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Moduli space of polynomial maps

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In the study of the dynamics of a polynomial map f, the eigenvalues of the fixed points of f play a very important role to characterize the original map f. In this paper, we shall study how many affine conjugacy classes of polynomial maps are there when the eigenvalues of their fixed points are specified.

For a natural number d with $d \geq 2$, we denote the moduli space of polynomial maps of degree d by

$$\widetilde{P}_d := \left\{ f \in \mathbb{C}[z] \mid \deg f = d \right\} /\sim,$$

where $\sim$ denotes the affine conjugacy of polynomial maps, i.e., for $f, g \in \mathbb{C}[z]$, $f \sim g$ holds if and only if there exists an affine transformation $\gamma(z) = az + b$ $(a, b \in \mathbb{C},\ a \neq 0)$ such that $f = \gamma \circ g \circ \gamma^{-1}$. We put

$$\mathrm{Fix}(f) := \left\{ \zeta \in \mathbb{C} \mid f(\zeta) = \zeta \right\}$$

for $f \in \mathbb{C}[z]$, where $\mathrm{Fix}(f)$ is considered counted with multiplicity. Hence we always have $\#\left(\mathrm{Fix}(f)\right) = \deg f$.

Proposition 1 (Fixed point theorem). *Let d be a natural number with $d \geq 2$ and suppose that a polynomial map $f \in P_d$ has no multiple fixed point. Then we have the equality*

$$\sum_{\zeta \in \mathrm{Fix}(f)} \frac{1}{1 - f'(\zeta)} = 0.$$

We define the parameter spaces

$$\Lambda_d := \left\{ (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \;\middle|\; \textstyle\sum_{i=1}^{d} \prod_{j \neq i} (1 - \lambda_j) = 0 \right\}$$

and $\widetilde{\Lambda}_d := \Lambda_d / \mathfrak{S}_d$, and denote by pr the projection map $pr : \Lambda_d \to \widetilde{\Lambda}_d$. We can define the map $\Phi_d : \widetilde{P}_d \to \widetilde{\Lambda}_d$ by

$$f \mapsto \left(f'(\zeta)\right)_{\zeta \in \mathrm{Fix}(f)}.$$

The aim of this paper is to analyze the structure of the map Φ_d.

Theorem 2. *In the case $d = 2$ or 3, the map Φ_d is bijective.*

This theorem is well-known and easy to prove. By this theorem, polynomial maps $f \in \widetilde{P}_d$ are completely parameterized by their fixed-point eigenvalues in the case $d = 2$ or 3. Historically, making use of this parameterization, John Milnor [2] started to study complex dynamics in the case of cubic polynomials.

In the main theorems of this paper, we investigate the map Φ_d for $d \geq 4$ in detail on the domain where polynomial maps have no multiple fixed points. We prepare two more symbols:

$$V_d := \left\{(\lambda_1, \ldots, \lambda_d) \in \Lambda_d \mid \lambda_i \neq 1 \text{ for any } 1 \leq i \leq d\right\},$$
$$\widetilde{V}_d := V_d/\mathfrak{S}_d.$$

We denote by $\bar{\lambda}$ the equivalent class of $\lambda \in \Lambda_d$ in $\widetilde{\Lambda}_d$.

Main Theorem 1. *Let d be a natural number with $d \geq 4$, and suppose that $\lambda = (\lambda_1, \ldots, \lambda_d)$ is an element of V_d. Then*

1. *we always have the inequalities* $0 \leq \#\left(\Phi_d^{-1}(\bar{\lambda})\right) \leq (d-2)!$.
2. *The cardinality* $\#\left(\Phi_d^{-1}(\bar{\lambda})\right)$ *is computed in finite steps from the two combinatorial data*

$$\mathcal{I}(\lambda) := \left\{ I \subsetneq \{1, 2, \ldots, d\} \;\middle|\; \sum_{i \in I} \frac{1}{1-\lambda_i} = 0 \right\},$$
$$\mathcal{K}(\lambda) := \left\{K \subseteq \{1, 2, \ldots, d\} \mid i, j \in K \Rightarrow \lambda_i = \lambda_j\right\}.$$

3. *If* $\mathcal{I}(\lambda) \subseteq \mathcal{I}(\lambda')$ *and* $\mathcal{K}(\lambda) \subseteq \mathcal{K}(\lambda')$ *for* $\lambda, \lambda' \in V_d$, *then we have* $\#\left(\Phi_d^{-1}(\bar{\lambda})\right) \geq \#\left(\Phi_d^{-1}(\bar{\lambda}')\right)$.
4. *The equality* $\#\left(\Phi_d^{-1}(\bar{\lambda})\right) = (d-2)!$ *holds if and only if* $\mathcal{I}(\lambda) = \emptyset$ *and* $\lambda_1, \ldots, \lambda_d$ *are mutually distinct.*
5. *If there exist* $c_1, \ldots, c_d \in \mathbb{Z} \setminus \{0\}$ *such that* $\sum_{i=1}^d |c_i| \leq 2(d-2)$ *and* $\frac{1}{1-\lambda_1} : \cdots : \frac{1}{1-\lambda_d} = c_1 : \cdots : c_d$, *then we have* $\Phi_d^{-1}(\bar{\lambda}) = \emptyset$.
6. *In the case* $d \leq 7$, *the converse of the assertion 5 holds.*

We are recently informed that Masayo Fujimura [1] also has studied the similar theme as Main Theorem 1 independently. She completely studied the map Φ_d for $d = 4$, and showed that Φ_d is not surjective for $d \geq 4$.

The local fiber structure of the map Φ_d is also determined by the combinatorial data $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$.

Main Theorem 2.

1. *For any* $\lambda, \lambda' \in V_d$ *with* $\mathcal{I}(\lambda) = \mathcal{I}(\lambda')$ *and* $\mathcal{K}(\lambda) = \mathcal{K}(\lambda')$, *there exist open neighborhoods* $\widetilde{U} \ni \bar{\lambda}$, $\widetilde{U}' \ni \bar{\lambda}'$ *in* $\widetilde{V}_d$ *and biholomorphic maps* $\mathfrak{L} : \Phi_d^{-1}(\widetilde{U}) \to \Phi_d^{-1}(\widetilde{U}')$, $\widetilde{L} : \widetilde{U} \to \widetilde{U}'$ *and* $L : U \to U'$ *such that the following conditions (1a) and (1b) are satisfied, where* U, U' *are the connected components of* $pr^{-1}(U)$, $pr^{-1}(U')$ *containing* λ, λ' *respectively.*
   (a) *The equalities* $\Phi_d \circ \mathfrak{L} = \widetilde{L} \circ \Phi_d$ *and* $pr \circ L = \widetilde{L} \circ pr$ *hold.*
   (b) *For any* $\lambda'' \in U$, *the equalities* $\mathcal{I}(\lambda'') = \mathcal{I}(L(\lambda''))$ *and* $\mathcal{K}(\lambda'') = \mathcal{K}(L(\lambda''))$ *hold.*

2. *For each combinatorial data* $\mathcal{I}, \mathcal{K} \subseteq \{I \mid I \subseteq \{1, \ldots, d\}\}$, *we define the parameter subspaces*

$$V(\mathcal{I}, \mathcal{K}) := \left\{\bar{\lambda} \in \widetilde{V}_d \mid \lambda \in V_d,\ \mathcal{I}(\lambda) = \mathcal{I} \text{ and } \mathcal{K}(\lambda) = \mathcal{K}\right\},$$
$$V(\mathcal{I}, *) := \left\{\bar{\lambda} \in \widetilde{V}_d \mid \lambda \in V_d,\ \mathcal{I}(\lambda) = \mathcal{I}\right\}$$

   and

$$V(*, \mathcal{K}) := \left\{\bar{\lambda} \in \widetilde{V}_d \mid \lambda \in V_d,\ \mathcal{K}(\lambda) = \mathcal{K}\right\}.$$

   Then for any $\mathcal{I}, \mathcal{K} \subseteq \{I \mid I \subseteq \{1, \ldots, d\}\}$ *we have the following:*
   (a) *the map* $\Phi_d|_{\Phi_d^{-1}(V(\mathcal{I},*))} : \Phi_d^{-1}(V(\mathcal{I}, *)) \to V(\mathcal{I}, *)$ *is proper.*
   (b) *The map* $\Phi_d|_{\Phi_d^{-1}(V(*,\mathcal{K}))} : \Phi_d^{-1}(V(*, \mathcal{K})) \to V(*, \mathcal{K})$ *is locally homeomorphic.*
   (c) *For each connected component* X *of* $\Phi_d^{-1}(V(\mathcal{I}, \mathcal{K}))$, *the map* $\Phi_d|_X : X \to V(\mathcal{I}, \mathcal{K})$ *is an unbranched covering.*

To state the computation of $\#\left(\Phi_d^{-1}(\bar{\lambda})\right)$ explicitly, we prepare the definition.

Definition 3. Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be an element of V_d. Then

- we define the set

$$\mathfrak{I}(\lambda) := \left\{ \{I_1, \ldots, I_l\} \;\middle|\; \begin{array}{c} I_1 \amalg \cdots \amalg I_l = \{1, \ldots, d\} \\ \sum_{i \in I_u} \frac{1}{1-\lambda_i} = 0 \text{ for any } 1 \le u \le l \\ I_u \neq \emptyset \text{ for any } 1 \le u \le l \\ l \ge 2 \end{array} \right\},$$

  where $I_1 \amalg \cdots \amalg I_l$ denotes the disjoint union of $I_1, \ldots, I_l$. Note that $\mathfrak{I}(\lambda)$ is completely determined by $\mathcal{I}(\lambda)$. The partial order $\prec$ in $\mathfrak{I}(\lambda)$ is defined by the refinement of sets.

- We denote by $K_1, \dots, K_q$ the collection of maximal elements of $\mathcal{K}(\lambda)$. Note that the equality $K_1 \amalg \cdots \amalg K_q = \{1, \dots, d\}$ holds. We put $\kappa_w := \#(K_w)$ for $1 \le w \le q$ and denote by g_w the greatest common divisor of $\kappa_1, \dots, \kappa_{w-1}, \kappa_w - 1, \kappa_{w+1}, \dots, \kappa_q$ for $1 \le w \le q$.

- We put $\beta(\lambda_i) := \frac{1}{1-\lambda_i}$ for $\lambda_i \in \mathbb{C} \setminus \{1\}$.

- We may assume λ to be in the form

$$\lambda = (\underbrace{\lambda_1, \dots, \lambda_1}_{\kappa_1}, \dots, \underbrace{\lambda_q, \dots, \lambda_q}_{\kappa_q}),$$

where $\lambda_1, \dots, \lambda_q$ are mutually distinct. For each $1 \le w \le q$ and for each divisor t of g_w with $t \ge 2$, we put $d(t) := \frac{d-1}{t} + 1$ and denote by $\lambda(t)$ the element of $V_{d(t)}$ such that

$$\lambda(t) := (\underbrace{\beta^{-1}(t\beta(\lambda_1)), \dots, \beta^{-1}(t\beta(\lambda_1))}_{\frac{\kappa_1}{t}}, \dots,$$
$$\underbrace{\beta^{-1}(t\beta(\lambda_w)), \dots, \beta^{-1}(t\beta(\lambda_w))}_{\frac{(\kappa_w)-1}{t}}, \dots, \underbrace{\beta^{-1}(t\beta(\lambda_q)), \dots, \beta^{-1}(t\beta(\lambda_q))}_{\frac{\kappa_q}{t}}, \lambda_w).$$

Note that $\mathcal{I}(\lambda(t))$ is completely determined by $\mathcal{I}(\lambda), \mathcal{K}(\lambda)$ and t.

Main Theorem 3. *Let $\lambda = (\lambda_1, \dots, \lambda_d)$ be an element of V_d. Then the cardinality $\#\left(\Phi_d^{-1}(\bar{\lambda})\right)$ is computed in the following steps.*

- *For each $\mathbb{I} = \{I_1, \dots, I_l\} \in \mathfrak{I}(\lambda)$, we define the number $e_{\mathbb{I}}(\lambda)$ satisfying the equality*

$$e_{\mathbb{I}}(\lambda) := \left(\prod_{u=1}^{l} (\#(I_u) - 1)!\right) - \sum_{\substack{\mathbb{I}' \in \mathfrak{I}(\lambda) \\ \mathbb{I}' \succ \mathbb{I},\, \mathbb{I}' \neq \mathbb{I}}} \left(e_{\mathbb{I}'}(\lambda) \cdot \prod_{u=1}^{l} \left(\prod_{k=\#(I_u)-\chi_u(\mathbb{I}')+1}^{\#(I_u)-1} k \right)\right),$$

where we put $\chi_u(\mathbb{I}') := \#(\{I' \in \mathbb{I}' \,|\, I' \subseteq I_u\})$ for $\mathbb{I}' \succ \mathbb{I}$.

- *We define the number $s_d(\lambda)$ to be*

$$s_d(\lambda) := (d-2)! - \sum_{\mathbb{I} \in \mathfrak{I}(\lambda)} \left(e_{\mathbb{I}}(\lambda) \cdot \prod_{k=d-\#(\mathbb{I})+1}^{d-2} k \right).$$

- *For each* $1 \leq w \leq q$ *and for each divisor* t *of* g_w *with* $t \geq 2$, *we define the number* $c_t(\lambda)$ *satisfying the equality*

$$\sum_{t|b,\ b|g_w} \frac{t}{b} c_b(\lambda) = \frac{s_{d(t)}(\lambda(t))}{\left(\frac{\kappa_1}{t}\right)! \cdots \left(\frac{\kappa_{(w-1)}}{t}\right)! \left(\frac{(\kappa_w)-1}{t}\right)! \left(\frac{\kappa_{(w+1)}}{t}\right)! \cdots \left(\frac{\kappa_q}{t}\right)!},$$

where $t|b$ *denotes that* t *divides* b. *Moreover we define the number* $c_1(\lambda)$ *satisfying the equality*

$$c_1(\lambda) + \sum_{w=1}^{q} \left(\sum_{t|g_w,\ t \geq 2} \frac{1}{t} c_t(\lambda) \right) = \frac{s_d(\lambda)}{\kappa_1! \cdots \kappa_q!}.$$

- *Then the numbers* $e_{\mathrm{I}}(\lambda)$, $s_d(\lambda)$ *and* $c_t(\lambda)$ *are non-negative integers. Moreover we have*

$$\#\left(\Phi_d^{-1}(\bar{\lambda})\right) = \sum_t c_t(\lambda) = c_1(\lambda) + \sum_{w=1}^{q} \left(\sum_{t|g_w,\ t \geq 2} c_t(\lambda) \right).$$

References

[1] Fujimura, Masayo. Projective moduli space for the polynomials. To appear in *Dynamics of Continuous , Discrete and Impulsive Systems*

[2] Milnor, John. Remarks on iterated cubic maps. *Experiment. Math.* 1 (1992), no. 1, 5–24.