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Simultaneous linearization and its application

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Abstract

This note gives a proof of Ueda's simultaneous linearization theorem with real multipliers and its simple application to quadratic dynamics. This note is based on my talk at RIMS on 5 October 2006, titled "A proof of simultaneous linearization with a polylog estimate."

1 Simultaneous linearization

Here we give an alternative proof of Ueda's simultaneous linearization in a simplified setting. For $R \geq 0$, let $E_R$ denote the region $\{z \in \mathbb{C} : \text{Re } z \geq R\}$.

Theorem 1.1 (Simultaneous Linearization) For $\epsilon \in [0,1]$, let $\{f_{\epsilon}\}$ be a family of holomorphic maps on $\{|z| \geq R > 0\}$ such that

$$f_{\epsilon}(z) = \tau_{\epsilon} z + 1 + O(1/z)$$
$$\rightarrow f_0(z) = z + 1 + O(1/z)$$

uniformly as $\epsilon \to 0$ where $\tau_{\epsilon} = 1 + \epsilon$. If $R \gg 0$, then for any $\epsilon \in [0,1]$ there exists a holomorphic map $u_{\epsilon} : E_R \to \mathbb{C}$ such that

$$u_{\epsilon}(f_{\epsilon}(z)) = \tau_{\epsilon} u_{\epsilon}(z) + 1$$

and $u_{\epsilon} \to u_0$ uniformly on compact sets of $E_R$. 
Indeed, a similar theorem holds for any radial (= non-tangential) convergence \( \tau_\epsilon \to 1 \) outside the unit disk. See Ueda's original proof ([Ue1], [Ue2]). Moreover, the error term \( O(1/z) \) can be replaced by \( O(|z|^{-\sigma}) \) with \( 0 < \sigma \leq 1 \). (See [Ka2].) Here we present a simplified proof only for real \( \tau_\epsilon \to 1 \) based on the argument of [Mi, Lemma 10.10]. The idea can be traced back at least to Leau's work on the Abel equation [L]. We first check:

**Lemma 1.2** If \( R \gg 0 \), there exists \( M > 0 \) such that \( |f_\epsilon(z) - (\tau_\epsilon z + 1)| \leq M/|z| \) on \( \{|z| \geq R\} \) and \( \text{Re} f_\epsilon(z) \geq \text{Re} z + 1/2 \) on \( E_R \) for any \( \epsilon \in [0, 1] \).

**Proof.** The first inequality and the existence of \( M \) is obvious. By replacing \( R \) by larger one, we have \( |f_\epsilon(z) - (\tau_\epsilon z + 1)| \leq M/R < 1/2 \) on \( E_R \). Then

\[
\text{Re} f_\epsilon(z) \geq \text{Re} (\tau_\epsilon z + 1) - 1/2 \geq \text{Re} z + 1/2.
\]

Let us fix such an \( R \gg 0 \). Next we show:

**Lemma 1.3** For any \( \epsilon \in [0, 1] \) and \( z_1, z_2 \in E_{2S} \) with \( S > R \), we have:

\[
\left| \frac{f_\epsilon(z_1) - f_\epsilon(z_2)}{z_1 - z_2} - \tau_\epsilon \right| \leq \frac{M}{S^2}.
\]

**Proof.** Set \( g_\epsilon(z) := f_\epsilon(z) - (\tau_\epsilon z + 1) \). For any \( |z| \geq 2S \) and \( w \in B(z, S) \), we have \( |w| > S \). This implies \( |g_\epsilon(w)| \leq M/|w| < M/S \) and thus \( g_\epsilon \) maps \( B(z, S) \) into \( B(0, M/S) \). By the Cauchy integral formula or the Schwarz lemma, it follows that

\[
|g_\epsilon'(z)| \leq (M/S)/S = M/S^2 \text{ on } \{|z| \geq S\}.
\]

(Note that the segment \( [z_2, z_1] \) is contained in \( E_{2S} \subset \{|z| \geq 2S\} \).)

**Proof of Theorem 1.1.** Set \( z_n := f_\epsilon^n(z) \) for \( z \in E_{2R} \). Note that such \( z_n \) satisfies

\[
|z_n| \geq \text{Re} z_n \geq \text{Re} z + \frac{n}{2} \geq 2R + \frac{n}{2}
\]

by Lemma 1.2. Now we fix \( a \in E_{2R} \) and define \( u_{n,a} = u_n : E_{2R} \to \mathbb{C} \) (\( n \geq 0 \)) by

\[
u_n(z) := \frac{z_n - a}{\tau_\epsilon^n}.
\]
Then we have
\[
|\frac{u_{n+1}(z)}{u_{n}(z)} - 1| = \left| \frac{z_{n+1} - a_{n+1}}{\tau_{\epsilon}(z_{n} - a_{n})} - 1 \right| = \frac{1}{\tau_{\epsilon}} \left| \frac{f_{\epsilon}(z_{n}) - f_{\epsilon}(a_{n})}{z_{n} - a_{n}} - \tau_{\epsilon} \right|.
\]
We apply Lemma 1.3 with \(2S = 2R + n/2\). Then
\[
\left| \frac{u_{n+1}(z)}{u_{n}(z)} - 1 \right| \leq \frac{M}{\tau_{\epsilon}(R+n/4)^2} \leq \frac{C}{(n+1)^2},
\]
where \(C = 16M\) and we may assume \(R > 1/4\). Set \(P := \prod_{n \geq 1}(1 + C/n^2)\). Since \(|u_{n+1}(z)/u_{n}(z)| \leq 1 + C/(n+1)^2\), we have
\[
|u_{n}(z)| = \left| \frac{u_{n}(z)}{u_{n-1}(z)} \right| \cdots \left| \frac{u_{1}(z)}{u_{0}(z)} \right| \cdot |u_{0}(z)| \leq P|z-a|.
\]
Hence
\[
|u_{n+1}(z) - u_{n}(z)| = \left| \frac{u_{n+1}(z)}{u_{n}(z)} - 1 \right| \cdot |u_{n}(z)| \leq \frac{CP}{(n+1)^2} \cdot |z-a|.
\]
This implies that \(u_{\epsilon} = u_{0} + (u_{1} - u_{0}) + \cdots = \lim u_{n}\) converges uniformly on compact subsets of \(E_{2R}\) and for all \(\epsilon \in [0,1]\). The univalence of \(u_{\epsilon}\) is shown in the same way as [Mi, Lemma 10.10].

Next we check that \(u_{\epsilon}(f_{\epsilon}(z)) = \tau_{\epsilon}u_{\epsilon}(z) + C_{\epsilon}\) with \(C_{\epsilon} \to 1\) as \(\epsilon \to 0\). One can easily check that \(u_{n}(f_{\epsilon}(z)) = \tau_{\epsilon}u_{n+1}(z) + C_{n}\) where
\[
C_{n} = \frac{a_{n+1} - a_{n}}{\tau_{\epsilon}^{n}} = \frac{(\tau_{\epsilon} - 1)a_{n}}{\tau_{\epsilon}^{n}} + \frac{1 + g_{\epsilon}(a_{n})}{\tau_{\epsilon}^{n}}.
\]
When \(\tau_{\epsilon} = 1\), \(C_{n}\) tends to 1 since \(|g_{\epsilon}(a_{n})| \leq M/|a_{n}| \leq M/(2R + n/2)\). When \(\tau_{\epsilon} > 1\), the last term of the equation above tends to 0. For \(n \geq 1\), we have
\[
a_{n} = \tau_{\epsilon}^{n}a + \frac{\tau_{\epsilon}^{n} - 1}{\tau_{\epsilon} - 1} + \sum_{k=0}^{n-1} \tau_{\epsilon}^{n-1-k}g_{\epsilon}(a_{k}).
\]
Thus
\[
\frac{(\tau_{\epsilon} - 1)a_{n}}{\tau_{\epsilon}^{n}} = (\tau_{\epsilon} - 1) \left( a + \frac{g_{\epsilon}(a)}{\tau_{\epsilon}} + \sum_{k=1}^{n-1} \frac{g_{\epsilon}(a_{k})}{\tau_{\epsilon}^{k+1}} \right) + 1 - \frac{1}{\tau_{\epsilon}^{n}}.
\]
Since \(|g_{\epsilon}(a_{k})| \leq M/(2R + k/2) \leq 2M/k\), we have
\[
\left| \sum_{k=1}^{n-1} \frac{g_{\epsilon}(a_{k})}{\tau_{\epsilon}^{k+1}} \right| \leq \frac{2M}{\tau_{\epsilon}} \sum_{k=1}^{n-1} \frac{1}{k\tau_{\epsilon}^{k}} \leq -2M \log(1 - \frac{1}{\tau_{\epsilon}}).
\]
and this implies that the sums above converge as \( n \to \infty \). Hence \( C_n \to C_\epsilon = 1 + O(\epsilon \log \epsilon) \).

Finally, by taking additional linear coordinate change by \( z \mapsto z/C_\epsilon \), \( u_\epsilon \) gives a desired holomorphic map.

**Notes.**

- One can check that \( u_\epsilon(z) = z(C_\epsilon^{-1} + o(1)) \) (\( \Re z \to \infty \)).

- There is a quasiconformal version of linearization theorem by McMullen. [Mc, §8].

## 2 Applications.

This section is devoted for a worked out example to explain my personal motivation to consider the simultaneous linearization theorem.

**Cauliflower.** In the family of quadratic maps, the simplest parabolic fixed point is given by \( g(z) = z + z^2 \). Now we consider its perturbation of the form \( f(z) = \lambda z + z^2 \) with \( \lambda \nearrow 1 \). According to [Mi, §8 and §10], we have the following fact:

**Proposition 2.1 (Königs and Fatou coordinates)** Let \( K_f \) and \( K_g \) be the filled Julia sets of \( f \) and \( g \). Then we have the following:

1. There exists a unique holomorphic branched covering map \( \phi_f : K_f^* \to \mathbb{C} \) satisfying the Schröder equation \( \phi_f(f(z)) = \lambda \phi_f(z) \) and \( \phi_f(0) = \phi_f(-\lambda/2) - 1 = 0 \). \( \phi_f \) is univalent near \( z = 0 \).

2. There exists a unique holomorphic branched covering map \( \phi_g : K_g^* \to \mathbb{C} \) satisfying the Abel equation \( \phi_g(g(z)) = \phi_g(z) + 1 \) and \( \phi_g(-1/2) = 0 \). \( \phi_g \) is univalent on a disk \( |z + r| < r \) with small \( r > 0 \).

Note that \( -\lambda/2 \) and \( -1/2 \) are the critical points of \( f \) and \( g \) respectively.

**Observation.** Set \( w = \phi_f(z) \). Now the proposition above asserts that the action of \( f|_{K_f^*} \) is semiconjugated to \( w \mapsto \lambda w \) by \( \phi_f \). Let us consider a Möbius map \( W = S_f(W) = \lambda(w - 1)/(\lambda - 1)w \) that sends \( \{0, 1, \lambda\} \) to \( \{-1, 0, 1\} \) respectively. By taking
the conjugation by $S_f$, the action of $w \mapsto \lambda w$ is viewed as $W \mapsto W/\lambda + 1$. Let us set $W = \Phi_f(z) := S_f \circ \phi_f(z)$. Now we have

$$\Phi_f(f(z)) = \Phi_f(z)/\lambda + 1 \quad \text{and} \quad \Phi_f(-\lambda/2) = 0.$$ 

On the other hand, by setting $W = \Phi_g(z) := \phi_g(z)$, we can view the action of $g|_{K^o_g}$ as $W \mapsto W + 1$. Thus we have

$$\Phi_g(g(z)) = \Phi_g(z) + 1 \quad \text{and} \quad \Phi_g(-1/2) = 0.$$ 

If $\lambda$ tends to 1, that is, $f \rightarrow g$, the semiconjugated action in $W$-coordinate converges uniformly on compact sets. However, as one can see by referring the proof of the proposition in [Mi, §8 and §10], $\phi_f$ and $\phi_g$ are given in completely different ways thus we cannot conclude the convergence $\Phi_f \to \Phi_g$ a priori.

Figure 1: Semiconjugation inside the filled Julia sets

But there is another evidence that support this observation. Figure 1 shows the equipotential curves of $\phi_f$ and $\phi_g$ in the filled Julia sets. Obviously similar patterns appear and it seems one converges to the other.

Actually, we have the following fact:
Proposition 2.2 For any compact set $E \subset K^2_g$, 

1. $E \subset K^2_g$ for all $f \approx g$; and 

2. $\Phi_f \to \Phi_g$ uniformly on $E$ as $f \to g$.

Here $f \approx g$ means that $f$ is sufficiently close to $g$, equivalently, $\lambda$ sufficiently close to 1. See [Kal1, Theorem 5.5] for more general version of this proposition, which is one of the key result to show the continuity of tessellation and pinching semiconjugacies constructed in [Kal1].

Proof. Let us take a general expression $f_\lambda(z) = \lambda z + z^2$ with $0 < \lambda \leq 1$ (thus $f_1 = g$). By looking at the action of $f_\lambda$ through a new coordinate $w = \chi_\lambda(z) = -\lambda^2/z$, we have 

$$\chi_\lambda \circ f_\lambda \circ \chi_\lambda^{-1}(w) = w/\lambda + 1 + O(1/w)$$

near $\infty$. Now we can set $\tau_\epsilon := 1/\lambda = 1 + \epsilon$ and $f_\epsilon := \chi_\lambda \circ f_\lambda \circ \chi_\lambda^{-1}$ to have the same setting as Theorem 1.1. We consider that $f$ and $g$ are parameterized by $\lambda$ or $\epsilon$. (It is convenient to use both parameterization.)

Let us show (1): For any compact $E \subset K^2_g$ and small $r > 0$, there exists $N \gg 0$ such that $g^N(E) \subset P_r = \{|z + r| < r\}$. (For instance, one can show this fact by existence of the Fatou coordinate.) By uniform convergence, we have $f^N(E) \subset P_r$ for all $f \approx g$. To show $E \subset K^2_g$, it is enough to show that $f(P_r) \subset P_r$ for all $f \approx g$. Since $\chi_\lambda(P_r) = E_R$ for some $R \gg 0$, Lemma 1.2 implies that $E_R \subset f_\epsilon(E_R)$ independently of $\epsilon$. This is equivalent to $f_\lambda(P_r) \subset P_r$ in a different coordinate. Thus we have (1).

Next let us check (2): Set $\Phi_\epsilon := \Phi_f$ and $\Phi_0 := \Phi_g$. Then we have $\Phi_\epsilon(f_\lambda(z)) = \tau_\epsilon \Phi_0(z) + 1$. On the other hand, by simultaneous linearization, we have a uniform convergence $u_\epsilon \to u_0$ on $E_R$ that satisfies $u_\epsilon(f_\lambda(w)) = \tau_\epsilon u_\epsilon(w) + 1$. By setting $\Psi_\epsilon(z) := u_\epsilon \circ \chi_\lambda(z)$, we have $\Psi_\epsilon \to \Psi_0$ compact uniformly on $P_r$, and $\Psi_\epsilon(f_\lambda(z)) = \tau_\epsilon \Psi_\epsilon(z) + 1$.

We need to adjust the images of critical orbits mapped by $\Phi_\epsilon$ and $\Psi_\epsilon$. Since $g^n(-1/2) \to 0$ along the real axis, there is an $M \gg 0$ such that $g^M(-1/2) = a_0 \in P_r$. By uniform convergence, we also have $f^M(-\lambda/2) =: a_\epsilon \in P_r$ and $a_\epsilon \to a_0$ as $\epsilon \to 0$. Set $b_\epsilon := \Psi_\epsilon(a_\epsilon)$ and $c_\epsilon := \Phi_\epsilon(a_\epsilon)$ for all $\epsilon \geq 0$. Set also $\ell_\epsilon(W) = \tau_\epsilon W + 1$, then we have $c_\epsilon = \ell_\epsilon^M(0) = \tau_\epsilon^{M-1} + \cdots + \tau_\epsilon + 1$ and $c_\epsilon \to c_0 = M$ as $\epsilon \to 0$. When $\epsilon > 0$, we take an affine map $T_\epsilon$ that fixes $1/(1 - \tau_\epsilon)$ and sends $b_\epsilon$ to $c_\epsilon$. When $\epsilon = 0$, we take an affine map $T_0$ that is the translation by $b_0 - c_0$. Then one can check that $T_\epsilon \to T_0$. 

compact uniformly on the plane and $\Phi_{\epsilon} := T_{\epsilon} \circ \Psi_{\epsilon}$ satisfies $\Phi_{\epsilon} \to \tilde{\Phi}_{0}$ on any compact sets of $P_r$. Moreover, $\Phi_{\epsilon}$ still satisfies $\Phi_{\epsilon}(f_{\lambda}(z)) = \tau_{\epsilon}\Phi_{\epsilon}(z) + 1$ and the images of the critical orbit by $\Phi_{\epsilon}$ and $\Phi_{\epsilon}$ agree. Finally by uniqueness of $\phi_f$ and $\phi_g$, one can easily check that $\Phi_{\epsilon} = \tilde{\Phi}_{\epsilon}$ on $P_r$.

Since 

$$\Phi_f(z) = \ell_{\epsilon}^{-N} \circ \tilde{\Phi}_{\epsilon}(f^N(z)) \to \ell_{0}^{-N} \circ \tilde{\Phi}_{0}(g^N(z)) = \Phi_g(z)$$

uniformly on $E$, we have (2). □

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References


