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# Moduli space of polynomial maps

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In the study of the dynamics of a polynomial map $f$, the eigenvalues of the fixed points of $f$ play a very important role to characterize the original map $f$. In this paper, we shall study how many affine conjugacy classes of polynomial maps are there when the eigenvalues of their fixed points are specified.

For a natural number $d$ with $d \geq 2$, we denote the moduli space of polynomial maps of degree $d$ by

$$\widetilde{P}_d := \{f \in \mathbb{C}[z] \mid \deg f = d\} /\sim,$$

where $\sim$ denotes the affine conjugacy of polynomial maps, i.e., for $f, g \in \mathbb{C}[z]$, $f \sim g$ holds if and only if there exists an affine transformation $\gamma(z) = az + b$ ($a, b \in \mathbb{C}$, $a \neq 0$) such that $f = \gamma \circ g \circ \gamma^{-1}$. We put

$$\mathrm{Fix}(f) := \{\zeta \in \mathbb{C} \mid f(\zeta) = \zeta\}$$

for $f \in \mathbb{C}[z]$, where $\mathrm{Fix}(f)$ is considered counted with multiplicity. Hence we always have $\#(\mathrm{Fix}(f)) = \deg f$.

**Proposition 1** (Fixed point theorem). *Let $d$ be a natural number with $d \geq 2$ and suppose that a polynomial map $f \in P_d$ has no multiple fixed point. Then we have the equality*

$$\sum_{\zeta \in \mathrm{Fix}(f)} \frac{1}{1 - f'(\zeta)} = 0.$$

We define the parameter spaces

$$\Lambda_d := \left\{(\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \;\middle|\; \textstyle\sum_{i=1}^{d} \prod_{j \neq i} (1 - \lambda_j) = 0\right\}$$

and $\widetilde{\Lambda}_d := \Lambda_d / \mathfrak{S}_d$, and denote by $pr$ the projection map $pr : \Lambda_d \to \widetilde{\Lambda}_d$. We can define the map $\Phi_d : \widetilde{P}_d \to \widetilde{\Lambda}_d$ by

$$f \mapsto (f'(\zeta))_{\zeta \in \mathrm{Fix}(f)}.$$

The aim of this paper is to analyze the structure of the map $\Phi_d$.

**Theorem 2.** *In the case $d = 2$ or $3$, the map $\Phi_d$ is bijective.*

This theorem is well-known and easy to prove. By this theorem, polynomial maps $f \in \widetilde{P}_d$ are completely parameterized by their fixed-point eigenvalues in the case $d = 2$ or 3. Historically, making use of this parameterization, John Milnor [2] started to study complex dynamics in the case of cubic polynomials.

In the main theorems of this paper, we investigate the map $\Phi_d$ for $d \geq 4$ in detail on the domain where polynomial maps have no multiple fixed points. We prepare two more symbols:

$$\begin{aligned} V_d &:= \left\{(\lambda_1, \ldots, \lambda_d) \in \Lambda_d \;\middle|\; \lambda_i \neq 1 \text{ for any } 1 \leq i \leq d\right\}, \\ \widetilde{V}_d &:= V_d / \mathfrak{S}_d. \end{aligned}$$

We denote by $\bar{\lambda}$ the equivalent class of $\lambda \in \Lambda_d$ in $\widetilde{\Lambda}_d$.

**Main Theorem 1.** *Let d be a natural number with $d \geq 4$, and suppose that $\lambda = (\lambda_1, \ldots, \lambda_d)$ is an element of $V_d$. Then*

1. *we always have the inequalities* $0 \leq \#\left(\Phi_d^{-1}(\bar{\lambda})\right) \leq (d-2)!$.
2. *The cardinality $\#\left(\Phi_d^{-1}(\bar{\lambda})\right)$ is computed in finite steps from the two combinatorial data*

$$\begin{aligned} \mathcal{I}(\lambda) &:= \left\{ I \subsetneq \{1, 2, \ldots, d\} \;\middle|\; \sum_{i \in I} \frac{1}{1-\lambda_i} = 0 \right\}, \\ \mathcal{K}(\lambda) &:= \left\{ K \subseteq \{1, 2, \ldots, d\} \;\middle|\; i, j \in K \Rightarrow \lambda_i = \lambda_j \right\}. \end{aligned}$$

3. *If $\mathcal{I}(\lambda) \subseteq \mathcal{I}(\lambda')$ and $\mathcal{K}(\lambda) \subseteq \mathcal{K}(\lambda')$ for $\lambda, \lambda' \in V_d$, then we have $\#\left(\Phi_d^{-1}(\bar{\lambda})\right) \geq \#\left(\Phi_d^{-1}(\bar{\lambda}')\right)$.*
4. *The equality $\#\left(\Phi_d^{-1}(\bar{\lambda})\right) = (d-2)!$ holds if and only if $\mathcal{I}(\lambda) = \emptyset$ and $\lambda_1, \ldots, \lambda_d$ are mutually distinct.*
5. *If there exist $c_1, \ldots, c_d \in \mathbb{Z} \setminus \{0\}$ such that $\sum_{i=1}^{d} |c_i| \leq 2(d-2)$ and $\frac{1}{1-\lambda_1} : \cdots : \frac{1}{1-\lambda_d} = c_1 : \cdots : c_d$, then we have $\Phi_d^{-1}(\bar{\lambda}) = \emptyset$.*
6. *In the case $d \leq 7$, the converse of the assertion 5 holds.*

We are recently informed that Masayo Fujimura [1] also has studied the similar theme as Main Theorem 1 independently. She completely studied the map $\Phi_d$ for $d = 4$, and showed that $\Phi_d$ is not surjective for $d \geq 4$.

The local fiber structure of the map $\Phi_d$ is also determined by the combinatorial data $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$.

**Main Theorem 2.**

1. *For any $\lambda, \lambda' \in V_d$ with $\mathcal{I}(\lambda) = \mathcal{I}(\lambda')$ and $\mathcal{K}(\lambda) = \mathcal{K}(\lambda')$, there exist open neighborhoods $\widetilde{U} \ni \bar{\lambda}$, $\widetilde{U}' \ni \bar{\lambda}'$ in $\widetilde{V}_d$ and biholomorphic maps $\mathfrak{L} : \Phi_d^{-1}(\widetilde{U}) \to \Phi_d^{-1}(\widetilde{U}')$, $\widetilde{L} : \widetilde{U} \to \widetilde{U}'$ and $L : U \to U'$ such that the following conditions (1a) and (1b) are satisfied, where $U, U'$ are the connected components of $pr^{-1}(U)$, $pr^{-1}(U')$ containing $\lambda, \lambda'$ respectively.*

   (a) *The equalities $\Phi_d \circ \mathfrak{L} = \widetilde{L} \circ \Phi_d$ and $pr \circ L = \widetilde{L} \circ pr$ hold.*

   (b) *For any $\lambda'' \in U$, the equalities $\mathcal{I}(\lambda'') = \mathcal{I}(L(\lambda''))$ and $\mathcal{K}(\lambda'') = \mathcal{K}(L(\lambda''))$ hold.*

2. *For each combinatorial data $\mathcal{I}, \mathcal{K} \subseteq \{I \mid I \subseteq \{1, \dots, d\}\}$, we define the parameter subspaces*

$$V(\mathcal{I}, \mathcal{K}) := \left\{\bar{\lambda} \in \widetilde{V}_d \mid \lambda \in V_d,\ \mathcal{I}(\lambda) = \mathcal{I} \text{ and } \mathcal{K}(\lambda) = \mathcal{K}\right\},$$
$$V(\mathcal{I}, *) := \left\{\bar{\lambda} \in \widetilde{V}_d \mid \lambda \in V_d,\ \mathcal{I}(\lambda) = \mathcal{I}\right\}$$

   *and*

$$V(*, \mathcal{K}) := \left\{\bar{\lambda} \in \widetilde{V}_d \mid \lambda \in V_d,\ \mathcal{K}(\lambda) = \mathcal{K}\right\}.$$

   *Then for any $\mathcal{I}, \mathcal{K} \subseteq \{I \mid I \subseteq \{1, \dots, d\}\}$ we have the following:*

   (a) *the map $\Phi_d|_{\Phi_d^{-1}(V(\mathcal{I},*))} : \Phi_d^{-1}(V(\mathcal{I}, *)) \to V(\mathcal{I}, *)$ is proper.*

   (b) *The map $\Phi_d|_{\Phi_d^{-1}(V(*,\mathcal{K}))} : \Phi_d^{-1}(V(*, \mathcal{K})) \to V(*, \mathcal{K})$ is locally homeomorphic.*

   (c) *For each connected component $X$ of $\Phi_d^{-1}(V(\mathcal{I}, \mathcal{K}))$, the map $\Phi_d|_X : X \to V(\mathcal{I}, \mathcal{K})$ is an unbranched covering.*

To state the computation of $\#\left(\Phi_d^{-1}(\bar{\lambda})\right)$ explicitly, we prepare the definition.

**Definition 3.** Let $\lambda = (\lambda_1, \dots, \lambda_d)$ be an element of $V_d$. Then

- we define the set

$$\mathfrak{I}(\lambda) := \left\{ \{I_1, \dots, I_l\} \;\middle|\; \begin{array}{c} I_1 \amalg \cdots \amalg I_l = \{1, \dots, d\} \\ \sum_{i \in I_u} \frac{1}{1-\lambda_i} = 0 \text{ for any } 1 \le u \le l \\ I_u \neq \emptyset \text{ for any } 1 \le u \le l \\ l \ge 2 \end{array} \right\},$$

  where $I_1 \amalg \cdots \amalg I_l$ denotes the disjoint union of $I_1, \dots, I_l$. Note that $\mathfrak{I}(\lambda)$ is completely determined by $\mathcal{I}(\lambda)$. The partial order $\prec$ in $\mathfrak{I}(\lambda)$ is defined by the refinement of sets.

- We denote by $K_1, \ldots, K_q$ the collection of maximal elements of $\mathcal{K}(\lambda)$. Note that the equality $K_1 \amalg \cdots \amalg K_q = \{1, \ldots, d\}$ holds. We put $\kappa_w := \#(K_w)$ for $1 \leq w \leq q$ and denote by $g_w$ the greatest common divisor of $\kappa_1, \ldots, \kappa_{w-1}, \kappa_w - 1, \kappa_{w+1}, \ldots, \kappa_q$ for $1 \leq w \leq q$.
- We put $\beta(\lambda_i) := \frac{1}{1-\lambda_i}$ for $\lambda_i \in \mathbb{C} \setminus \{1\}$.
- We may assume $\lambda$ to be in the form

$$\lambda = (\underbrace{\lambda_1, \ldots, \lambda_1}_{\kappa_1}, \ldots, \underbrace{\lambda_q, \ldots, \lambda_q}_{\kappa_q}),$$

where $\lambda_1, \ldots, \lambda_q$ are mutually distinct. For each $1 \leq w \leq q$ and for each divisor $t$ of $g_w$ with $t \geq 2$, we put $d(t) := \frac{d-1}{t} + 1$ and denote by $\lambda(t)$ the element of $V_{d(t)}$ such that

$$\lambda(t) := (\underbrace{\beta^{-1}(t\beta(\lambda_1)), \ldots, \beta^{-1}(t\beta(\lambda_1))}_{\frac{\kappa_1}{t}}, \ldots,$$
$$\underbrace{\beta^{-1}(t\beta(\lambda_w)), \ldots, \beta^{-1}(t\beta(\lambda_w))}_{\frac{(\kappa_w)-1}{t}}, \ldots, \underbrace{\beta^{-1}(t\beta(\lambda_q)), \ldots, \beta^{-1}(t\beta(\lambda_q))}_{\frac{\kappa_q}{t}}, \lambda_w).$$

Note that $\mathcal{I}(\lambda(t))$ is completely determined by $\mathcal{I}(\lambda), \mathcal{K}(\lambda)$ and $t$.

**Main Theorem 3.** *Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be an element of $V_d$. Then the cardinality $\#\left(\Phi_d^{-1}(\bar{\lambda})\right)$ is computed in the following steps.*

- *For each $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$, we define the number $e_{\mathbb{I}}(\lambda)$ satisfying the equality*

$$e_{\mathbb{I}}(\lambda) := \left(\prod_{u=1}^{l} (\#(I_u) - 1)!\right) - \sum_{\substack{\mathbb{I}' \in \mathfrak{I}(\lambda) \\ \mathbb{I}' \succ \mathbb{I},\, \mathbb{I}' \neq \mathbb{I}}} \left( e_{\mathbb{I}'}(\lambda) \cdot \prod_{u=1}^{l} \left( \prod_{k=\#(I_u)-\chi_u(\mathbb{I}')+1}^{\#(I_u)-1} k \right)\right),$$

*where we put $\chi_u(\mathbb{I}') := \#(\{I' \in \mathbb{I}' \mid I' \subseteq I_u\})$ for $\mathbb{I}' \succ \mathbb{I}$.*

- *We define the number $s_d(\lambda)$ to be*

$$s_d(\lambda) := (d-2)! - \sum_{\mathbb{I} \in \mathfrak{I}(\lambda)} \left( e_{\mathbb{I}}(\lambda) \cdot \prod_{k=d-\#(\mathbb{I})+1}^{d-2} k \right).$$

- *For each* $1 \leq w \leq q$ *and for each divisor* $t$ *of* $g_w$ *with* $t \geq 2$, *we define the number* $c_t(\lambda)$ *satisfying the equality*

$$\sum_{t|b,\ b|g_w} \frac{t}{b} c_b(\lambda) = \frac{s_{d(t)}(\lambda(t))}{\left(\frac{\kappa_1}{t}\right)! \cdots \left(\frac{\kappa_{(w-1)}}{t}\right)! \left(\frac{(\kappa_w)-1}{t}\right)! \left(\frac{\kappa_{(w+1)}}{t}\right)! \cdots \left(\frac{\kappa_q}{t}\right)!},$$

*where* $t|b$ *denotes that* $t$ *divides* $b$. *Moreover we define the number* $c_1(\lambda)$ *satisfying the equality*

$$c_1(\lambda) + \sum_{w=1}^{q} \left( \sum_{t|g_w,\ t\geq 2} \frac{1}{t} c_t(\lambda) \right) = \frac{s_d(\lambda)}{\kappa_1! \cdots \kappa_q!}.$$

- *Then the numbers* $e_{\mathrm{I}}(\lambda)$, $s_d(\lambda)$ *and* $c_t(\lambda)$ *are non-negative integers. Moreover we have*

$$\#\left(\Phi_d^{-1}(\bar{\lambda})\right) = \sum_t c_t(\lambda) = c_1(\lambda) + \sum_{w=1}^{q} \left( \sum_{t|g_w,\ t\geq 2} c_t(\lambda) \right).$$

# References


[1] Fujimura, Masayo. Projective moduli space for the polynomials. To appear in *Dynamics of Continuous , Discrete and Impulsive Systems*

[2] Milnor, John. Remarks on iterated cubic maps. *Experiment. Math.* 1 (1992), no. 1, 5–24.