1.2)
\end{aligned}$$

where $u_0^\pm(\xi) = v_0^\pm(x)$, ν is outward normal to Γ of Ω^+, and $U^\pm(u^\pm, \pi^\pm)$, $E^\pm(u^\pm)$ and $\tilde{E}^\pm(u^\pm)$ are nonlinear terms of the following forms:

$$U^\pm(u^\pm, \pi^\pm) = V_1^\pm\Big(\int_0^t \nabla u^\pm\, d\tau\Big)\nabla u^\pm + V_2^\pm\Big(\int_0^t \nabla u^\pm\, d\tau\Big)\pi^\pm,$$

$$E^\pm(u^\pm) = V_3^\pm\Big(\int_0^t \nabla u^\pm\, d\tau\Big)\nabla u^\pm, \quad \tilde{E}^\pm(u^\pm) = V_4^\pm\Big(\int_0^t \nabla u^\pm\, d\tau\Big)u^\pm$$

with some polynomials $V_j^\pm(\cdot)$ such as $V_j^\pm(0) = 0$ $(j = 1, 2, 3, 4)$.

The linearized problem of (1.2) is the following:

$$\begin{aligned}
&\partial_t u^\pm - \operatorname{Div} S^\pm(u^\pm, \pi^\pm) = f && \text{in } \Omega^\pm,\ t > 0 \\
&\operatorname{div} u^\pm = g^\pm = \operatorname{div} \tilde{g} && \text{in } \Omega^\pm,\ t > 0 \\
&[S^+(u^+, \pi^+) - S^-(u^-, \pi^-)]\nu|_\Gamma = h^+ - h^-|_\Gamma, \quad u^+|_\Gamma = u^-|_\Gamma && t > 0 \\
&u|_{\partial\Omega} = 0, \qquad u^\pm|_{t=0} = u_0. && \qquad (1.3)
\end{aligned}$$

First of all, we introduce function spaces and some symbols. For any domain D in $\mathbb{R}^n$, integer m and $1 \leqq q \leqq \infty$, $L_q(D)$ and $W_q^m(D)$ denote the usual Lebesgue space and Sobolev space of functions defined on D with norms: $\|\cdot\|_{L_q(D)}$ and $\|\cdot\|_{W_q^m(D)}$, respectively. And also, for any Banach space X, interval I, integer ℓ and $1 \leqq p \leqq \infty$, $L_p(I, X)$ and $W_p^\ell(I, X)$ denote the usual Lebesgue space and Sobolev space of the X-valued functions defined on I with norms: $\|\cdot\|_{L_p(I,X)}$ and $\|\cdot\|_{W_p^\ell(I,X)}$, respectively. Set

$$
\begin{aligned}
&W_{q,p}^{\ell,m}(D \times I) = L_p(I, W_q^\ell(D)) \cap W_p^m(I, L_q(D)),\\
&\qquad \|u\|_{W_{q,p}^{\ell,m}(D\times I)} = \|u\|_{L_p(I,W_q^\ell(D))} + \|u\|_{W_p^m(I,L_q(D))},\\
&W_q^0(D) = L_q(D), \quad W_p^0(I, X) = L_p(I, X),\\
&W_{p,0}^\ell((0,T), X) = \{u \in W_p^\ell((-\infty, T), X) \mid u = 0 \text{ for } t < 0\},\\
&W_{p,0}^0((0,T), X) = L_{p,0}((0,T), X).
\end{aligned}
$$

Given $\alpha \in \mathbb{R}$, we set

$$
\begin{aligned}
&< D_t >^\alpha u(t) = \mathcal{F}^{-1}[(1+s^2)^{\alpha/2}\mathcal{F}u(s)](t),\\
&H_p^\alpha(\mathbb{R}, X) = \{u \in L_p(\mathbb{R}, X) \mid < D_t >^\alpha u \in L_p(\mathbb{R}, X)\},\\
&\qquad \|u\|_{H_p^\alpha(\mathbb{R},X)} = \| < D_t >^\alpha u\|_{L_p(\mathbb{R},X)} + \|u\|_{L_p(\mathbb{R},X)}.
\end{aligned}
$$

Here and hereafter, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively. Set

$$
\begin{aligned}
&H_{q,p}^{1,1/2}(D \times \mathbb{R}) = H_p^{1/2}(\mathbb{R}, L_q(D)) \cap L_p(\mathbb{R}, W_q^1(D)),\\
&\qquad \|u\|_{H_{q,p}^{1,1/2}(D\times\mathbb{R})} = \|u\|_{H_p^{1/2}(\mathbb{R},L_q(D))} + \|u\|_{L_p(\mathbb{R},W_q^1(D))},\\
&H_{q,p,0}^{1,1/2}(D \times (0,\infty)) = \{u \in H_{q,p}^{1,1/2}(D \times \mathbb{R}) \mid u = 0 \text{ for } t < 0\}.
\end{aligned}
$$

Finally, given $0 < T \leqq \infty$ we set

$$
\begin{aligned}
&H_{q,p,0}^{1,1/2}(D \times (0,T))\\
&\qquad = \{u \mid {}^\exists v \in H_{q,p,0}^{1,1/2}(D \times (0,\infty)),\ u = v \text{ on } D \times (0,T)\},\\
&\qquad \|u\|_{H_{q,p,0}^{1,1/2}(D\times(0,T))} = \inf\{\|v\|_{H_{q,p}^{1,1/2}(D\times\mathbb{R})} \mid\\
&\qquad\qquad v \in H_{q,p,0}^{1,1/2}(D \times (0,\infty)) \text{ with } v = u \text{ on } D \times (0,T)\}.
\end{aligned}
$$

Given Banach space X with norm $\|\cdot\|_X$, we set

$$
X^n = \{v = (v_1, \ldots, v_n)^* \mid v_j \in X\}, \quad \|v\|_X = \sum_{j=1}^n \|v_j\|_X.
$$

The dot $\cdot$ denotes the inner-product of $\mathbb{R}^n$. $F = (F_{ij})$ means the $n \times n$ matrix whose i-th row and j-th column component is F_{ij}. For the differentiation of the $n \times n$ matrix of functions $F = (F_{ij})$, the n-vector of functions $u = (u_1, \ldots, u_n)^*$ and the scalar function θ, we use the following symbols: $\theta_t = \partial_t\theta = \partial\theta/\partial t$, $\partial_j\theta = \partial\theta/\partial x_j$,

$$
\nabla\theta = (\partial_1\theta, \ldots, \partial_n\theta)^*, \quad u_t = \partial_t u = (\partial_t u_1, \ldots, \partial_t u_n), \quad \nabla u = (\partial_i u_j),
$$

$$\operatorname{div} u = \sum_{j=1}^{n} \partial_j u_j, \quad \operatorname{Div} F = (\sum_{j=1}^{n} \partial_j F_{1j}, \ldots, \sum_{j=1}^{n} \partial_j F_{nj})^*.$$

The inner products $(\cdot,\cdot)_D$ and $(\cdot,\cdot)_{\partial D}$ are defined by

$$(u,v)_D = \int_D u(x)\cdot v(x)\,dx, \quad (u,v)_{\partial D} = \int_{\partial D} u(x)\cdot v(x)\,d\sigma,$$

where $d\sigma$ denotes the surface element of ∂D. We denote by C a generic constant and $C_{a,b,\ldots}$ denotes a constant depending on the quantities a, b, ... The constants C and $C_{a,b,\ldots}$ may change from line to line.

We refer previous results concerning free interface problems. Tanaka [18] proved the global in time solvability of (1.2) in $W_2^{2+\alpha}$ with $\alpha \in (\frac{1}{2}, 1)$ for $n = 3$ and sufficiently small data with surface tension case. Takahashi [17] proved the global in time existence of weak solutions of (1.1) in the spaces such that the first derivative of the velocity in L_p with $p > 2(n+1)$ with respect to time and space provided that ν^+ is close to ν^- without surface tension case, based on the result of the linearized problem by Giga and Takahashi [9]. Nouri and Poupand [10] proved the local in time existence of a weak solution of the Navier-Stokes equation describing a multi fluid flow for arbitrary initial data without surface tension case.

When Ω is the whole space, Denisova [4] proved the local in time unique solvability for arbitrary initial data in $W_2^{2+\alpha, 1+\frac{\alpha}{2}}$ with $\alpha \in (\frac{1}{2}, 1)$ with or without surface tension case for $n = 3$ by using the local in time unique solvability result of the linearized problem in [2] and [5]. Denisova and Solonnikov [7] proved the local in time unique solvability for arbitrary initial data in the Hölder spaces with a power-like weight for $n = 3$ with surface tension case by using the local in time unique solvability result of the linearized problem in [3] and [6]. Abels [1] proved the global in time existence of varifold and measure-valued varifold solutions for singular free interfaces with or without surface tension case.

Our goal is to show global and local in time unique solvability of (1.2) in the class of the anisotropic Sobolev space $W_{q,p}^{2,1}$. In this paper, as the important step of our approach, we would like to show L_p-L_q maximal regularity of (1.3) global and local in time. We consider this problem by analytic semigroup approach using the same method as Shibata and Shimizu [13, 14, 15]. One of our main issues is to use $\mathcal{R}$-boundedness and operator valued Fourier multiplier theorem which are recently developed by Weis [20] and Denk, Hieber and Prüss [8] to show L_p-L_q maximal regularity of (1.3).

We start with an analytic semigroup approach to the initial-boundary value problem:

$$\begin{aligned}
&\partial_t u^\pm - \operatorname{Div} S^\pm(u^\pm, \pi^\pm) = 0, \qquad \operatorname{div} u^\pm = 0 \quad \text{in } \Omega^\pm \\
&[S^+(u^+,\pi^+) - S^-(u^-,\pi^-)]\nu|_\Gamma = 0, \quad u^+|_\Gamma = u^-|_\Gamma \\
&u|_{\partial\Omega} = 0 \qquad u|_{t=0} = u_0.
\end{aligned} \tag{1.4}$$

Set

$$\begin{aligned}
J_q(\Omega) &= \{u \in L_q(\Omega)^n \mid \operatorname{div} u = 0 \text{ in } \Omega,\ \nu_\Omega \cdot u|_{\partial\Omega} = 0\}, \\
G_q(\Omega) &= \{\nabla\pi \mid \pi \in W_q^1(\Omega),\ \int_\Omega \pi\,dx = 0\}.
\end{aligned}$$

We use the Helmholtz decomposition:

$$L_q(\Omega)^n = J_q(\Omega) \oplus G_q(\Omega)$$

for $1 < q < \infty$, where $\oplus$ is the direct sum. Let P_q denote a continuous projection from $L_q(\Omega)^n$ into $J_q(\Omega)$ along $G_q(\Omega)$ and we consider the resolvent problem corresponding to (1.4):

$$\begin{aligned}
&\lambda u^{\pm} - \operatorname{Div} S^{\pm}(u^{\pm}, \pi^{\pm}) = P_q u_0, \qquad \operatorname{div} u^{\pm} = 0 \quad \text{in } \Omega^{\pm} \\
&[S^{+}(u^{+}, \pi^{+}) - S^{-}(u^{-}, \pi^{-})]\nu|_{\Gamma} = 0, \quad u^{+}|_{\Gamma} = u^{-}|_{\Gamma} \\
&u|_{\partial\Omega} = 0.
\end{aligned} \tag{1.5}$$

Since we want to get the evolution equation only for velocity u, we have to eliminate pressure π. If we apply the divergence to the first equation, multiply the third equation by ν, apply the divergence to the third equation, and multiply the first equation by ν_Ω in (1.5), then we obtain

$$\begin{aligned}
&\Delta\pi^{\pm} = 0 \qquad \text{in } \Omega^{\pm} \\
&\pi^{+} - \pi^{-}|_{\Gamma} = \nu \cdot [\mu^{+} D(u^{+}) - \mu^{-} D(u^{-})]\nu - \operatorname{div} u^{+} + \operatorname{div} u^{-}|_{\Gamma} \\
&\partial_\nu \pi^{+} - \partial_\nu \pi^{-}|_{\Gamma} = -\nu \cdot [\mu^{+} \operatorname{Div} D(u^{+}) - \mu^{-} \operatorname{Div} D(u^{-})] \\
&\qquad\qquad\qquad - 2\mu^{+}\partial_\nu \operatorname{div} u^{+} + 2\mu^{-}\partial_\nu \operatorname{div} u^{-}|_{\Gamma} \\
&\partial_{\nu_\Omega}\pi^{-}|_{\partial\Omega} = \nu_\Omega \cdot \mu^{-}[\Delta u^{-} - \nabla(\operatorname{div} u^{-})|_{\partial\Omega}.
\end{aligned} \tag{1.6}$$

For $u^{\pm} \in W_q^2(\Omega^{\pm})$ with $u \in W_q^1(\Omega)$, under the condition $\int_\Omega \pi\, dx = 0$, there exists a unique solution $\pi^{\pm} \in W_q^1(\Omega^{\pm})$ of (1.6). Let us define the map $K^{\pm} : W_q^2(\Omega^{\pm}) \to W_q^1(\Omega^{\pm})$ by $\pi^{\pm} = K^{\pm}(u)$.

(1.5) is reduced to the reduced Stokes equation:

$$\begin{aligned}
&\lambda u^{\pm} - \operatorname{Div} S^{\pm}(u^{\pm}, K^{\pm}(u)) = P_q u_0 \qquad \text{in } \Omega^{\pm} \\
&[S^{+}(u^{+}, K^{+}(u)) - S^{-}(u^{-}, K^{-}(u))]\nu|_{\Gamma} = 0, \quad u^{+}|_{\Gamma} = u^{-}|_{\Gamma} \\
&u|_{\partial\Omega} = 0.
\end{aligned} \tag{1.7}$$

Inversely, assume that u solves (1.7). Then, since $\operatorname{div} u^{\pm}$ satisfies

$$\begin{aligned}
&(\lambda - 2\mu^{\pm}\Delta)\operatorname{div} u^{\pm} = 0 \qquad \text{in } \Omega^{\pm} \\
&\operatorname{div} u^{+}|_{\Gamma} = \operatorname{div} u^{-}|_{\Gamma} \\
&2\mu_{+}\partial_\nu \operatorname{div} u^{+}|_{\Gamma} = 2\mu_{-}\partial_\nu \operatorname{div} u^{-}|_{\Gamma} \\
&\partial_{\nu_\Omega} \operatorname{div} u^{-}|_{\partial\Omega} = 0,
\end{aligned}$$

we obtain $\operatorname{div} u^{\pm} = 0$ in $\Omega^{\pm}$. Therefore (1.6) is equivalent to (1.7).

Let us define the reduced Stokes operator A_q:

$$\begin{aligned}
A_q u &= -\operatorname{Div} S^{\pm}(u^{\pm}, K^{\pm}(u)) \qquad u \in \mathcal{D}(A_q), \\
\mathcal{D}(A_q) &= \{u \in W_q^1(\Omega)^n \cap J_q(\Omega) \mid u^{\pm} \in W_q^2(\Omega^{\pm})^n,\ u|_{\partial\Omega} = 0, \\
&\qquad u^{+}|_{\Gamma} = u^{-}|_{\Gamma},\ [S^{+}(u^{+}, K^{+}(u)) - S^{-}(u^{-}, K^{-}(u))]\nu|_{\Gamma} = 0\}.
\end{aligned}$$

From Shibata and Shimizu [12], we know the following theorem.

Proposition 1.1. *Let $1 < p < \infty$, $0 < \epsilon < \pi/2$ and*

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \leqq \pi - \epsilon\}.$$

Then, there exists $\sigma > 0$ such that for every $u_0 \in L_q(\Omega)^n$, $\lambda \in \mathbb{C} \setminus (-\infty, -\sigma)$, (1.5) admits a unique solution $u^{\pm} \in W_q^2(\Omega^{\pm})^n$ with $u \in W_q^1(\Omega)^n$ which enjoys the estimate:

$$|\lambda|\|u\|_{L_q(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{L_q(\Omega)} + \sum_{+-} \|u^{\pm}\|_{W_q^2(\Omega^{\pm})} \leqq C_{q,\epsilon,\sigma}\|u_0\|_{L_q(\Omega)}$$

for every $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$.

Combining the fact that A_q is a densely defined closed operator and Proposition 1.1, we obtain the following proposition.

Proposition 1.2. *Let $1 < q < \infty$. Then A_q generates an analytic semigroup $\{e^{-A_q t}(t)\}_{t \geqq 0}$ on $J_q(\Omega)$ which decays exponentially:*

$$\|e^{-tA_q}u_0\|_{J_q(\Omega)} \leqq C_{\gamma,q}\, e^{-\gamma t}\|u_0\|_{J_q(\Omega)} \quad \text{for } \gamma > 0.$$

Set

$$\mathcal{D}_{q,p}(\Omega) = [J_q(\Omega), \mathcal{D}(A_q)]_{1-1/p,p}$$

where $[\cdot,\cdot]_{\theta,p}$ denotes the real interpolation functor. Let us define the Besov spaces $B^{2(1-1/p)}_{q,p}(\Omega^\pm)$ and $B^{1-1/p}_{q,p}(\Omega)$ by the real interpolation

$$B^{2(1-1/p)}_{q,p}(\Omega^\pm) = [L_q(\Omega^\pm), W^2_q(\Omega^\pm)]_{1-1/p,p}$$
$$B^{1-1/p}_{q,p}(\Omega) = [L_q(\Omega), W^1_q(\Omega)]_{1-1/p,p}.$$

Then by Triebel [19] or Steiger [16] we obtain

$$\mathcal{D}_{q,p}(\Omega) =$$
$$\begin{cases} \{u \in B^{1-1/p}_{q,p}(\Omega),\ u^\pm \in B^{2(1-1/p)}_{q,p}(\Omega^\pm) \mid \operatorname{div} u = 0 \text{ in } \Omega, \\ \quad u|_{\partial\Omega} = 0,\ u^+|_\Gamma = u^-|_\Gamma,\ [S^+(u^+, K^+(u)) - S^-(u^-, K^-(u))]\nu|_\Gamma = 0\} \\ \qquad\qquad \text{when } 2(1-1/p) > 1 + 1/q \\ \{u \in B^{1-1/p}_{q,p}(\Omega),\ u^\pm \in B^{2(1-1/p)}_{q,p}(\Omega^\pm) \mid \operatorname{div} u = 0 \text{ in } \Omega, \\ \qquad\qquad u|_{\partial\Omega} = 0,\ u_+|_\Gamma = u_-|_\Gamma\} \\ \qquad\qquad \text{when } 1/q < 2(1-1/p) < 1 + 1/q \\ \{u \in B^{1-1/p}_{q,p}(\Omega),\ u^\pm \in B^{2(1-1/p)}_{q,p}(\Omega^\pm) \mid \operatorname{div} u = 0 \text{ in } \Omega\} \\ \qquad\qquad \text{when } 2(1-1/p) < 1/q. \end{cases}$$

Now, we shall state the main result which shows a unique existence theorem of solutions to (1.3) on the whole time interval $\mathbb{R}_+$ which decays exponentially when t goes to ∞.

Theorem 1.3. *Let $1 < p, q < \infty$. Then there exists a $\gamma_0 > 0$ such that if u_0, f, $\tilde{g}$, $g^\pm$, $h^\pm$ for* (1.3) *satisfy the conditions*

$$u_0 \in \mathcal{D}_{q,p}(\Omega),\ e^{\gamma t} f \in L_p(\mathbb{R}_+, L_q(\Omega))^n,\ e^{\gamma t}\tilde{g} \in W^1_{p,0}(\mathbb{R}_+, L_q(\Omega))^n$$
$$e^{\gamma t} g^\pm \in L_p(\mathbb{R}_+, W^1_q(\Omega^\pm)),\ e^{\gamma t} h^\pm \in H^{1,1/2}_{q,p,0}(\Omega^\pm \times \mathbb{R}_+)^n$$

for some $\gamma \in [0, \gamma_0]$, and

$$\nu \cdot (\tilde{g}^+ - \tilde{g}^-)|_\Gamma = 0, \quad \int_\Omega g\, dx = 0, \quad \int_{\partial\Omega} \nu_\Omega \cdot \tilde{g}\, d\sigma = 0$$

for $t > 0$, then (1.3) *admits a unique solution*

$$(u^\pm, \pi^\pm) \in W^{2,1}_{q,p}(\Omega^\pm \times \mathbb{R}_+)^n \times L_p(\mathbb{R}_+, W^1_q(\Omega^\pm))$$

with $u \in W^{1,1}_{q,p}(\Omega \times \mathbb{R}_+)$ and $\int_\Omega \pi\, dx = 0$. Moreover there exist $\tilde{\pi}^\pm \in H^{1,1/2}_{q,p}(\Omega_\pm \times \mathbb{R}_+)$ such that $\tilde{\pi}^\pm|_\Gamma = \pi^\pm|_\Gamma$ for $t > 0$. For the solution the estimate

$$\sum_{+-}(\|e^{\gamma t}u^\pm\|_{W^{2,1}_{q,p}(\Omega^\pm\times\mathbb{R}_+)} + \|e^{\gamma t}\pi^\pm\|_{L_p(\mathbb{R}_+,W^1_q(\Omega^\pm))} + \|e^{\gamma t}\tilde{\pi}^\pm\|_{H^{1,1/2}_{q,p}(\Omega^\pm\times\mathbb{R})}) + \|e^{\gamma t}u\|_{W^{1,1}_{q,p}(\Omega\times\mathbb{R}_+)}$$
$$\leqq C\{\|u_0\|_{\mathcal{D}_{q,p}(\Omega)} + \|e^{\gamma t}f\|_{L_p(\mathbb{R}_+,L_q(\Omega))} + \|e^{\gamma t}\tilde{g}\|_{W^1_p(\mathbb{R}_+,L_q(\Omega))}$$
$$+\sum_{+-}(\|e^{\gamma t}g^\pm\|_{L_p(\mathbb{R}_+,W^1_q(\Omega^\pm))} + \|e^{\gamma t}h^\pm\|_{H^{1,1/2}_{q,p}(\Omega^\pm\times\mathbb{R})})\}$$

holds.

The theorem which follows is concerned with a unique existence theorem of solutions to (1.3) on the finite time interval $(0,T)$.

Theorem 1.4. *Let $1<p,q<\infty$. If u_0, f, $\tilde{g}$, $g^\pm$, $h^\pm$ for (1.3) satisfy the conditions*

$$u_0 \in \mathcal{D}_{q,p}(\Omega),\ f\in L_p((0,T),L_q(\Omega))^n,\ \tilde{g}\in W^1_{p,0}((0,T),L_q(\Omega))^n,$$
$$g^\pm \in L_p((0,T),W^1_q(\Omega^\pm)),\ h^\pm \in H^{1,1/2}_{q,p,0}(\Omega^\pm\times(0,T))^n$$

and

$$\nu\cdot(\tilde{g}^+-\tilde{g}^-)|_\Gamma = 0,\quad \int_\Omega g\,dx = 0,\quad \int_{\partial\Omega}\nu_\Omega\cdot\tilde{g}\,d\sigma = 0$$

for $t>0$, then (1.3) admits a unique solution

$$(u^\pm,\pi^\pm)\in W^{2,1}_{q,p}(\Omega^\pm\times(0,T))^n\times L_p((0,T),W^1_q(\Omega^\pm))$$

with $u\in W^{1,1}_{q,p}(\Omega\times(0,T))$ and $\int_\Omega \pi\,dx=0$. Moreover there exist $\tilde{\pi}^\pm\in H^{1,1/2}_{q,p}(\Omega_\pm\times(0,T))$ such that $\tilde{\pi}^\pm|_\Gamma = \pi^\pm|_\Gamma$ for $t>0$. For the solution the estimate

$$\sum_{+-}(\|u^\pm\|_{W^{2,1}_{q,p}(\Omega^\pm\times(0,T))} + \|\pi^\pm\|_{L_p((0,T),W^1_q(\Omega^\pm))} + \|\tilde{\pi}^\pm\|_{H^{1,1/2}_{q,p}(\Omega^\pm\times(0,T))}) + \|u\|_{W^{1,1}_{q,p}(\Omega\times(0,T))}$$
$$\leqq C\{\|u_0\|_{\mathcal{D}_{q,p}(\Omega)} + \|f\|_{L_p((0,T),L_q(\Omega))} + \|\tilde{g}\|_{W^1_p((0,T),L_q(\Omega))}$$
$$+\sum_{+-}(\|g^\pm\|_{L_p((0,T),W^1_q(\Omega^\pm))} + \|h^\pm\|_{H^{1,1/2}_{q,p}(\Omega^\pm\times(0,T))})\}$$

holds, where C is independent of T.

2 An idea of our proof of Theorems 1.3 and 1.4

Roughly speaking, our proof is divided into the following three steps. First of all, we show the L_p-L_q maximal regularity of solutions to the model problems in the whole space, in the half-space and in the whole space with interface $x_n=0$ by using the operator valued Fourier multiplier theorem due to Weis [20]. The key observation for this is to show the $\mathcal{R}$ boundedness of the family of solution operators to such model problems.

Second of all, we consider the problem (1.3) with $u_0=\tilde{g}=g^\pm=h^\pm=0$ in a bounded domain Ω and we use the usual localization procedure to reduce the problem to the model problems in the whole space and half-space. To estimate the perturbation terms appearing in this procedure we use an exponential decay estimate of the analytic semigroup generated by the generalized Stokes operator associated with interface condition and no slip boundary condition.

Third of all, using a solution to the Laplace equation with zero Neumann boundary condition:

$$\Delta z = g \text{ in } \Omega,\quad \left.\frac{\partial z}{\partial\nu_\Omega}\right|_{\partial\Omega} = 0,\quad \int_\Omega g\,dx = 0,$$

we reduce the problem (1.3) to the case where $\tilde{g} = g^{\pm} = 0$. After this procedure, to treat the equation (1.3) with non-zero $h^{\pm}$ we use a solution to the dual problem with $u_0 = \tilde{g} = g^{\pm} = h^{\pm} = 0$ which was discussed in the second step.

In this section, we show an idea of our proof of the L_p-L_q maximal regularity of a model problem in the whole space with interface $x_n = 0$. The detail of the proof of Theorems 1.3 and 1.4 will be given in the forthcoming paper.

Let us consider the following model problem in $\mathbb{R}^n$ with $x_n = 0$:

$$\begin{aligned}
&\partial_t v^{\pm} - \operatorname{Div} S^{\pm}(v^{\pm}, \theta^{\pm}) = 0 \quad \text{in } \mathbb{R}^n_{\pm} \times \mathbb{R}_+ \\
&\operatorname{div} v^{\pm} = 0 \qquad\qquad\qquad\quad\ \, \text{in } \mathbb{R}^n_{\pm} \times \mathbb{R}_+ \\
&[S^+(v^+, \theta^+) - S^-(v^-, \theta^-)](0, \ldots, 0, -1)|_{\mathbb{R}^n_0} = -h^+ + h^-|_{\mathbb{R}^n_0} \\
&v^+|_{\mathbb{R}^n_0} = v^-|_{\mathbb{R}^n_0}, \qquad v^{\pm}|_{t=0} = 0,
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
&\mathbb{R}^n_{\pm} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \pm x_n > 0\}, \\
&\mathbb{R}^n_0 = \partial \mathbb{R}^n_+ = \{x = (x', 0) \in \mathbb{R}^n \mid x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\}
\end{aligned}$$

and $h^{\pm} \in H^{1,1/2}_{q,p}(\mathbb{R}^n_{\pm} \times \mathbb{R}_+)$.

We would like to show the L_p-L_q maximal regularity estimate of (2.1):

$$\|e^{-\gamma t} \partial_t v^+\|_{L_p(\mathbb{R}_+, L_q(\mathbb{R}^n_+))} \leqq C \sum_{+-} \|e^{-\gamma t} h^{\pm}\|_{H^{1,1/2}_{q,p}(\mathbb{R}^n_{\pm} \times \mathbb{R})}. \tag{2.2}$$

In the course of our proof, we may assume that $h^{\pm} \in C_0^{\infty}(\mathbb{R}^n_{\pm} \times \mathbb{R}_+)^n$ by the denseness of $C_0^{\infty}(\mathbb{R}^n_{\pm} \times \mathbb{R}_+)$ in $H^{1,1/2}_{q,p,0}(\mathbb{R}^n_{\pm} \times \mathbb{R}_+)$. We shall use the partial Laplace-Fourier transform with respect to (x', t) variables and its inversion formula which are defined by the relations:

$$\begin{aligned}
\mathcal{L}[g](\xi', x_n, \lambda) = \hat{g}(\xi', x_n, \lambda) &= \iint_{\mathbb{R}^n} e^{-\lambda t - ix' \cdot \xi'} g(x', x_n, t)\, dx' dt = \mathcal{F}_{x',t}[e^{-\gamma t} g(x', x_n, t)](\xi', \tau), \\
\mathcal{L}^{-1}[h](x', x_n, t) &= \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^n} e^{\lambda t + ix' \cdot \xi'} h(\xi', x_n, \lambda)\, d\xi' d\tau = e^{\gamma t} \mathcal{F}^{-1}_{\xi', \tau}[h(\xi', x_n, \gamma + i\tau)](x', t),
\end{aligned}$$

where $\lambda = \gamma + i\tau$, and $\mathcal{F}_{x',t}$ and $\mathcal{F}^{-1}_{\xi',\tau}$ denote the Fourier transform and its inversion formula with respect to (x', t) and (ξ', τ), respectively, and $\xi' = (\xi_1, \ldots, \xi_{n-1})$. Applying the partial Laplace-Fourier transform with respect to (x', t) variable to (2.1), we have

$$\begin{aligned}
&[(\lambda + \mu^{\pm}|\xi'|^2) - \mu^{\pm}\partial_n^2]\hat{v}_j^{\pm}(\xi', x_n, \lambda) + i\xi_j \hat{\theta}^{\pm}(\xi', x_n, \lambda) = 0, && \pm x_n > 0 \\
&[(\lambda + \mu^{\pm}|\xi'|^2) - \mu^{\pm}\partial_n^2]\hat{v}_n^{\pm}(\xi', x_n, \lambda) + \partial_n \hat{\theta}(\xi', x_n, \lambda) = 0, && \pm x_n > 0 \\
&\sum_{j=1}^{n-1} i\xi_j \hat{v}_j^{\pm}(\xi', x_n, \lambda) + \partial_n \hat{v}_n^{\pm}(\xi', x_n, \lambda) = 0, && \pm x_n > 0 \\
&\mu^+(i\xi_j \hat{v}_n^+(\xi', 0, \lambda) + \partial_n \hat{v}_j^+(\xi', 0, \lambda)) \\
&\qquad - \mu^-(i\xi_j \hat{v}_n^-(\xi', 0, \lambda) + \partial_n \hat{v}_j^-(\xi', 0, \lambda)) = \hat{h}_j^+(\xi', 0, \lambda) - \hat{h}_j^-(\xi', 0, \lambda) \\
&(2\mu^+ \partial_n \hat{v}_n^+(\xi', 0, \lambda) - \hat{\theta}^+(\xi', 0, \lambda)) \\
&\qquad - (2\mu^- \partial_n \hat{v}_n^-(\xi', 0, \lambda) - \hat{\theta}^-(\xi', 0, \lambda)) = \hat{h}_n^+(\xi', 0, \lambda) - \hat{h}_n^-(\xi', 0, \lambda) \\
&\hat{v}^+(\xi', 0, \lambda) - \hat{v}^-(\xi', 0, \lambda) = 0,
\end{aligned} \tag{2.3}$$

where $j = 1, \dots, n-1$. Writing $A = |\xi'|$ and $B^{\pm} = B^{\pm}_{\lambda}(\xi') = \sqrt{\lambda/\mu^{\pm} + |\xi'|^2}$, we shall look for solutions to (2.3) of the form:

$$\begin{aligned}
&\hat{v}^+(\xi', x_n, \lambda) = \alpha^+(e^{-Ax_n} - e^{-B^+x_n}) + \beta^+ e^{-B^+x_n}, \quad \hat{\theta}^+(\xi', x_n, \lambda) = \gamma^+ e^{-Ax_n}, \quad x_n > 0\\
&\hat{v}^-(\xi', x_n, \lambda) = \alpha^-(e^{Ax_n} - e^{B^-x_n}) + \beta^- e^{B^-x_n}, \qquad \hat{\theta}^-(\xi', x_n, \lambda) = \gamma^- e^{Ax_n}, \qquad x_n < 0. \quad (2.4)
\end{aligned}$$

Inserting (2.4) into (2.3), we obtain the explicit expression of the solutions $\hat{v}_j^{\pm}$ $(j = 1, \dots, n-1)$, $\hat{v}_n^{\pm}$ and $\theta^{\pm}$:

$$\begin{aligned}
&\hat{v}_j^+(\xi', x_n, \lambda)\\
&= -\xi_j \frac{e^{-B^+x_n} - e^{-Ax_n}}{B^+ - A} \frac{1}{f(A, B^+, B^-)}[(\mu^+ - \mu^-)AB^+ + A(\mu^+B^+ + \mu^-B^-)\\
&\qquad + \mu^-B^-(B^+ + B^-)]\, \tilde{\xi}' \cdot (\hat{h}^{+\prime}(\xi', 0, \lambda) - \hat{h}^{-\prime}(\xi', 0, \lambda))\\
&\quad - i\xi_j \frac{e^{-B^+x_n} - e^{-Ax_n}}{B^+ - A} \frac{1}{f(A, B^+, B^-)} [\mu^+(A^2 + (B^+)^2) - \mu^-(A^2 + (B^-)^2)\\
&\qquad + \mu^-(A + B^-)(B^+ + B^-)]\, (\hat{h}_n^+(\xi', 0, \lambda) - \hat{h}_n^-(\xi', 0, \lambda))\\
&\quad + e^{-B^+x_n} \frac{\xi_j}{\mu^+B^+ + \mu^-B^-} \frac{1}{f(A, B^+, B^-)}\\
&\qquad \times [2(\mu^+ - \mu^-)A(\mu^+B^+ - \mu^-B^-) + (\mu^+ + \mu^-)A(\mu^+B^+ + \mu^-B^-)\\
&\qquad + \mu^+\mu^-(B^+ + B^-)^2 - (\mu^+ - \mu^-)^2A^2]\, \tilde{\xi}' \cdot (\hat{h}^{+\prime}(\xi', 0, \lambda) - \hat{h}^{-\prime}(\xi', 0, \lambda))\\
&\quad + e^{-B^+x_n} \frac{i\xi_j}{\mu^+B^+ + \mu^-B^-} \frac{1}{f(A, B^+, B^-)}\\
&\qquad \times [(\mu^+ + \mu^-)\{\mu^+(A^2 + (B^+)^2) - \mu^-(A^2 + (B^-)^2)\} - \mu^+\mu^-((B^+)^2 - (B^-)^2)\\
&\qquad - (\mu^+ - \mu^-)A\{\mu^+(A + B^+) + \mu^-(A + B^-)\}]\, (\hat{h}_n^+(\xi', 0, \lambda) - \hat{h}_n^-(\xi', 0, \lambda))\\
&\quad - e^{-B^+x_n} \frac{1}{\mu^+B^+ + \mu^-B^-} (\hat{h}_j^+(\xi', 0, \lambda) - \hat{h}_j^-(\xi', 0, \lambda)),
\end{aligned}$$

$$\begin{aligned}
&\hat{v}_j^-(\xi', x_n, \lambda)\\
&= -\xi_j \frac{e^{B^-x_n} - e^{Ax_n}}{B^- - A} \frac{1}{f(A, B^+, B^-)}[(\mu^- - \mu^+)AB^- + A(\mu^+B^+ + \mu^-B^-)\\
&\qquad + \mu^+B^+(B^+ + B^-)]\, \tilde{\xi}' \cdot (\hat{h}^{+\prime}(\xi', 0, \lambda) - \hat{h}^{-\prime}(\xi', 0, \lambda))\\
&\quad - i\xi_j \frac{e^{B^-x_n} - e^{Ax_n}}{B^- - A} \frac{1}{f(A, B^+, B^-)} [\mu^+(A^2 + (B^+)^2) - \mu^-(A^2 + (B^-)^2)\\
&\qquad - \mu^+(A + B^+)(B^+ + B^-)]\, (\hat{h}_n^+(\xi', 0, \lambda) - \hat{h}_n^-(\xi', 0, \lambda))\\
&\quad + e^{B^-x_n} \frac{\xi_j}{\mu^+B^+ + \mu^-B^-} \frac{1}{f(A, B^+, B^-)}\\
&\qquad \times [2(\mu^+ - \mu^-)A(\mu^+B^+ - \mu^-B^-) + (\mu^+ + \mu^-)A(\mu^+B^+ + \mu^-B^-)\\
&\qquad + \mu^+\mu^-(B^+ + B^-)^2 - (\mu^+ - \mu^-)^2A^2]\, \tilde{\xi}' \cdot (\hat{h}^{+\prime}(\xi', 0, \lambda) - \hat{h}^{-\prime}(\xi', 0, \lambda))\\
&\quad + e^{B^-x_n} \frac{i\xi_j}{\mu^+B^+ + \mu^-B^-} \frac{1}{f(A, B^+, B^-)}\\
&\qquad \times [(\mu^+ + \mu^-)\{\mu^+(A^2 + (B^+)^2) - \mu^-(A^2 + (B^-)^2)\} - \mu^+\mu^-((B^+)^2 - (B^-)^2)
\end{aligned}$$

$$
\begin{aligned}
&\quad - (\mu^+ - \mu^-)A\{\mu^+(A+B^+) + \mu^-(A+B^-)\}] \, (\hat{h}_n^+(\xi',0,\lambda) - \hat{h}_n^-(\xi',0,\lambda)) \\
&- e^{B^- x_n} \frac{1}{\mu^+ B^+ + \mu^- B^-} \, (\hat{h}_j^+(\xi',0,\lambda) - \hat{h}_j^-(\xi',0,\lambda)),
\end{aligned}
$$

$$
\begin{aligned}
&\hat{v}_n^+(\xi',x_n,\lambda) \\
&= \frac{e^{-B^+x_n} - e^{-Ax_n}}{B^+ - A} A \, \frac{-i}{f(A,B^+,B^-)} [(\mu^+ - \mu^-)AB^+ + A(\mu^+B^+ + \mu^-B^-) \\
&\qquad + \mu^- B^-(B^+ + B^-)] \, \tilde{\xi}' \cdot (\hat{h}^{+'}(\xi',0,\lambda) - \hat{h}^{-'}(\xi',0,\lambda)) \\
&+ \frac{e^{-B^+x_n} - e^{-Ax_n}}{B^+ - A} A \, \frac{1}{f(A,B^+,B^-)} [\mu^+(A^2 + (B^+)^2) - \mu^-(A^2 + (B^-)^2) \\
&\qquad + \mu^-(A+B^-)(B^+ + B^-)] \, (\hat{h}_n^+(\xi',0,\lambda) - \hat{h}_n^-(\xi',0,\lambda)) \\
&+ e^{-B^+x_n} \frac{iA[\mu^+(A-B^+) - \mu^-(A-B^-)]}{f(A,B^+,B^-)} \, \tilde{\xi}' \cdot (\hat{h}^{+'}(\xi',0,\lambda) - \hat{h}^{-'}(\xi',0,\lambda)) \\
&- e^{-B^+x_n} \frac{A[\mu^+(A+B^+) + \mu^-(A+B^-)]}{f(A,B^+,B^-)} \, (\hat{h}_n^+(\xi',0,\lambda) - \hat{h}_n^-(\xi',0,\lambda)),
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
&\hat{v}_n^-(\xi',x_n,\lambda) \\
&= \frac{e^{B^-x_n} - e^{Ax_n}}{B^- - A} A \, \frac{i}{f(A,B^+,B^-)} [(\mu^- - \mu^+)AB^- + A(\mu^+B^+ + \mu^-B^-) \\
&\qquad + \mu^+ B^+(B^+ + B^-)] \, \tilde{\xi}' \cdot (\hat{h}^{+'}(\xi',0,\lambda) - \hat{h}^{-'}(\xi',0,\lambda)) \\
&+ \frac{e^{B^-x_n} - e^{Ax_n}}{B^- - A} A \, \frac{1}{f(A,B^+,B^-)} [-\mu^+(A^2 + (B^+)^2) + \mu^-(A^2 + (B^-)^2) \\
&\qquad + \mu^+(A+B^+)(B^+ + B^-)] \, (\hat{h}_n^+(\xi',0,\lambda) - \hat{h}_n^-(\xi',0,\lambda)) \\
&+ e^{B^-x_n} \frac{iA[\mu^+(A-B^+) - \mu^-(A-B^-)]}{f(A,B^+,B^-)} \, \tilde{\xi}' \cdot (\hat{h}^{+'}(\xi',0,\lambda) - \hat{h}^{-'}(\xi',0,\lambda)) \\
&+ e^{B^-x_n} \frac{-A[\mu^+(A+B^+) + \mu^-(A+B^-)]}{f(A,B^+,B^-)} \, (\hat{h}_n^+(\xi',0,\lambda) - \hat{h}_n^-(\xi',0,\lambda)),
\end{aligned}
$$

$$
\begin{aligned}
\hat{\theta}^+(\xi',x_n,\lambda) &= e^{-Ax_n} \frac{\mu^+(A+B^+)}{f(A,B^+,B^-)} \\
&\times [i\{(\mu^+ - \mu^-)AB^+ + A(\mu^+B^+ + \mu^-B^-) + \mu^-B^-(B^+ + B^-)\} \\
&\qquad \times \tilde{\xi}' \cdot (\hat{h}^{+'}(\xi',0,\lambda) - \hat{h}^{-'}(\xi',0,\lambda)) \\
&- \{\mu^+(A^2 + (B^+)^2) - \mu^-(A^2 + (B^-)^2) + \mu^-(A+B^-)(B^+ + B^-)\} \\
&\qquad \times (\hat{h}_n^+(\xi',0,\lambda) - \hat{h}_n^-(\xi',0,\lambda))],
\end{aligned}
$$

$$
\begin{aligned}
\hat{\theta}^-(\xi',x_n,\lambda) &= e^{Ax_n} \frac{\mu^-(A+B^-)}{f(A,B^+,B^-)} \\
&\times [i\{(\mu^- - \mu^+)AB^- + A(\mu^+B^+ + \mu^-B^-) + \mu^+B^+(B^+ + B^-)\} \\
&\qquad \times \tilde{\xi}' \cdot (\hat{h}^{+'}(\xi',0,\lambda) - \hat{h}^{-'}(\xi',0,\lambda))
\end{aligned}
$$

$$
\begin{aligned}
&- \{\mu^+(A^2+(B^+)^2) - \mu^-(A^2+(B^-)^2) - \mu^+(A+B^+)(B^+ + B^-)\} \\
&\qquad\qquad\qquad\qquad \times (\hat{h}_n^+(\xi',0,\lambda) - \hat{h}_n^-(\xi',0,\lambda))],
\end{aligned}
$$

where $\tilde{\xi}' = (\tilde{\xi}_1, \cdots, \tilde{\xi}_{n-1})$, $\tilde{\xi}_j = \xi_j/|\xi'|$ and

$$
\begin{aligned}
f(A,B^+,B^-) = &- (\mu^+ - \mu^-)^2 A^3 \\
&+ \{(3\mu^+ - \mu^-)\mu^+ B^+ + (3\mu^- - \mu^+)\mu^- B^-\} A^2 \\
&+ \{(\mu^+ B^+ + \mu^- B^-)^2 + \mu^+\mu^-(B^+ + B^-)^2\} A \\
&+ (\mu^+ B^+ + \mu^- B^-)(\mu^+(B^+)^2 + \mu^-(B^-)^2).
\end{aligned}
$$

Set

$$
\begin{aligned}
\mathbb{C}_+ &= \{\lambda = \gamma + i\tau \mid \gamma \geqq 0, \ \tau \in \mathbb{R}\setminus\{0\}\}, \\
\omega' &= \{(\xi',\lambda) \mid \xi' \in \mathbb{R}^{n-1}\setminus\{0\}, \ \lambda \in \mathbb{C}_+\}, \\
\omega &= \{(\xi',x_n,\lambda) \mid (\xi',\lambda) \in \omega', x_n \geqq 0\}.
\end{aligned}
$$

By Shibata-Shimizu [12], we obtained the following lemma.

Lemma 2.1. *We have the following two inequalities:*

$$
\begin{aligned}
\mathrm{Re}\, B_\pm &\geqq c(\mu^\pm)(|\lambda|^{\frac12} + |\xi'|), \\
|f(A,B^+,B^-)| &\geqq c(\mu^+,\mu^-)(|\lambda|^{\frac12} + |\xi'|)^3
\end{aligned}
$$

for every $(\xi',\lambda) \in \omega'$ with suitable positive constants $c(\mu^+,\mu^-)$ and $c(\mu_\pm)$ independent of λ and ξ'.

We estimate the term:

$$
\partial_t v_1^+(x,t) = -\mathcal{L}^{-1}\big[Ae^{-B_\lambda^+(\xi')x_n}\frac{\lambda[\mu^+(A+B^+) + \mu^-(A+B^-)]}{f(A,B^+,B^-)}(\hat{h}_n^+(\xi',0,\lambda) - \hat{h}_n^-(\xi',0,\lambda))\big](x',t)
$$

which is the partial Laplace-Fourier inverse transform of the time derivative of the fourth term of the right member of $\hat{v}_n^+(\xi',x_n,\lambda)$ in (2.5). Other terms are estimated similarly. If we put

$$
\ell(\xi',\lambda) = \frac{\lambda[\mu^+(A+B^+) + \mu^-(A+B^-)]}{f(A,B^+,B^-)},
$$

then noting $A = |\xi'|$ we write $\partial_t v_1^+(x,t)$ as follows:

$$
\begin{aligned}
\partial_t v_1^+(x,t) &= -\mathcal{L}^{-1}[|\xi'|e^{-B_\lambda^+(\xi')x_n}\ell(\xi',\lambda)(\hat{h}^+(\xi',0,\lambda) - \hat{h}^-(\xi',0,\lambda))](x',t) \\
&= \int_0^\infty \partial y_n \mathcal{L}^{-1}[|\xi'|e^{-B_\lambda^+(\xi')(x_n+y_n)}\ell(\xi',\lambda) \\
&\qquad\qquad \times \mathcal{L}[h^+(\xi',y_n,\lambda) - h^-(\xi',y_n,\lambda)](x',t)\, dy_n \\
&= e^{\gamma t}\int_0^\infty \mathcal{F}^{-1}_{\xi',\tau}[|\xi'|e^{-B_\lambda^+(\xi')(x_n+y_n)}\ell(\xi',\lambda) \\
&\qquad \times \{B_\lambda^+(\xi')\mathcal{L}[h^+ - h^-](y_n) - \mathcal{L}[\partial_n h^+ - \partial_n h^-](y_n)\}](x',t)\, dy_n.
\end{aligned}
$$

If we set

$$
h^{\pm\gamma}(x,t) = \mathcal{F}^{-1}_{\xi',\tau}[B_\lambda^+(\xi')\mathcal{L}[h^\pm] - \mathcal{L}[\partial_n h^\pm]](x',t),
$$

$$k_\tau(x) = \mathcal{F}_{\xi'}^{-1}[|\xi'|e^{-B_\lambda^+(\xi')x_n}\ell(\xi',\lambda)](x'),$$
$$[K_\tau f](x) = \int_{\mathbb{R}^n_+} k_\tau(x'-y', x_n+y_n)f(y)\,dy,$$

then we obtain

$$e^{-\gamma t}\partial_t v_1^+(x,t) = \int_0^\infty \mathcal{F}_\tau^{-1}[\mathcal{F}_{\xi'}^{-1}[|\xi'|e^{-B_\lambda^+(\xi')(x_n+y_n)}\ell(\xi',\lambda)] * \mathcal{F}_t[h^{+\gamma}-h^{-\gamma}](x')](t)\,dy_n$$
$$= \mathcal{F}_\tau^{-1}[K_\tau\mathcal{F}_t[h^{+\gamma}-h^{-\gamma}]](t).$$

Let us denote the space of X-valued C^∞- functions with compact support by $\mathcal{D}(\mathbb{R},X)$ and let $\mathcal{D}'(\mathbb{R},X) = \mathcal{B}(\mathcal{D}(\mathbb{R}),X)$ denote the space of X-valued distributions. The X-valued Schwartz spaces $\mathcal{S}(\mathbb{R},X)$ and $\mathcal{S}'(\mathbb{R},X)$ are defined similarly. Given $M \in L_{1,\mathrm{loc}}(\mathbb{R},\mathcal{B}(X,Y))$, we may define an operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R},X) \to \mathcal{S}'(\mathbb{R},Y)$ by means of

$$T_M\phi = \mathcal{F}^{-1}M\mathcal{F}\phi \quad \text{for all } \mathcal{F}\phi \in \mathcal{D}(\mathbb{R},X) \tag{2.6}$$

Since $\mathcal{F}^{-1}\mathcal{D}(\mathbb{R},X)$ is dense in $L_p(\mathbb{R},X)$, we see that T_M is well-defined and linear on a dense subset of $L_p(\mathbb{R},X)$. Concerning the boundedness of the operator T_M, the following theorem was proved by Weis [20].

Theorem 2.2. *Suppose that X and Y are UMD Banach spaces and let $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R}\setminus\{0\},\mathcal{B}(X,Y))$ such that the following conditions are satisfied:*

$$\mathcal{R}(\{M(\tau) \mid \tau \in \mathbb{R}\setminus\{0\}\}) = a_0 < \infty,$$
$$\mathcal{R}(\{\tau M'(\tau) \mid \tau \in \mathbb{R}\setminus\{0\}\}) = a_1 < \infty.$$

Then, the operator T_M defined by (2.6) is extended to a bounded linear operator from $L_p(\mathbb{R},X)$ into $L_p(\mathbb{R},Y)$ with norm

$$\|T_M\|_{\mathcal{B}(L_p(\mathbb{R},X),L_p(\mathbb{R},Y))} \leqq C(a_0+a_1),$$

where $C > 0$ depends only on p, X and Y.

For

$$e^{-\gamma t}\partial_t v_1^+(x,t) = \mathcal{F}_\tau^{-1}[K_\tau\mathcal{F}_t[h^{+\gamma}-h^{-\gamma}]](t),$$

if $\{K_\tau \mid \tau \in \mathbb{R}\setminus\{0\}\}$ and $\{\tau\partial_\tau K_\tau \mid \tau \in \mathbb{R}\setminus\{0\}\}$ are $\mathcal{R}$-bounded on $\mathcal{B}(L_q(\mathbb{R}^n_+))$, then by Theorem 2.2 we obtain

$$\|e^{-\gamma t}\,\partial_t v_1^+\|_{L_p(\mathbb{R}_+,L_q(\mathbb{R}^n_+))} \leqq C_{p,q}\sum_{+-}\|h^{\pm\gamma}\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_\pm))}$$
$$\leqq C\sum_{+-}(\|e^{-\gamma t}\,h^\pm\|_{H_p^{1/2}(\mathbb{R},L_q(\mathbb{R}^n_\pm))} + \|e^{-\gamma t}\,h^\pm\|_{L_p(\mathbb{R},W_q^1(\mathbb{R}^n_\pm))})$$
$$\leqq C\sum_{+-}\|e^{-\gamma t}\,h^\pm\|_{H_{q,p}^{1,1/2}(\mathbb{R}^n_\pm\times\mathbb{R})}, \tag{2.7}$$

because

$$h^{\pm\gamma}(x,t) = \mathcal{F}_{\xi',\tau}^{-1}[(\lambda+|\xi'|^2)^{1/2}\mathcal{F}_{\xi',\tau}[e^{-\gamma t}h^\pm] - \mathcal{F}_{\xi',\tau}[e^{-\gamma t}\partial_n h^\pm]](x',t).$$

Therefore what we have to do to obtain (2.7) is that we show the $\mathcal{R}$-boundedness of the families $\{K_\tau \mid \tau \in \mathbb{R}\setminus\{0\}\}$ and $\{\tau\partial_\tau K_\tau \mid \tau \in \mathbb{R}\setminus\{0\}\}$ on $\mathcal{B}(L_q(\mathbb{R}^n_+))$ for $1 < q < \infty$. To do this, we shall use the following proposition due to Denk, Hieber and Prüss [8].

Proposition 2.3. *Let $1 < q < \infty$. Let G be a domain in $\mathbb{R}^n$ and $\mathcal{T} = \{T_\tau \mid \tau \in \mathcal{M}\} \subset \mathcal{B}(L_q(G))$ be a family of the kernel operators:*

$$T_\tau f(x) = \int_G k_\tau(x,y) f(y)\,dy$$

for $x \in G$ and $f \in L_q(G)$. Suppose that there exists a $k_0(x,y)$ such that

$$|k_\tau(x,y)| \leqq k_0(x,y)$$

for almost all x, $y \in G$ and any $\tau \in \mathcal{M}$. Set

$$T_0 f(x) = \int_G k_0(x,y) f(y)\,dy$$

If $T_0 \in \mathcal{B}(L_q(G))$, then $\mathcal{T}$ is $\mathcal{R}$-bounded on $\mathcal{B}(L_q(G))$, whose $\mathcal{R}$-bound is less than or equal to $C_{n,q,G}\|T_0\|_{\mathcal{B}(L_q(G))}$.

If we set

$$[K_0 f](x) = \int_{\mathbb{R}^n_+} \frac{c_0}{[|x'-y'|^2 + (x_n+y_n)^2]^{n/2}} f(y)\,dy$$

then $K_0 \in \mathcal{B}(L_q(\mathbb{R}^n_+))$. Therefore if we can show

$$|k_\tau(x)| \leqq c_0|x|^{-n}, \quad |\tau\partial_\tau k_\tau(x)| \leqq c_0|x|^{-n}, \tag{2.8}$$

then by Proposition 2.3, we obtain that $\{K_\tau \mid \tau \in \mathbb{R}\setminus\{0\}\}$ and $\{\tau\partial_\tau K_\tau \mid \tau \in \mathbb{R}\setminus\{0\}\}$ are $\mathcal{R}$-bounded on $\mathcal{B}(L_q(\mathbb{R}^n_+))$.

In order to show (2.8), we use the following lemma ([11, Theorem 2.3]).

Lemma 2.4. *Let X be a Banach space and $\|\cdot\|_X$ its norm. Let a be a number $> -n$ and set $a = N + \sigma - n$, where $N \geqq 0$ is an integer and $0 < \sigma \leqq 1$. Let $f(\xi)$ be a function in $C^\infty(\mathbb{R}^n \setminus \{0\}, X)$ such that*

$$\partial_\xi^\alpha f(\xi) \in L_1(\mathbb{R}^n, X), \quad \forall\,|\alpha| \leqq N,$$
$$\|\partial_\xi^\alpha f(\xi)\|_X \leqq C_\alpha |\xi|^{a-|\alpha|}, \quad \forall \xi \neq 0,\ \forall \alpha \in \mathbb{N}_0^n.$$

Then we have

$$\|\mathcal{F}^{-1}[f](x)\|_X \leqq C_{n,a}(\max_{|\alpha|\leqq N+2} C_\alpha)|x|^{-(n+a)}, \quad \forall x \neq 0.$$

Since

$$|\partial_{\xi'}^{\alpha'} \ell(\xi',\lambda)]| \leqq C_{\alpha'}|\xi'|^{-|\alpha'|} \quad \text{for} \quad \alpha' \in \mathbb{N}_0^{n-1},$$

it follows that

$$|\partial_{\xi'}^{\alpha'}[|\xi'|e^{-B_\lambda^+(\xi')x_n}\ell(\xi',\lambda)]| \leqq C_{\alpha'}|\xi'|^{1-|\alpha'|}e^{-d|\xi'|x_n} \quad \forall \alpha' \in \mathbb{N}_0^{n-1}$$

with some $d > 0$, by Lemma 2.4 we have

$$|k_\tau(x)| \leqq c|x'|^{-(n-1+1)}.$$

By changing valuable $\xi' x_n = \eta'$ and putting $|\alpha'| = 0$

$$|k_\tau(x)| \leqq c\int_{\mathbb{R}^{n-1}} |\xi'|e^{-d|\xi'|x_n}\,d\xi' \leqq c|x_n|^{-n}\int_{\mathbb{R}^{n-1}} |\eta'|e^{-d\eta'}\,d\eta' \leqq c|x_n|^{-n}.$$

Similarly we prove $|\tau\partial_\tau k_\tau(x)| \leqq c_0|x|^{-n}$, thus we have showed (2.8).

References

[1] H. Abels, *On generalized solutions of two-phase flows for viscous incompressible fluids*, Interface and Free Boundaries, to appear.

[2] I. V. Denisova, *A priori estimates for the solution of a linear time-dependent problem connected with the motion of a drop in a fluid medium*, Trudy Mat. Inst. Steklov., **188** (1990) 3–21 (in Russian); English transl.: Proc. Steklov Inst. Math., **188** (1991) 1–24.

[3] I. V. Denisova, *Solvability in Hölder spaces of a linear problem concerning the motion of two fluids separated by a closed surface*, Algebra i Analiz, **5** (1993) 122–148 (in Russian); English transl.: St.Petersburg Math. J., **5** (1994) 765–787.

[4] I. V. Denisova, *Problem of the motion of two viscous incompressible fluids separated by a closed free interface*, Acta Appl. Math., **37** (1994) 31–40.

[5] I. V. Denisova and V. A. Solonnikov, *Solvability of a linearized problem on the motion of a drop in a fluid flow*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **171** (1989) 53–65 (in Russian); English transl.: J. Soviet Math., **56** (1991) 2309–2316.

[6] I. V. Denisova and V. A. Solonnikov, *Solvability in Hölder spaces of a model initial-boundary value problem generated by a problem on the motion of two fluids*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **181** (1991) 5–44 (in Russian); English transl.: J. Math. Sci., **70** (1994) 1717–1746.

[7] I. V. Denisova and V. A. Solonnikov, *Classical solvability of the problem on the the motion of two viscous incompressible fluids*, Algebra i Analiz **7** (1995) 101–142 (in Russian); English transl.: St.Petersburg Math. J., **7** (1996) 755–786.

[8] R. Denk, M. Hieber and J. Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Memoirs of AMS, Vol 166, No. 788, 2003.

[9] Y. Giga and Sh. Takahashi, *On global weak solutions of the nonstationary two-phase Stokes flow*, SIAM J. Math. Anal., **25** (1994) 876–893.

[10] A. Nouri and F. Poupaud *An existence theorem for the multifluid Navier-Stokes problem*, J. Differential Equations, **123** (1995) 71–88.

[11] Y. Shibata and S. Shimizu, *A decay property of the Fourier transform and its application to the Stokes problem*, J. Math. Fluid Mech. **3** (2001) 213-230.

[12] Y. Shibata and S. Shimizu, *On a resolvent estimate of the interface problem for the Stokes system in a bounded domain*, J. Differential Equations, **191** (2003) 408-444.

[13] Y. Shibata and S. Shimizu, *L_p-L_q maximal regularity and viscous incompressible flows with free surface*, Proc. Japan Acad. Ser. A Math. Sci., **81** (2005) 151–155.

[14] Y. Shibata and S. Shimizu, *A On the L_p-L_q maximal regularity of the Neumann problem for the Stokes equations in a bounded domain*, Preprint (2005).

[15] Y. Shibata and S. Shimizu, *On some free boundary problem for the Navier-Stokes equations*, Differential Integral Equations, to appear.

[16] O. Steiger, *On Navier-Stokes equations with first order boundary conditions*, Ph.D. thesis, Universität Zürich, 2004.

[17] Sh. Takahashi, *On global weak solutions of the nonstationary two-phase Navier-Stokes flow*, Adv. Math. Sci. Appl., **5** (1995) 321–342.

[18] N. Tanaka, *Global existence of two phase non-homogeneous viscous incompressible weak fluid flow*, Commun. Partial Differential Equations, **18** (1993) 41–81.

[19] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, 2nd edition, Johann Ambrosius Barth, Heidelberg, 1995.

[20] L. Weis, *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann., **319** (2001) 735–758.