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Discontinuity of the straightening map for a family of renormalizable polynomials

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Abstract

We study straightening maps of families of polynomial-like maps and give a necessary condition for a straightening map to be continuous. We also apply this result to a family of real renormalizable polynomials and show that its straightening map is discontinuous.

1 Polynomial-like maps and straightening maps

Definition (Polynomial-like maps). A map $f : U' \to U$ is called a *polynomial-like map* if it is proper and holomorphic, U' and U are topological disks and $U' \Subset U$.

The *filled Julia set* of a polynomial-like map $f : U' \to U$ is defined by $K(f; U', U) = \bigcap_{n>0} f^{-n}(U')$ and $J(f; U', U) = \partial K(f; U', U)$ is called the *Julia set.*

Definition (Hybrid equivalence). Two polynomial-like maps (or polynomials) $f : U' \to U$ and $g : V' \to V$ are said to be *hybrid equivalent* if there exists a quasiconformal map ψ defined between their filled Julia sets such that $g \circ \psi = \psi \circ f$ and $\bar{\partial}\psi \equiv 0$ a.e. on $K(f; U', U)$.

Although hybrid conjugacy ψ is not unique, it is uniquely determined if restricted to $K(f; U', U)$ (up to affine self conjugacy for g).

Douady and Hubbard [DH] proved the following theorem using quasiconformal surgery.

Theorem 1.1 (Straightening theorem). *Any polynomial-like map $f : U' \to U$ of degree d is hybrid equivalent to some polynomial g of degree d. Furthermore, if $K(f; U', U)$ is connected, then g is unique up to affine conjugacy.*

Consider a holomorphic family $(f_\lambda : U'_\lambda \to U_\lambda)_{\lambda \in \Lambda}$ of polynomial-like maps of degree d. Let

$$\mathcal{C}_\Lambda = \{\lambda \in \Lambda;\ K(f_\lambda; U'_\lambda, U_\lambda)\text{: connected}\}$$

be the *connectedness locus.* By the straightening theorem, we can define the *straight-*

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ening map $\mathcal{S}_\Lambda : \mathcal{C}_\Lambda \to \mathcal{C}_d$ by $\mathcal{S}_\Lambda(\lambda) = g \in \mathcal{C}_d$ if $f_\lambda : U'_\lambda \to U_\lambda$ is hybrid equivalent to g where $\mathcal{C}_d$ is the connectedness locus of affine conjugacy classes of polynomials of degree d.

In [DH], Douady and Hubbard showed that if the degree $d = 2$, the straightening map is continuous. However, they also constructed an example to show that straightening maps are not continuous in general when $d \geq 3$, by using parabolic implosion.

Hence in view of the study of parameter spaces of polynomials, it is natural to ask whether straightening maps are continuous or not for a polynomial-like family which is a restriction of a family of polynomials (straightening maps for renormalizable family of polynomials):

Question. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of polynomial of degree d' and assume $\{f_\lambda : U'_\lambda \to U_\lambda\}_{\lambda\in\Lambda}$ is a holomorphic family of polynomial-like maps of degree $d < d'$. Then is the straightening map $\mathcal{S}_\Lambda : \mathcal{C}_\Lambda \to \mathcal{C}_d$ continuous?

Here we give an example of a discontinuous straightening map for a family of cubic polynomial-like restrictions of polynomials of degree 5.

2 Parabolic implosion

We refer some basic facts about parabolic implosion here.

Assume 0 is a non-degenerate 1-parabolic periodic point of period p for a polynomial f_0 of degree $d \geq 2$, i.e., $f_0^n(0) \neq 0$ for $0 < n < p$, $f_0^p(0) = 0$, $(f_0^p)'(0) = 1$ and $(f_0^p)''(0) \neq 0$. By taking linear conjugacy, we may assume f_0^p has the form

$$f_0^p(z) = z + z^2 + O(z^3).$$

near 0. Then for sufficiently small $\varepsilon > 0$, there exist conformal maps $\Phi_{f_0,\mathrm{attr}} : D_{f_0,\mathrm{attr}} = \{|z+\varepsilon| < \varepsilon\} \to \mathbb{C}$ and $\Phi_{f_0,\mathrm{rep}} : D_{f_0,\mathrm{rep}} = \{|z-\varepsilon| < \varepsilon\} \to \mathbb{C}$ such that

$$\Phi_{f_0,*}(f_0^p(z)) = \Phi_{f_0,*}(z) + 1 \quad (* = \mathrm{attr}, \mathrm{rep}). \tag{1}$$

We call $\Phi_{f_0,\mathrm{attr}}$ (resp. $\Phi_{f_0,\mathrm{rep}}$) an *attracting Fatou coordinate* (resp. *repelling Fatou coordinate*) for f_0. Fatou coordinates are unique up to post-composition by translation. By using the functional equation (1), $\Phi_{f_0,\mathrm{attr}}$ can be extended on the whole basin of attraction B_0 of 0 and $\Phi_{f_0,\mathrm{rep}}^{-1}$ can be extended on the whole plane $\mathbb{C}$. For $c \in \mathbb{C}$, let us denote $g_{f_0,c}(z) = \Phi_{f_0,\mathrm{rep}}^{-1}(\Phi_{f_0,\mathrm{attr}}(z) + c) : B_0 \to \mathbb{C}$. We call $g_{f_0,c}$ a *Lavaurs map* of f and c the *phase* of $g_{f_0,c}$. By definition, $f \circ g_{f_0,c} = g_{f_0,c} \circ f$ on B_0.

Let f be a polynomial near f_0. By taking an affine conjugacy, we may assume that 0 is still a p-periodic point for f. Let us denote by $(f^p)'(0) = \exp(2\pi i\alpha)$ with $|\alpha|$ small. Let x be the other p-periodic point near 0. If $|\arg(\alpha)| < \frac{\pi}{4}$ (or $|\arg(\alpha) - \pi| < \frac{\pi}{4}$), then for small $\varepsilon > 0$, there exist two disks D_{attr} and $D_{f,\mathrm{rep}}$ of radius ε whose boundaries contain 0 and x (we assume D_{attr} intersects the negative real axis and $D_{f,\mathrm{rep}}$ intersects the positive real axis), and

$$f(\partial D_{f,\mathrm{attr}} \setminus \overline{D_{f,\mathrm{rep}}}) \subset D_{f,\mathrm{attr}},$$
$$\partial D_{f,\mathrm{rep}} \setminus \overline{D_{f,\mathrm{attr}}} \subset f(D_{f,\mathrm{rep}}).$$

Furthermore, there exists a conformal map Φ_f defined on $D_{\mathrm{attr}} \cup D_{\mathrm{rep}}$ such that $\Phi_f(f^p(z)) = \Phi_f(z) + 1$. This is called a *Fatou coordinate* for f. It is also unique up to post-composition by translation. Furthermore, Fatou coordinates depend continuously on parameters if we normalize properly. For example, if $f_n \to f_0$, then there exist sequences c_n and C_n such that $\Phi_{f_n} + c_n \to \Phi_{f,\mathrm{attr}}$ on $D_{f,\mathrm{attr}}$ and $\Phi_{f_n} + C_n \to \Phi_{f,\mathrm{rep}}$ on $D_{f,\mathrm{rep}}$ as $n \to \infty$.

Hence

$$\begin{aligned} f_n^m &= \Phi_{f_n}^{-1}(\Phi_{f_n}(z) + m) \\ &= \Phi_{f_n}^{-1}(\Phi_{f_n}(z) + c_n + (m - c_n + C_n) - C_n). \end{aligned}$$

By passing to a subsequence, we may assume that $c_n - C_n$ converges in $\mathbb{C}/\mathbb{Z}$ as $n \to \infty$, namely, there exists some sequence $m_n \in \mathbb{Z}$ such that $\lim_{n\to\infty} m_n - c_n + C_n = c$. Therefore,

$$\begin{aligned} f_n^{m_n}(z) &\to \Phi_{f_0,\mathrm{rep}}^{-1}(\Phi_{f_0,\mathrm{attr}}(z) + c) \\ &= g_{f_0,c}. \end{aligned}$$

For such a case, we denote $f_n \xrightarrow{\mathrm{geom}} (f_0, g_{f_0,c})$ and we say that f_n *geometrically converges to* $(f_0, g_{f_0,c})$.

3 Continuous straightening maps

Theorem 3.1. *Let $(f_\lambda : U'_\lambda \to U_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of polynomial-like maps of degree $d \geq 2$. Assume*

(i) *$0 \in \mathcal{C}_\Lambda$ and the straightening map $\mathcal{S}_\Lambda$ is continuous on $\mathcal{C}_\Lambda$;*
(ii) *for any $\lambda \in \Lambda$, 0 is a periodic point of period p for f_λ. It is 1-parabolic and non-degenerate for f_0;*
(iii) *α_λ is a repelling periodic point for λ;*
(iv) *ω_λ and ω'_λ are distinct critical points for f_λ. They lie in the basin of 0 for f_0 and there exist $N, N' \geq 0$ such that $f^N(\omega_0) = f^{N'}(\omega'_0)$;*
(v) *for any $\varepsilon > 0$, there exist some λ_0 and some sequence $\lambda_n \to \lambda_0$ such that*
   - *$\lambda_0, \lambda_n \in \mathcal{C}_\Lambda$.*
   - *$|\lambda_0| < \varepsilon$.*
   - *$0 < |f_{\lambda_0}^N(\omega_{\lambda_0}) - f_{\lambda_0}^{N'}(\omega'_{\lambda_0})| < \varepsilon$.*
   - *0 is 1-parabolic and non-degenerate for f_{λ_0}.*
   - *$f_{\lambda_n} \xrightarrow{\mathrm{geom}} (f_{\lambda_0}, g)$ with $g(f_{\lambda_0}^N(\omega_{\lambda_0})) = \alpha_{\lambda_0}$.*

Then

$$|\operatorname{mult}_{f_0}(\alpha_0)| = |\operatorname{mult}_{f_0}(\alpha(P_0))| \tag{2}$$

where $P_0 = \mathcal{S}_\Lambda(f_0) \in \mathcal{C}_d$*,* $\alpha(P_0) = \psi_0(\alpha_0)$*,* ψ_0 *is a hybrid conjugacy from* f_0 *to* P_0*, and* $\operatorname{mult}_f(\alpha)$ *is the multiplier for* f *at* α ($\operatorname{mult}_f(\alpha) = f^p(\alpha)$ *if* α *is a periodic point of period* p)*.*

4 Discontinuity of straightening maps

In this section, we give an application of Theorem 3.1 that answers the question in Section 1. Here, we only consider real polynomials, so we always assume Fatou coordinates and Lavaurs maps are also real (i.e., the real axis is mapped to the real axis).

Let

$$P_0(z) = 1 - 1.5645...z^2 - 0.30368...z^3,$$
$$\tilde{f}_0(z) = -z - 1.2558...z^3 + 2.8793...z^4 + z^5.$$

Then

- P_0 has a quadratic-like restriction hybrid equivalent to $Q(z) = z^2 - 1.75$, which has a non-degenerate 1-parabolic orbit of period 3. Let $p(Q) = -1.7469...$ be the smallest point of the periodic orbit and let $p(P_0)$ be the corresponding periodic point for P_0.
- $\operatorname{Crit}(P_0) = \{\omega(P_0), \omega'(P_0)\} \subset \mathbb{R}$ with $\omega(P_0) < \omega'(P_0)$ and
$$P_0(\omega(P_0)) = P_0^2(\omega'(P_0)).$$
- The fixed point $\alpha(P_0)$ for P_0 which corresponds to the α-fixed point of Q is repelling and contained in the domain of definition of the repelling Fatou coordinates $\Phi_{P_0,\mathrm{rep}}$ for P_0 at $p(P_0)$ and $\Phi_{P_0,\mathrm{rep}}(\alpha(P_0)) \in \mathbb{R}$. Therefore, $\alpha(P_0)$ is contained in the image of the real axis by a Lavaurs map $g_{f_0,c}$ for any $c \in \mathbb{R}$.
- $\tilde{f}_0$ is topologically conjugate to P_0 on the real axis.
- The fixed point $\alpha(\tilde{f}_0)$ corresponding to $\alpha(P_0)$ has multiplier -1.
- There exists f_0 arbitrarily close to $\tilde{f}_0$ such that f_0 has a real cubic-like restriction hybrid equivalent to P_0.

Furthermore, we can verify that there exists some neighborhood Λ of f_0 in Poly_5 such that we can restrict Λ to a holomorphic family of cubic-like maps which satisfies all the assumption of Theorem 3.1 except the continuity of the straightening map. However, since $\operatorname{mult}_{f_0}(\alpha_0)$ is arbitrarily close to $\operatorname{mult}_{\tilde{f}_0}(\alpha(\tilde{f}_0)) = -1$, $|\operatorname{mult}_{f_0}(\alpha_0)| \neq |\operatorname{mult}_{P_0}(\alpha(P_0))$. Therefore, $\mathcal{S}_\Lambda$ is not continuous on arbitrarily small neighborhood of f_0.

Hence we have proved the following:

Theorem 4.1. *There exist a polynomial f_0 of degree $d' \geq 4$ and a neighborhood Λ such that*

- *any $f \in \Lambda$ has a polynomial-like restriction $f : U'_f \to U_f$ of degree $d \geq 3$ and this forms a holomorphic family of polynomial-like maps;*
- *$f_0 : U'_{f_0} \to U_{f_0}$ lies in the connectedness locus.*
- *the straightening map $\mathcal{S}_\Lambda$ is not continuous on any neighborhood of f_0.*

Remark 1. It is likely that for any repelling periodic points, there exists a good perturbation to apply Theorem 3.1. If this is true, then for any repelling periodic point x of f_0, the moduli of multipliers of x and the corresponding periodic points for P_0 must coincide. Then, by the result of Prado [Pr] and Przytycki and Urbanski [PU], $f_0 : U'_{f_0} \to U_{f_0}$ and P_0 are conformally conjugate.

5 Proof of Theorem 3.1

First, we introduce a conformal invariant defined by the difference between Fatou coordinates of critical orbits.

Definition. Let f be a polynomial or a polynomial-like map having a Fatou coordinate Φ. Assume there exist two (marked) critical points ω, ω' and $N, N' \geq 0$ such that $f^N(\omega)$ and $f^{N'}(\omega)$ lie in the domain of definition of Φ. Then define $\theta(f) = \theta(f, \Phi) \in \mathbb{C}/\mathbb{Z}$ by

$$\theta(f) = \Phi(f^N(\omega)) - \Phi(f^{N'}(\omega')). \tag{3}$$

Note that $\theta(f)$ does not depend on the choice of a Fatou coordinate because it is unique up to post-composition by translation and it is canceled by taking the difference. It also does not depend on N and N' because of the functional equation $\Phi(f(z)) = \Phi(z) + 1$ (as long as the orbit of ω or ω' does not escape from the domain of definition of Φ).

Under the assumption of Theorem 3.1, we consider (attracting) Fatou coordinates near 0 and N and N' as in iv of Theorem 3.1. In the following, let us denote simply $\Phi_{\lambda,\mathrm{attr}}$, $\Phi_{\lambda,\mathrm{rep}}$ and Φ_λ instead of $\Phi_{f_\lambda,\mathrm{attr}}$, $\Phi_{f_\lambda,\mathrm{rep}}$ and Φ_{f_λ}. For example,

$$\theta(f_{\lambda_0}) = \Phi_{\lambda_0,\mathrm{attr}}(f_{\lambda_0}^N(\omega_{\lambda_0})) - \Phi_{\lambda_0,\mathrm{attr}}(f_{\lambda_0}^{N'}(\omega'_{\lambda_0}))$$
$$\theta(f_{\lambda_n}) = \Phi_{\lambda_n}(f_{\lambda_n}^N(\omega_{\lambda_n})) - \Phi_{\lambda_n}(f_{\lambda_n}^{N'}(\omega'_{\lambda_n})).$$

Now suppose the assumption of Theorem 3.1 holds. Take λ, $\lambda_n \to \lambda$ which satisfy the assumption v in Theorem 3.1 for small $\varepsilon > 0$. For $\lambda \in \mathcal{C}_\Lambda$, let $P_\lambda = \mathcal{S}_\Lambda(\lambda)$ be the polynomial of degree d hybrid equivalent to $f_\lambda : U'_\lambda \to U_\lambda$ with a hybrid conjugacy ψ_λ. By assumption, P_λ depends continuously on λ. Let us denote by Φ^P_λ ($\Phi^P_{\lambda,\mathrm{attr}}$, or $\Phi^P_{\lambda,\mathrm{rep}}$) be a Fatou coordinate(s) for P_λ, and by $\omega(P_\lambda) = \psi_\lambda(\omega_\lambda)$, $\omega'(P_\lambda) = \psi_\lambda(\omega'_\lambda)$ the corresponding critical points for P_λ.

Since ψ_{λ_0} is holomorphic in the basin of 0, an attracting Fatou coordinate $\Phi^P_{\lambda_0,\mathrm{attr}}$ for P_{λ_0} can be written as $\Phi^P_{\lambda_0,\mathrm{attr}} = \Phi_{\lambda_0,\mathrm{attr}} \circ \psi^{-1}_{\lambda_0}$. Therefore,

$$\begin{aligned}\theta(P_{\lambda_0}) &= \Phi^P_{\lambda_0,\mathrm{attr}}(P^N_{\lambda_0}(\omega(P_{\lambda_0}))) - \Phi^P_{\lambda_0,\mathrm{attr}}(P^{N'}_{\lambda_0}(\omega'(P_{\lambda_0})))\\ &= \Phi_{\lambda_0,\mathrm{attr}} \circ \psi^{-1}_{\lambda_0}(P^N_{\lambda_0}(\psi_{\lambda_0}(\omega_{\lambda_0}))) - \Phi_{\lambda_0,\mathrm{attr}} \circ \psi^{-1}_{\lambda_0}(P^{N'}_{\lambda_0}(\psi_{\lambda_0}(\omega'_{\lambda_0})))\\ &= \Phi_{\lambda_0,\mathrm{attr}}(f^N_{\lambda_0}(\omega_{\lambda_0})) - \Phi_{\lambda_0,\mathrm{attr}}(f^{N'}_{\lambda_0}(\omega'_{\lambda_0}))\\ &= \theta(f_{\lambda_0}).\end{aligned}$$

By the continuity of Fatou coordinates, we may assume $\Phi^P_{\lambda_n} \to \Phi^P_{\lambda_0,\mathrm{attr}}$. Thus, since $P_{\lambda_n} \to P_{\lambda_0}$, we have

$$\begin{aligned}\theta(P_n) &= \Phi^P_{\lambda_n}(P^N_{\lambda_n}(\omega(P_{\lambda_n}))) - \Phi^P_{\lambda_n}(P^{N'}_{\lambda_n}(\omega'(P_{\lambda_n})))\\ &\xrightarrow{n\to\infty} \theta(P_{\lambda_0}) = \theta(f_{\lambda_0}).\end{aligned}$$

On the other hand, since $f_{\lambda_n} \xrightarrow{\text{geom}} (f_{\lambda_0}, g)$, there exists a sequence (m_n) such that $f^{m_n}_{\lambda_n} \to g = \Phi^{-1}_{\lambda_0,\mathrm{rep}} \circ \Phi_{\lambda_0,\mathrm{attr}}$ (we may assume the phase is equal to zero by replacing the Fatou coordinates). Hence

$$\begin{aligned}\theta(f_{\lambda_0}) &= \Phi_{\lambda_0,\mathrm{attr}}(f^N_{\lambda_0}(\omega_{\lambda_0})) - \Phi_{\lambda_0,\mathrm{attr}}(f^{N'}_{\lambda_0}(\omega'_{\lambda_0}))\\ &= \Phi_{\lambda_0,\mathrm{rep}} \circ g(f^N_{\lambda_0}(\omega_{\lambda_0})) - \Phi_{\lambda_0,\mathrm{rep}} \circ g(f^{N'}_{\lambda_0}(\omega'_{\lambda_0}))\\ &= \Phi_{\lambda_0,\mathrm{rep}}(\alpha_{\lambda_0}) - \Phi_{\lambda_0,\mathrm{rep}} \circ g(f^{N'}_{\lambda_0}(\omega'_{\lambda_0})).\end{aligned}$$

By passing to a further subsequence, we may assume $\psi_{\lambda_n} \to \varphi$, which is a quasiconformal conjugacy between f_{λ_0} and P_{λ_0} (note that we may assume ψ_λ is uniformly K-quasiconformal for some K, by using the tubing construction in [DH]). Furthermore, since there exists a sequence C_n such that $\Phi_{\lambda_n} + C_n \to \Phi_{\lambda_0,\mathrm{rep}}$ on the repelling side, we have

$$\begin{aligned}\theta(P_n) &= (\Phi^P_{\lambda_n}(P^N_{\lambda_n}(\omega(P_{\lambda_n}))) + m_n) - (\Phi^P_{\lambda_n}(P^{N'}_{\lambda_n}(\omega'(P_{\lambda_n}))) + m_n)\\ &= \Phi^P_{\lambda_n} \circ P^{m_n}_{\lambda_n}(P^N_{\lambda_n}(\psi_{\lambda_n}(\omega_{\lambda_n}))) - \Phi^P_{\lambda_n} \circ P^{m_n}_{\lambda_n}(P^{N'}_{\lambda_n}(\psi_{\lambda_n}(\omega'_{\lambda_n})))\\ &= (\Phi^P_{\lambda_n} \circ \psi_{\lambda_n} \circ f^{m_n}_{\lambda_n}(f^N_{\lambda_n}(\omega_{\lambda_n})) + C_n) - (\Phi^P_{\lambda_n} \circ \psi_{\lambda_n} \circ f^{m_n}_{\lambda_n}(f^{N'}_{\lambda_n}(\omega'_{\lambda_n})) - C_n)\\ &\xrightarrow{n\to\infty} \Phi^P_{\lambda_0,\mathrm{rep}} \circ \varphi \circ g(f^N_{\lambda_0}(\omega_{\lambda_0})) - \Phi^P_{\lambda_0,\mathrm{rep}} \circ \varphi \circ g(f^{N'}_{\lambda_0}(\omega'_{\lambda_0}))\\ &= \Phi^P_{\lambda_0,\mathrm{rep}} \circ \varphi(\alpha_{\lambda_0}) - \Phi^P_{\lambda_0,\mathrm{rep}} \circ \varphi \circ g(f^{N'}_{\lambda_0}(\omega'_{\lambda_0})).\end{aligned}$$

Therefore, we have proved

Lemma 5.1.

$$\Phi_{\lambda_0,\mathrm{rep}}(\alpha_{\lambda_0}) - \Phi_{\lambda_0,\mathrm{rep}}(w) = \Phi^P_{\lambda_0,\mathrm{rep}} \circ \varphi(\alpha_{\lambda_0}) - \Phi^P_{\lambda_0,\mathrm{rep}} \circ \varphi(w). \tag{4}$$

where $w = g(f^{N'}_{\lambda_0}(\omega'_{\lambda_0}))$ tends to α_{λ_0} as $\varepsilon \to 0$.

For a map h defined near a point z_0, let us denote

$$\operatorname{dist}_{z_0}(h) = \lim_{z \to z_0} \left| \frac{h(z) - h(z_0)}{z - z_0} \right|$$

if the limit exists.

It is easy to see the following lemma:

Lemma 5.2. *If* $|\operatorname{mult}_{f_{\lambda_0}}(\alpha_{\lambda_0})| \neq |\operatorname{mult}_{P_{\lambda_0}}(\alpha(P_{\lambda_0}))|$, *then the distortion of* φ *or* φ^{-1} *diverges. Namely,*

$$\lim_{z \to \alpha_{\lambda_0}} \left| \frac{\varphi(z) - \varphi(\alpha_{\lambda_0})}{z - \alpha_{\lambda_0}} \right| = 0 \text{ or } \infty. \tag{5}$$

Since $\Phi_{\lambda_0,\mathrm{rep}}$, g and $\Phi^P_{\lambda_0,\mathrm{rep}}$ is conformal near α_{λ_0}, $f^N_{\lambda_0}(\omega_{\lambda_0})$ and $\alpha(P_{\lambda_0})$ respectively, their distortion $\operatorname{dist}_{\alpha_{\lambda_0}}(\Phi_{\lambda_0,\mathrm{rep}})$, $\operatorname{dist}_{f^N_{\lambda_0}(\omega_{\lambda_0})}(g)$ and $\operatorname{dist}_{\alpha(P_{\lambda_0})}(\Phi^P_{\lambda_0,\mathrm{rep}})$ are bounded away from zero and infinity (note that this estimate does not depend on λ_0 because of the continuity of Fatou coordinates). By the equality (4), this implies that $\operatorname{dist}_{\alpha_{\lambda_0}}(\varphi)$ is uniformly bounded away from zero and infinity. Therefore, by Lemma 5.2, we have for $\varepsilon > 0$ sufficiently small,

$$|\operatorname{mult}_{f_{\lambda_0}}(\alpha_{\lambda_0})| = |\operatorname{mult}_{P_{\lambda_0}}(\alpha(P_{\lambda_0}))|. \tag{6}$$

Therefore, we have proved that there exists λ_0 arbitrarily close to 0 with 6. Therefore, we have proved Theorem 3.1.

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