Moduli space of polynomial maps

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In the study of the dynamics of a polynomial map $f$, the eigenvalues of the fixed points of $f$ play a very important role to characterize the original map $f$. In this paper, we shall study how many affine conjugacy classes of polynomial maps are there when the eigenvalues of their fixed points are specified.

For a natural number $d$ with $d \geq 2$, we denote the moduli space of polynomial maps of degree $d$ by

$$\tilde{P}_d := \{ f \in \mathbb{C}[z] \mid \deg f = d \} / \sim,$$

where $\sim$ denotes the affine conjugacy of polynomial maps, i.e., for $f, g \in \mathbb{C}[z]$, $f \sim g$ holds if and only if there exists an affine transformation $\gamma(z) = az + b$ $(a, b \in \mathbb{C}, a \neq 0)$ such that $f = \gamma \circ g \circ \gamma^{-1}$. We put

$$\text{Fix}(f) := \{ \zeta \in \mathbb{C} \mid f(\zeta) = \zeta \}$$

for $f \in \mathbb{C}[z]$, where $\text{Fix}(f)$ is considered counted with multiplicity. Hence we always have $\#(\text{Fix}(f)) = \deg f$.

**Proposition 1 (Fixed point theorem).** Let $d$ be a natural number with $d \geq 2$ and suppose that a polynomial map $f \in P_{d}$ has no multiple fixed point. Then we have the equality

$$\sum_{\zeta \in \text{Fix}(f)} \frac{1}{1-f'(\zeta)} = 0.$$

We define the parameter spaces

$$\Lambda_d := \left\{ (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \mid \sum_{i=1}^{d} \prod_{j \neq i} (1 - \lambda_j) = 0 \right\}$$

and $\widetilde{\Lambda}_d := \Lambda_d / S_d$, and denote by $\text{pr}$ the projection map $\text{pr} : \Lambda_d \to \widetilde{\Lambda}_d$. We can define the map $\Phi_d : \tilde{P}_d \to \widetilde{\Lambda}_d$ by

$$f \mapsto (f'((\zeta)))_{\zeta \in \text{Fix}(f)}.$$

The aim of this paper is to analyze the structure of the map $\Phi_d$.

**Theorem 2.** In the case $d = 2$ or $3$, the map $\Phi_d$ is bijective.
This theorem is well-known and easy to prove. By this theorem, polynomial maps \( f \in \tilde{P}_d \) are completely parameterized by their fixed-point eigenvalues in the case \( d = 2 \) or \( 3 \). Historically, making use of this parameterization, John Milnor [2] started to study complex dynamics in the case of cubic polynomials.

In the main theorems of this paper, we investigate the map \( \Phi_d \) for \( d \geq 4 \) in detail on the domain where polynomial maps have no multiple fixed points. We prepare two more symbols:

\[
V_d := \{(\lambda_1, \ldots, \lambda_d) \in \Lambda_d \mid \lambda_i \neq 1 \text{ for any } 1 \leq i \leq d\},
\]
\[
\tilde{V}_d := V_d/\mathcal{G}_d.
\]

We denote by \( \bar{\lambda} \) the equivalent class of \( \lambda \in \Lambda_d \) in \( \tilde{\Lambda}_d \).

**Main Theorem 1.** Let \( d \) be a natural number with \( d \geq 4 \), and suppose that \( \lambda = (\lambda_1, \ldots, \lambda_d) \) is an element of \( V_d \). Then

1. we always have the inequalities \( 0 \leq \#(\Phi_d^{-1}(\overline{\lambda})) \leq (d - 2)! \).

2. The cardinality \( \#(\Phi_d^{-1}(\overline{\lambda})) \) is computed in finite steps from the two combinatorial data

\[
\mathcal{I}(\lambda) := \left\{ I \subseteq \{1, 2, \ldots, d\} \mid \sum_{i \in I} \frac{1}{1 - \lambda_i} = 0 \right\},
\]
\[
\mathcal{K}(\lambda) := \{ K \subseteq \{1, 2, \ldots, d\} \mid i, j \in K \Rightarrow \lambda_i = \lambda_j \}.
\]

3. If \( \mathcal{I}(\lambda) \subseteq \mathcal{I}(\lambda') \) and \( \mathcal{K}(\lambda) \subseteq \mathcal{K}(\lambda') \) for \( \lambda, \lambda' \in V_d \), then we have \( \#(\Phi_d^{-1}(\overline{\lambda})) \geq \#(\Phi_d^{-1}(\overline{\lambda}')) \).

4. The equality \( \#(\Phi_d^{-1}(\overline{\lambda})) = (d - 2)! \) holds if and only if \( \mathcal{I}(\lambda) = \emptyset \) and \( \lambda_1, \ldots, \lambda_d \) are mutually distinct.

5. If there exist \( c_1, \ldots, c_d \in \mathbb{Z} \setminus \{0\} \) such that \( \sum_{i=1}^d |c_i| \leq 2(d - 2) \) and \( \frac{1}{1 - \lambda_1} : \cdots : \frac{1}{1 - \lambda_d} = c_1 : \cdots : c_d \), then we have \( \Phi_d^{-1}(\overline{\lambda}) = \emptyset \).

6. In the case \( d \leq 7 \), the converse of the assertion 5 holds.

We are recently informed that Masayo Fujimura [1] also has studied the similar theme as Main Theorem 1 independently. She completely studied the map \( \Phi_d \) for \( d = 4 \), and showed that \( \Phi_d \) is not surjective for \( d \geq 4 \).

The local fiber structure of the map \( \Phi_d \) is also determined by the combinatorial data \( \mathcal{I}(\lambda) \) and \( \mathcal{K}(\lambda) \).
Main Theorem 2.

1. For any $\lambda, \lambda' \in V_d$ with $I(\lambda) = I(\lambda')$ and $K(\lambda) = K(\lambda')$, there exist open neighborhoods $\tilde{U} \ni \lambda, \tilde{U}' \ni \lambda'$ in $\tilde{V}_d$ and biholomorphic maps $\Sigma : \Phi_d^{-1}(\tilde{U}) \to \Phi_d^{-1}(\tilde{U}')$, $L : \tilde{U} \to \tilde{U}'$ and $L : U \to U'$ such that the following conditions (1a) and (1b) are satisfied, where $U, U'$ are the connected components of $pr^{-1}(U)$, $pr^{-1}(U')$ containing $\lambda, \lambda'$ respectively.

(a) The equalities $\Phi_d \circ \Sigma = \tilde{L} \circ \Phi_d$ and $pr \circ L = \tilde{L} \circ pr$ hold.
(b) For any $\lambda'' \in U$, the equalities $I(L(\lambda'')) = I(\lambda'')$ and $K(L(\lambda'')) = K(\lambda'')$ hold.

2. For each combinatorial data $I, K \subseteq \{I \mid I \subseteq \{1, \ldots, d\}\}$, we define the parameter subspaces

$$V(I, K) := \left\{ \overline{\lambda} \in \tilde{V}_d \mid \lambda \in V_d, I(\lambda) = I \text{ and } K(\lambda) = K \right\},$$

$$V(I, *) := \left\{ \overline{\lambda} \in \tilde{V}_d \mid \lambda \in V_d, I(\lambda) = I \right\}$$

and

$$V(*, K) := \left\{ \overline{\lambda} \in \tilde{V}_d \mid \lambda \in V_d, K(\lambda) = K \right\}.$$

Then for any $I, K \subseteq \{I \mid I \subseteq \{1, \ldots, d\}\}$ we have the following:

(a) the map $\Phi_d|_{\Phi_d^{-1}(V(I, *))} : \Phi_d^{-1}(V(I, *)) \to V(I, *)$ is proper.
(b) The map $\Phi_d|_{\Phi_d^{-1}(V(*, K))} : \Phi_d^{-1}(V(*, K)) \to V(*, K)$ is locally homeomorphic.
(c) For each connected component $X$ of $\Phi_d^{-1}(V(I, K))$, the map $\Phi_d|_X : X \to V(I, K)$ is an unbranched covering.

To state the computation of $\#(\Phi_d^{-1}(\overline{\lambda}))$ explicitly, we prepare the definition.

**Definition 3.** Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be an element of $V_d$. Then

- we define the set

$$I(\lambda) := \left\{ \{I_1, \ldots, I_l\} \mid \begin{array}{c} I_1 \amalg \cdots \amalg I_l = \{1, \ldots, d\} \\ \sum_{i \in I_u} \frac{1}{1-\lambda_i} = 0 \text{ for any } 1 \leq u \leq l \\ I_u \neq \emptyset \text{ for any } 1 \leq u \leq l \\ l \geq 2 \end{array} \right\},$$

where $I_1 \amalg \cdots \amalg I_l$ denotes the disjoint union of $I_1, \ldots, I_l$. Note that $I(\lambda)$ is completely determined by $I(\lambda)$. The partial order $<$ in $I(\lambda)$ is defined by the refinement of sets.
We denote by $K_1, \ldots, K_q$ the collection of maximal elements of $\mathcal{K}(\lambda)$. Note that the equality $K_1 \cdots \cup K_q = \{1, \ldots, d\}$ holds. We put $\kappa_w := \#(K_w)$ for $1 \leq w \leq q$ and denote by $g_w$ the greatest common divisor of $\kappa_1, \ldots, \kappa_{w-1}, \kappa_{w} - 1, \kappa_{w+1}, \ldots, \kappa_q$ for $1 \leq w \leq q$.

We put $\beta(\lambda_i) := \frac{1}{1-\lambda_i}$ for $\lambda_i \in \mathbb{C} \setminus \{1\}$.

We may assume $\lambda$ to be in the form

$$\lambda = (\underbrace{\lambda_1, \ldots, \lambda_1}_{\kappa_1}, \ldots, \underbrace{\lambda_q, \ldots, \lambda_q}_{\kappa_q}),$$

where $\lambda_1, \ldots, \lambda_q$ are mutually distinct. For each $1 \leq w \leq q$ and for each divisor $t$ of $g_w$ with $t \geq 2$, we put $d(t) := \frac{d-1}{t} + 1$ and denote by $\lambda(t)$ the element of $V_{d(t)}$ such that

$$\lambda(t) := \underbrace{\beta^{-1}(t\beta(\lambda_1))}, \ldots, \underbrace{\beta^{-1}(t\beta(\lambda_1))}, \ldots,$$

$$\underbrace{\beta^{-1}(t\beta(\lambda_w))}, \ldots, \underbrace{\beta^{-1}(t\beta(\lambda_w))}, \ldots, \underbrace{\beta^{-1}(t\beta(\lambda_q))}, \ldots, \underbrace{\beta^{-1}(t\beta(\lambda_q))}, \lambda_w).$$

Note that $\mathcal{I}(\lambda(t))$ is completely determined by $\mathcal{I}(\lambda), \mathcal{K}(\lambda)$ and $t$.

**Main Theorem 3.** Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be an element of $V_d$. Then the cardinality $\#(\Phi_d^{-1}(\overline{\lambda}))$ is computed in the following steps.

- For each $I = \{I_1, \ldots, I_l\} \in \mathcal{I}(\lambda)$, we define the number $e_I(\lambda)$ satisfying the equality

$$e_I(\lambda) := \left( \prod_{u=1}^{l} (\#(I_u) - 1)! \right) - \sum_{I' \in \mathcal{I}(\lambda)} \left( e_{I'}(\lambda) \cdot \prod_{u=1}^{l} \left( \prod_{k = \#(I_u) - \chi_u(I') + 1}^{\#(I_u) - 1} k \right) \right),$$

where we put $\chi_u(I') := \#(\{I' \in I' | I' \subseteq I_u\})$ for $I' \succ I$.

- We define the number $s_d(\lambda)$ to be

$$s_d(\lambda) := (d - 2)! - \sum_{I \in \mathcal{I}(\lambda)} \left( e_I(\lambda) \cdot \prod_{k = d - \#(I) + 1}^{d-2} k \right).$$
For each $1 \leq w \leq q$ and for each divisor $t$ of $g_w$ with $t \geq 2$, we define the number $c_t(\lambda)$ satisfying the equality

$$\sum_{t|b, b|g_w} \frac{t}{b} c_b(\lambda) = \frac{s_{d(t)}(\lambda(t))}{\binom{\kappa_1}{t}! \cdots \binom{\kappa(w-1)}{t}! \binom{\kappa(w+1)}{t}! \cdots \binom{\kappa_q}{t}!},$$

where $t|b$ denotes that $t$ divides $b$. Moreover we define the number $c_1(\lambda)$ satisfying the equality

$$c_1(\lambda) + \sum_{w=1}^{q} \left( \sum_{t|g_w, t \geq 2} \frac{1}{t} c_t(\lambda) \right) = \frac{s_d(\lambda)}{\kappa_1! \cdots \kappa_q!}.$$

Then the numbers $e_1(\lambda), s_d(\lambda)$ and $c_t(\lambda)$ are non-negative integers. Moreover we have

$$\# \left( \Phi_d^{-1}(\bar{\lambda}) \right) = \sum_{t} c_t(\lambda) = c_1(\lambda) + \sum_{w=1}^{q} \left( \sum_{t|g_w, t \geq 2} c_t(\lambda) \right).$$

References
