Simultaneous linearization and its application

Tomoki Kawahira (川平 友規)
Graduate School of Mathematics
Nagoya University

Abstract

This note gives a proof of Ueda's simultaneous linearization theorem with real multipliers and its simple application to quadratic dynamics. This note is based on my talk at RIMS on 5 October 2006, titled “A proof of simultaneous linearization with a polylog estimate.”

1 Simultaneous linearization

Here we give an alternative proof of Ueda's simultaneous linearization in a simplified setting. For $R \geq 0$, let $E_R$ denote the region $\{z \in \mathbb{C} : \mathrm{Re} z \geq R\}$.

Theorem 1.1 (Simultaneous Linearization) For $\epsilon \in [0, 1]$, let $\{f_\epsilon\}$ be a family of holomorphic maps on $\{|z| \geq R > 0\}$ such that

\[ f_\epsilon(z) = \tau_\epsilon z + 1 + O(1/z) \]
\[ \rightarrow f_0(z) = z + 1 + O(1/z) \]

uniformly as $\epsilon \to 0$ where $\tau_\epsilon = 1 + \epsilon$. If $R \gg 0$, then for any $\epsilon \in [0, 1]$ there exists a holomorphic map $u_\epsilon : E_R \to \mathbb{C}$ such that

\[ u_\epsilon(f_\epsilon(z)) = \tau_\epsilon u_\epsilon(z) + 1 \]

and $u_\epsilon \to u_0$ uniformly on compact sets of $E_R$. 
Indeed, a similar theorem holds for any radial (= non-tangential) convergence $\tau_{\epsilon} \to 1$ outside the unit disk. See Ueda's original proof ([Ue1], [Ue2]). Moreover, the error term $O(1/z)$ can be replaced by $O(|z|^{-\sigma})$ with $0 < \sigma \leq 1$. (See [Ka2].) Here we present a simplified proof only for real $\tau_{\epsilon} \to 1$ based on the argument of [Mi, Lemma 10.10]. The idea can be traced back at least to Leau's work on the Abel equation [L]. We first check:

**Lemma 1.2** If $R \gg 0$, there exists $M > 0$ such that $|f_{\epsilon}(z) - (\tau_{\epsilon} z + 1)| \leq M/|z|$ on $\{|z| \geq R\}$ and $\text{Re} f_{\epsilon}(z) \geq \text{Re} z + 1/2$ on $E_{R}$ for any $\epsilon \in [0,1]$.

**Proof.** The first inequality and the existence of $M$ is obvious. By replacing $R$ by larger one, we have $|f_{\epsilon}(z) - (\tau_{\epsilon} z + 1)| \leq M/R < 1/2$ on $E_{R}$. Then

$$\text{Re} f_{\epsilon}(z) \geq \text{Re} (\tau_{\epsilon} z + 1) - 1/2 \geq \text{Re} z + 1/2.$$  

\[\blacksquare\]

Let us fix such an $R \gg 0$. Next we show:

**Lemma 1.3** For any $\epsilon \in [0,1]$ and $z_{1}$, $z_{2} \in E_{2S}$ with $S > R$, we have:

$$\left| \frac{f_{\epsilon}(z_{1}) - f_{\epsilon}(z_{2})}{z_{1} - z_{2}} - \tau_{\epsilon} \right| \leq \frac{M}{S^{2}}.$$  

**Proof.** Set $g_{\epsilon}(z) := f_{\epsilon}(z) - (\tau_{\epsilon} z + 1)$. For any $|z| \geq 2S$ and $w \in B(z, S)$, we have $|w| > S$. This implies $|g_{\epsilon}(w)| \leq M/|w| < M/S$ and thus $g_{\epsilon}$ maps $B(z, S)$ into $B(0, M/S)$. By the Cauchy integral formula or the Schwarz lemma, it follows that $|g_{\epsilon}'(z)| \leq (M/S)/S = M/S^{2}$ on $\{|z| \geq S\}$. Now the inequality easily follows by:

$$|g_{\epsilon}(z_{1}) - g_{\epsilon}(z_{2})| = \left| \int_{[z_{2}, z_{1}]} g_{\epsilon}'(z)dz \right| \leq \int_{[z_{2}, z_{1}]} |g_{\epsilon}'(z)||dz| \leq \frac{M}{S^{2}} |z_{1} - z_{2}|.$$  

(Note that the segment $[z_{2}, z_{1}]$ is contained in $E_{2S} \subset \{|z| \geq 2S\}$.)

\[\blacksquare\]

**Proof of Theorem 1.1.** Set $z_{n} := f_{\epsilon}^{n}(z)$ for $z \in E_{2R}$. Note that such $z_{n}$ satisfies

$$|z_{n}| \geq \text{Re} z_{n} \geq \text{Re} z + \frac{n}{2} \geq 2R + \frac{n}{2}$$

by Lemma 1.2. Now we fix $a \in E_{2R}$ and define $u_{n, \epsilon} = u_{n} : E_{2R} \to \mathbb{C}$ ($n \geq 0$) by

$$u_{n}(z) := \frac{z_{n} - a_{n}}{\tau_{\epsilon}^{n}}.$$
Then we have
\[ \left| \frac{u_{n+1}(z)}{u_{n}(z)} - 1 \right| = \left| \frac{z_{n+1} - a_{n+1}}{\tau_{\epsilon}(z_{n} - a_{n})} - 1 \right| = \frac{1}{\tau_{\epsilon}} \cdot \left| \frac{f_{\epsilon}(z_{n}) - f_{\epsilon}(a_{n})}{z_{n} - a_{n}} - \tau_{\epsilon} \right|. \]

We apply Lemma 1.3 with $2S = 2R + n/2$. Then
\[ \left| \frac{u_{n+1}(z)}{u_{n}(z)} - 1 \right| \leq \frac{M}{\tau_{\epsilon}(R + n/4)^{2}} \leq \frac{C}{(n+1)^{2}}, \]
where $C = 16M$ and we may assume $R > 1/4$. Set $P := \prod_{n \geq 1} (1 + C/n^2)$. Since $|u_{n+1}(z)/u_{n}(z)| \leq 1 + C/(n+1)^2$, we have
\[ |u_{n}(z)| = \left| \frac{u_{n}(z)}{u_{n-1}(z)} \right| \cdots \left| \frac{u_{1}(z)}{u_{0}(z)} \right| \cdot |u_{0}(z)| \leq P|z - a|. \]

Hence
\[ |u_{n+1}(z) - u_{n}(z)| = \left| \frac{u_{n+1}(z)}{u_{n}(z)} - 1 \right| \cdot |u_{n}(z)| \leq \frac{CP}{(n+1)^{2}} \cdot |z - a|. \]

This implies that $u_{\epsilon} = u_{0} + (u_{1} - u_{0}) + \cdots = \lim u_{n}$ converges uniformly on compact subsets of $E_{2R}$ and for all $\epsilon \in [0, 1]$. The univalence of $u_{\epsilon}$ is shown in the same way as [Mi, Lemma 10.10].

Next we check that $u_{\epsilon}(f_{\epsilon}(z)) = \tau_{\epsilon}u_{\epsilon}(z) + C_{\epsilon}$ with $C_{\epsilon} \to 1$ as $\epsilon \to 0$. One can easily check that $u_{n}(f_{\epsilon}(z)) = \tau_{\epsilon}u_{n+1}(z) + C_{n}$ where
\[ C_{n} = \frac{a_{n+1} - a_{n}}{\tau_{\epsilon}^{n}} = \frac{(\tau_{\epsilon} - 1)a_{n}}{\tau_{\epsilon}^{n}} + \frac{1 + g_{\epsilon}(a_{n})}{\tau_{\epsilon}^{n}}. \]

When $\tau_{\epsilon} = 1$, $C_{n}$ tends to 1 since $|g_{\epsilon}(a_{n})| \leq M/|a_{n}| \leq M/(2R + n/2)$. When $\tau_{\epsilon} > 1$, the last term of the equation above tends to 0. For $n \geq 1$, we have
\[ a_{n} = \tau_{\epsilon}^{n}a + \frac{\tau_{\epsilon}^{n-1} - 1}{\tau_{\epsilon} - 1} + \sum_{k=0}^{n-1} \tau_{\epsilon}^{n-1-k}g_{\epsilon}(a_{k}). \]

Thus
\[ \frac{(\tau_{\epsilon} - 1)a_{n}}{\tau_{\epsilon}^{n}} = (\tau_{\epsilon} - 1) \left( a + \frac{g_{\epsilon}(a)}{\tau_{\epsilon}} + \sum_{k=1}^{n-1} \frac{g_{\epsilon}(a_{k})}{\tau_{\epsilon}^{k+1}} \right) + 1 - \frac{1}{\tau_{\epsilon}^{n}}. \]

Since $|g_{\epsilon}(a_{k})| \leq M/(2R + k/2) \leq 2M/k$, we have
\[ \left| \sum_{k=1}^{n-1} \frac{g_{\epsilon}(a_{k})}{\tau_{\epsilon}^{k+1}} \right| \leq \frac{2M}{\tau_{\epsilon}} \sum_{k=1}^{n-1} \frac{1}{k \tau_{\epsilon}^{k}} \leq -2M \log(1 - \frac{1}{\tau_{\epsilon}}). \]
and this implies that the sums above converge as \( n \rightarrow \infty \). Hence \( C_n \rightarrow C_\epsilon = 1 + O(\epsilon \log \epsilon) \).

Finally, by taking additional linear coordinate change by \( z \mapsto z/C_\epsilon \), \( u_\epsilon \) gives a desired holomorphic map.

Notes.

- One can check that \( u_\epsilon(z) = z(C_\epsilon^{-1} + o(1)) \) (Re \( z \rightarrow \infty \)).

- There is a quasiconformal version of linearization theorem by McMullen. [Mc, §8].

2 Applications.

This section is devoted for a worked out example to explain my personal motivation to consider the simultaneous linearization theorem.

Cauliflower. In the family of quadratic maps, the simplest parabolic fixed point is given by \( g(z) = z + z^2 \). Now we consider its perturbation of the form \( f(z) = \lambda z + z^2 \) with \( \lambda \nearrow 1 \). According to [Mi, §8 and §10], we have the following fact:

**Proposition 2.1 (Königs and Fatou coordinates)** Let \( K_f \) and \( K_g \) be the filled Julia sets of \( f \) and \( g \). Then we have the following:

1. There exists a unique holomorphic branched covering map \( \phi_f : K_f^* \rightarrow \mathbb{C} \) satisfying the Schröder equation \( \phi_f(f(z)) = \lambda \phi_f(z) \) and \( \phi_f(0) = \phi_f(-\lambda/2) - 1 = 0 \). \( \phi_f \) is univalent near \( z = 0 \).

2. There exists a unique holomorphic branched covering map \( \phi_g : K_g^* \rightarrow \mathbb{C} \) satisfying the Abel equation \( \phi_g(g(z)) = \phi_g(z) + 1 \) and \( \phi_g(-1/2) = 0 \). \( \phi_g \) is univalent on a disk \( |z + r| < r \) with small \( r > 0 \).

Note that \(-\lambda/2\) and \(-1/2\) are the critical points of \( f \) and \( g \) respectively.

**Observation.** Set \( w = \phi_f(z) \). Now the proposition above asserts that the action of \( f|_{K_f^*} \) is semiconjugated to \( w \mapsto \lambda w \) by \( \phi_f \). Let us consider a Möbius map \( W = S_f(W) = \lambda(w - 1)/(\lambda - 1)w \) that sends \( \{0, 1, \lambda\} \) to \( \{\infty, 0, 1\} \) respectively. By taking
the conjugation by $S_f$, the action of $w \mapsto \lambda w$ is viewed as $W \mapsto W/\lambda + 1$. Let us set $W = \Phi_f(z) := S_f \circ \phi_f(z)$. Now we have

$$
\Phi_f(f(z)) = \Phi_f(z)/\lambda + 1 \quad \text{and} \quad \Phi_f(-\lambda/2) = 0.
$$

On the other hand, by setting $W = \Phi_g(z) := \phi_g(z)$, we can view the action of $g|_{K^o}$ as $W \mapsto W + 1$. Thus we have

$$
\Phi_g(g(z)) = \Phi_g(z) + 1 \quad \text{and} \quad \Phi_g(-1/2) = 0.
$$

If $\lambda$ tends to 1, that is, $f \to g$, the semiconjugated action in $W$-coordinate converges uniformly on compact sets. However, as one can see by referring the proof of the proposition in [Mi, §8 and §10], $\phi_f$ and $\phi_g$ are given in completely different ways thus we cannot conclude the convergence $\Phi_f \to \Phi_g$ a priori.

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Figure 1: Semiconjugation inside the filled Julia sets

But there is another evidence that support this observation. Figure 1 shows the equipotential curves of $\phi_f$ and $\phi_g$ in the filled Julia sets. Obviously similar patterns appear and it seems one converges to the other.

Actually, we have the following fact:
Proposition 2.2 For any compact set $E \subset K^2_g$,

(1) $E \subset K^2_g$ for all $f \approx g$; and

(2) $\Phi_f \to \Phi_g$ uniformly on $E$ as $f \to g$.

Here $f \approx g$ means that $f$ is sufficiently close to $g$, equivalently, $\lambda$ sufficiently close to 1. See [Kal1, Theorem 5.5] for a more general version of this proposition, which is one of the key results to show the continuity of tessellation and pinching semiconjugacies constructed in [Kal1].

Proof. Let us take a general expression $f_\lambda(z) = \lambda z + z^2$ with $0 < \lambda \leq 1$ (thus $f_1 = g$).

By looking at the action of $f_\lambda$ through a new coordinate $w = \chi_\lambda(z) = -\lambda^2/z$, we have

$$\chi_\lambda \circ f_\lambda \circ \chi_\lambda^{-1}(w) = w/\lambda + 1 + O(1/w)$$

near $\infty$. Now we can set $\tau_\epsilon := 1/\lambda = 1 + \epsilon$ and $f_\epsilon := \chi_\lambda \circ f_\lambda \circ \chi_\lambda^{-1}$ to have the same setting as Theorem 1.1. We consider that $f$ and $g$ are parameterized by $\lambda$ or $\epsilon$. (It is convenient to use both parameterization.)

Let us show (1): For any compact $E \subset K^2_g$ and small $r > 0$, there exists $N > 0$ such that $g^N(E) \subset P_r = \{|z + r| < r\}$. (For instance, one can show this fact by existence of the Fatou coordinate.) By uniform convergence, we have $f^N(E) \subset P_r$ for all $f \approx g$. To show $E \subset K^2_g$, it is enough to show that $f(P_r) \subset P_r$ for all $f \approx g$. Since $\chi_\lambda(P_r) = E_R$ for some $R > 0$, Lemma 1.2 implies that $E_R \subset f_\epsilon(E_R)$ independently of $\epsilon$. This is equivalent to $f_\epsilon(P_r) \subset P_r$ in a different coordinate. Thus we have (1).

Next let us check (2): Set $\Phi_\epsilon := \Phi_f$ and $\Phi_0 := \Phi_g$. Then we have $\Phi_\epsilon(f_\lambda(z)) = \tau_\epsilon \Phi_f(z) + 1$. On the other hand, by simultaneous linearization, we have a uniform convergence $u_\epsilon \to u_0$ on $E_R$ that satisfies $u_\epsilon(f_\lambda(w)) = \tau_\epsilon u_\epsilon(w) + 1$. By setting $\Psi_\epsilon(z) := u_\epsilon \circ \chi_\lambda(z)$, we have $\Psi_\epsilon \to \Psi_0$ compact uniformly on $P_r$, and $\Psi_\epsilon(f_\lambda(z)) = \tau_\epsilon \Psi_\epsilon(z) + 1$.

We need to adjust the images of critical orbits mapped by $\Phi_\epsilon$ and $\Psi_\epsilon$. Since $g^n(-1/2) \to 0$ along the real axis, there is an $M > 0$ such that $g^M(-1/2) =: a_0 \in P_r$. By uniform convergence, we also have $f^M(-1/2) =: a_\epsilon \in P_r$ and $a_\epsilon \to a_0$ as $\epsilon \to 0$.

Set $b_\epsilon := \Psi_\epsilon(a_\epsilon)$ and $c_\epsilon := \Phi_\epsilon(a_\epsilon)$ for all $\epsilon \geq 0$. Set also $\ell_\epsilon(W) = \tau_\epsilon W + 1$, then we have $c_\epsilon = \ell_\epsilon^M(0) = \tau_\epsilon^{M-1} + \cdots + \tau_\epsilon + 1$ and $c_\epsilon \to c_0 = M$ as $\epsilon \to 0$. When $\epsilon > 0$, we take an affine map $T_\epsilon$ that fixes $1/(1 - \tau_\epsilon)$ and sends $b_\epsilon$ to $c_\epsilon$. When $\epsilon = 0$, we take an affine map $T_0$ that is the translation by $b_0 - c_0$. Then one can check that $T_\epsilon \to T_0$.
compact uniformly on the plane and $\Phi_\epsilon := T_\epsilon \circ \Psi_\epsilon$ satisfies $\Phi_\epsilon \to \tilde{\Phi}_0$ on any compact sets of $P_r$. Moreover, $\Phi_\epsilon$ still satisfies $\Phi_\epsilon(f_\lambda(z)) = \tau_\epsilon \Phi_\epsilon(z) + 1$ and the images of the critical orbit by $\Phi_\epsilon$ and $\tilde{\Phi}_\epsilon$ agree. Finally by uniqueness of $\phi_f$ and $\phi_g$, one can easily check that $\Phi_\epsilon = \tilde{\Phi}_\epsilon$ on $P_r$.

Since

$$\Phi_f(z) = \ell_\epsilon^{-N} \circ \tilde{\Phi}_\epsilon(f^N(z)) \to \ell_0^{-N} \circ \tilde{\Phi}_0(g^N(z)) = \Phi_g(z)$$

uniformly on $E$, we have (2).

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### References


