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Perturbations of bivariate Chebyshev maps of $\mathbf{C}^2$

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Abstract

The Chebyshev map is a typical chaotic map. Few things are known about the dynamics of generalized Chebyshev maps in higher dimensions. Generalized Chebyshev maps are related to the theory of complex Lie algebras and affine Weyl groups.

In a former note we showed that bivariate Chebyshev maps of $\mathbf{C}^2$ are also chaotic and the support of the maximal entropy measure μ of each Chebyshev map is connected.

In this note, we perturb a bivariate Chebyshev map in a certain direction. Then, we show that the support μ of a perturbed map is a Cantor sets.

1 Introduction

We study perturbations of bivariate Chebyshev maps $\mathbf{C}^2$. Bivariate Chebyshev maps T_n from $\mathbf{C}^2$ to $\mathbf{C}^2$, $(n \in \mathbf{Z})$ are given by [V]. Uchimura [U] studies dynamics of bivariate Chebyshev maps.

$$T_n(x,y) = (g_n(x,y), g_n(y,x)),$$

where $g_n(x,y)$ is a generalized Chebyshev polynomial defined by [L].

$$\text{Let} \quad x = t_1+t_2+t_3, \quad y = t_1t_2+t_1t_3+t_2t_3, \quad 1 = t_1t_2t_3.$$

$$\text{Set} \quad g_n(x,y) := t_1^n + t_2^n + t_3^n. \quad \text{Hence}$$

$$g_n(y,x) = (1/t_1)^n + (1/t_2)^n + (1/t_3)^n = g_{-n}(x,y).$$

$$\text{For example,} \quad T_2(x,y) = (x^2-2y, y^2-2x),$$

$$T_3(x,y) = (x^3-3xy+3, y^3-3xy+3),$$

$$T_4(x,y) = (x^4-4x^2y+2y^2+4x, y^4-4xy^2+2x^2+4y).$$

We perturb bivariate Chebyshev maps. We introduce a new parameter c in $T_k(x,y)$.

We set

$$f_c^{(k)}(x,y) = (g_k(x,y,c), g_k(y,x,c))$$

where $g_k(x,y,c)$ is homogeneous in x,y,c and $g_k(x,y,1)=g_k(x,y)$. We call $f_c^k(x,y)$ a c-Chebyshev map.
For example, $g_3(x,y)=x^3-3xy+3$ and

$$f_c^{(3)}(x,y)=(x^3-3cxy+3c^3,y^3-3cxy+3c^3).$$

Theorem *Assume $c>1$. Then the support of the maximal entropy measure of a c-Chebyshev map $f_c^{(k)}(x,y)$ is a Cantor set for any $k\in\mathbf{Z}\setminus\{0,1,-1\}$.*

In one dimensional case, this is parallel to the following facts.
The map $f_c^{(2)}(x,y)$ restricted on the line $\{x=y\}$ is the map $q_c^{(2)}(x)=x^2-2cx$ which is conjugate to the map $p_\lambda(x)=x^2+\lambda$. Parameters correspond as follows:

$$1\le c \iff \lambda\le -2.$$

$p_{-2}(x)=x^2-2$ is a Chebyshev map. If $\lambda<-2$, $K(p_\lambda(x))$ is a Cantor set.
When $k=3$, the map $f_c^{(3)}(x,y)$ restricted on the line $\{x=y\}$ is the map

$$q_c^{(3)}(x)=x^3-3cx^2+3c^3.$$

Milnor[M] shows the moduli space of real cubic maps up to affine conjugate. In the space the set $\{q_c^{(3)}(x):c\in\mathbf{R}\}$ can be represented as a curve. When $c=1$, the curve is tangent to the horizontal line. When $c>1$, it lies in R_3.
From Theorem, we can easily see that if $c>1$, $K(q_c^{(3)})$ is a Cantor set on the real axis.

We review some properties of bivariate Chebyshev map (See [U]). From the definition of bivariate Chebyshev maps we have a branch covering map. The following diagram is commutative.

$$\begin{array}{ccc}
(\mathbf{C}-\{0\})^2 & \xrightarrow{\quad g_n \quad} & (\mathbf{C}-\{0\})^2 \\
(u,v)\ \Big\downarrow \Psi & & \Psi \Big\downarrow\ (u^n,v^n) \\
\mathbf{C}^2 & \xrightarrow{\quad T_k \quad} & \mathbf{C}^2 \\
(x=u+v+\frac{1}{uv}, & & (u^n+v^n+(\frac{1}{uv})^n, \\
y=\frac{1}{u}+\frac{1}{v}+uv) & & \frac{1}{u^n}+\frac{1}{v^n}+(uv)^n)
\end{array}$$

$\Psi:\mathbf{C}^{*2}-\Psi^{-1}(D)\to\mathbf{C}^2-D$ is a 6-sheeted covering map. And its branch locus D of Ψ is written as $x^2y^2-4x^3-4y^3+18xy-27=0$.

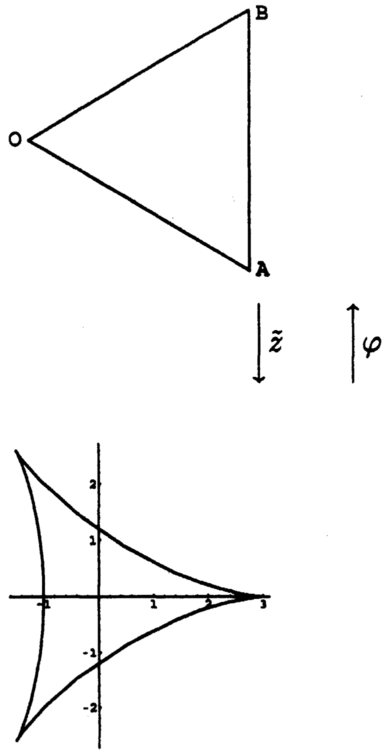
Next we study fundamental domains.
$T_n(x,y)$ restricted on $\{x=\overline{y}\}$ is a Chebyshev polynomial defined by [K] :

$$P_{n,0}^{-\frac{1}{2}}(z,\overline{z})=e^{in\sigma}+e^{-in\tau}+e^{i(n\tau-n\sigma)}.$$

$$\text{Set}\quad z(\sigma,\tau):=e^{i\sigma}+e^{-i\tau}+e^{i(\tau-\sigma)}=u+iv.$$

The mapping $z:(\sigma,\tau)\to(u,v)$ is a diffeomorphism from R_1 onto S , where S is a closed domain bounded by a hypocycloid and R_1 is a triangular region. R is a slight modification of R_1.

(s,t) $\qquad R$



(u,v) $\qquad S$

K sets and J_2 sets of T_k's have the following properties.

(1) $K(T_k) = \{|\ t_1\ |=|\ t_2\ |= 1\} = S \subset \{x = \overline{y}\}$.

(2) $J_2 = supp\ \mu = S$

(3) Repelling periodic points are equidistributed in S and J_2 is connected.
We describe Critical sets $C(T_k)$ and critical values $T_k(C)$.
(1) Any irreducible component of $C(T_k)$ is a rational curve of degree 2 or 4 :

$$x = t + \varepsilon^j t + \frac{1}{\varepsilon^j t^2}, \qquad (\varepsilon^k = 1)$$

$$y = \frac{1}{t} + \frac{1}{\varepsilon^j t} + \varepsilon^j t^2.$$

(2) Critical values $T_k(C)$ is written as

$$x = t^k + t^k + \frac{1}{t^{2k}}, \qquad y = \frac{1}{t^k} + \frac{1}{t^k} + t^{2k}.$$

We use the properties of Böttcher coordinates (see [BJ])

$$W^s(J_\Pi, T_n) = \{x \in \mathbf{P}^2 : d(T_n^j x, J_\Pi) \to 0, j \to \infty\},$$

where

$$J_\Pi = \{(x : y : 0) :|\ x\ |=|\ y\ |\}. \quad \text{(the Julia set in the line at infinity)}$$

There exists a homeomorphism Ψ such that

$$\Psi : W^s(J_\Pi, f_k) \to W^s(J_\Pi, T_k)$$

conjugating f_k to T_k where $f_k(x,y) = (x^k, y^k)$.

External rays $R(r,\phi,\theta)$ of T_k are calculated by Nakane [N]

$$x = re^{-2\pi i\theta} + \frac{1}{r}e^{2\pi i(\theta-\phi)} + e^{2\pi i\phi},$$

$$y = re^{2\pi i(\phi-\theta)} + \frac{1}{r}e^{2\pi i\theta} + e^{-2\pi i\phi}.$$

2 A main result

We state a main result of this note.

Theorem *Assume $c > 1$. Then the support of the maximal entropy measure μ of a c-Chebyshev map $f_c^{(k)}(x,y)$ is a Cantor set for any $k \in \mathbf{Z} \setminus \{0, 1, -1\}$.*

When $c = 1$, the support of μ of $f_1^{(k)}(x,y)(= T_k(x,y))$ is a connected set S on the plane $\{x = \overline{y}\}$. However if $c > 1$, the support of μ of $f_c^{(k)}(x,y)$ is not connected. This shows that a bifurcation occurs at $c = 1$.

One of the reason why we define a c-Chebyshev map in such a form is shown in the next fact. Let

$$x = c(t_1 + t_2 + t_3), \quad y = c(1/t_1 + 1/t_2 + 1/t_3)$$

and $\quad t_1 t_2 t_3 = 1. \quad$ Then

$$f_c(x,y) = (c^k(t_1^k + t_2^k + t_3^k), c^k(1/t_1^k + 1/t_2^k + 1/t_3^k)).$$

The critical set $C(f_c)$ and the critical value $f_c(C)$ are written as follows.

$$C(f_c) : x = c((1+\varepsilon)t + 1/(\varepsilon t^2)),$$

$$y = c(1/t + 1/(\varepsilon t) + \varepsilon t^2),$$

$$f_c(C) : x = c^k(2t^k + 1/t^{2k}), \quad y = c^k(2/t^k + t^{2k}),$$

where $\varepsilon^k = 1$ and $t \in \mathbf{C} \setminus \{0\}$.

3 Sketch of the proof of Theorem

The key observation of the proof of Theorem is the following proposition.

Proposition 3.1. *If $c > 1, K(f_c) \cap C(f_c) = \phi$.*

This is equivalent to the statement if $c > 1$, then $f_c^n(f_c(C)) \to \infty$, $(n \to \infty)$ with respect to Euclidean norm. The critical value $f_c(C)$ is parameterized as

$$x = c^k(2t^k + 1/t^{2k}), \quad y = c^k(2/t^k + t^{2k}), t \in \mathbf{C} \setminus \{0\}.$$

To prove Proposition 3.1, we will shrink the domain $\mathbf{C} \setminus \{0\}$ of t.

$$(u_n(t), v_n(t)) := f_c^n(u_0(t), u_0(1/t)), \quad \text{where } u_0(t) = c^k(2t^k + 1/t^{2k}).$$

$(u_n(t), v_n(t))$ represents an element of $f_c^n(f_c(C))$.
We can see that for any $n \in \mathbf{N}, v_n(1/t) = u_n(t)$.
Hence we can shrink the domain $\mathbf{C}^*$. We consider the domain $\bar{\mathbf{D}} \setminus \{0\}$.

Proposition 3.2. *For any $t \in \overline{\mathbf{D}} \setminus \{0\}, | v_n(t) | \to \infty$ as $n \to \infty$ where $\mathbf{D}$ denotes the unit disk.*

To prove this we need two steps.
(A) If $c > 1$, then $| v_n(t) |$ has its minimum value on the boundary $\partial\mathbf{D}$ for any $n \in \mathbf{N}$.
(B) If $c > 1$, then $| v_n(e^{i\theta}) | \to \infty$ as $n \to \infty$ for any θ in $[0, 2\pi)$.

For example, when $k = 3$,

$$v_1(t) = c^9t^{18} + (6c^9 - 6c^7)t^9 + 3c^3 - 15c^7 + 12c^9 + (8c^9 - 6c^7)\frac{1}{t^9},$$

$$u_1(t) = v_1(\frac{1}{t}) \quad .$$

Proof of (A)
(A) If $c > 1$, then $| v_n(t) |$ has its minimum value on $\partial\mathbf{D}$ for any $n \in \mathbf{N}$.

$v_n(t)$ is a rational function in t.
$| v_n(t) |$ has its minimum value on $\partial\mathbf{D}$, if
(A1) $v_n(t)$ is holomorphic in $\mathbf{D}^*$ and
(A2) $v_n(t)$ has no zeros in $\mathbf{D}^*$.
We prove A2 by Argument principle :

$$W(v_n(S^1), 0) = N - M.$$

These are proved by considering the map

$$g_c(z) := f_c(x, y) \mid \{x = \overline{y}\}$$

from $\{x = \overline{y}\}$ to $\{x = \overline{y}\}$.
When c is real, $f_c(x, y)$ admits an invariant plane $\{x = \overline{y}\}$. Then we consider the map $g_c(z)$ on the plane $\{x = \overline{y}\}$ given by

$$g_c(z) = f_c(x, y) \mid \{x = \overline{y}\}.$$

The map $g_c(z)$ may be viewed as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$.
The critical set of g_c is equal to the set

$$\{c(e^{i\theta} + \varepsilon e^{i\theta} + 1/(\varepsilon e^{2\theta i})) : 0 \leq \theta < 2\pi\}.$$

The critical value of g_c is equal to the set

$$\{c^k(2e^{k\theta i} + e^{-2k\theta i})) : 0 \leq \theta < 2\pi\}.$$

We state properties of the map $g_c(z)$.
(1) For any point z in the interior $c^k S^\circ$, $g_c^{-1}(z) \subset cS$ and $g_c^{-1}(z)$ consists of k^2 distinct elements.
(2) $g_c \mid \mathbf{C} \setminus cS : \mathbf{C} \setminus cS \to \mathbf{C} \setminus c^k S$ is a k-sheeted unbranched covering map and is sense preserving.
(3) $g_c \mid \partial cS : \partial cS \to \partial c^k S$ is a k−to−one map.
Then we use topological argument principle (see [Ne], p.350). Let $W(\Gamma, p)$ be the winding number of a closed curve Γ round a point p. Let S^1 denote the unit circle $\{e^{i\theta} : 0 \leq \theta < 2\pi\}$. We assume that a hypocycloid $c^k\partial S$ is oriented in an anti-clockwise sense.
Topological Argument Principle implies
$\sharp\{z \in \mathbf{C} : g_c(z) = p$ and z belongs to the interior of $\gamma\}$ (counted with their topological multiplicities) $= W(g_c(\gamma), p)$

Hence if $c > 1$, then $W(g_c^n(c^k\partial S), 0) = k^n$.

Hence if $c > 1$, then

$$W(u_n(S^1), 0) = -W(v_n(S^1), 0) = k^{n+1}.$$

The multiplicity of the pole at $t = 0$ of $v_n(t)$ is at most k^{n+1}.
We apply the argument principle to the rational function $v_n(t)$ on the unit circle S^1 : $W(v_n(S^1), 0) = N - M$ where N is the number of zeros in the unit disk $\mathbf{D}$ and M is the number of poles in $\mathbf{D}$. We conclude that $v_n(t)$ has no zeros in $\mathbf{D}$ and it is holomorphic in $\mathbf{D} \setminus \{0\}$. Then $v_n(t)$ has its minimum-modulus on the boundary $\partial\mathbf{D}$. $\square$

Proof of (B)
(B) If $c > 1$, then $\mid v_n(e^{i\theta}) \mid \to \infty$ as $n \to \infty$ for any $\theta \in [0, 2\pi)$.

To prove this, we introduce a new distance $d(z, \partial cS)$ from an element z in $\mathbf{C} \setminus cS$ to the boundary ∂cS and we show that there exists a number $a > 1$ such that

$$ad(z, \partial cS) < d(g_c(z), \partial cS).$$

To prove this we use Böttcher coordinates.
Based on Nakane[2004] , we consider the Böttcher coordinate Ψ restricted on the set $\{t_1 = t, \quad t_2 = 1/\bar{t}\}$. We denote the map by ψ. Since

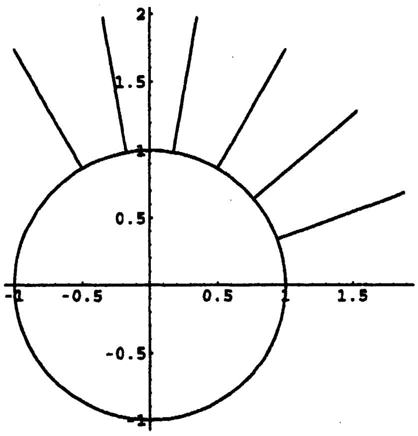
$$\Psi(t, 1/\bar{t}) = (t + 1/\bar{t} + \bar{t}/t, 1/t + \bar{t} + t/\bar{t}),$$

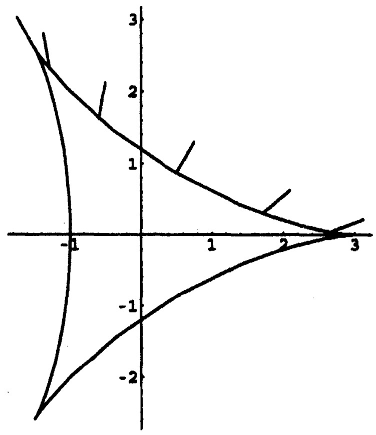
$$\psi(t) = t + 1/\bar{t} + \bar{t}/t$$

and $\Psi(t, 1/\bar{t})$ lies on the plane $\{x = \bar{y}\}$. Since

$$\psi(re^{i\theta}) = (r + 1/r)e^{i\theta} + e^{-2i\theta}, \quad (r > 1),$$

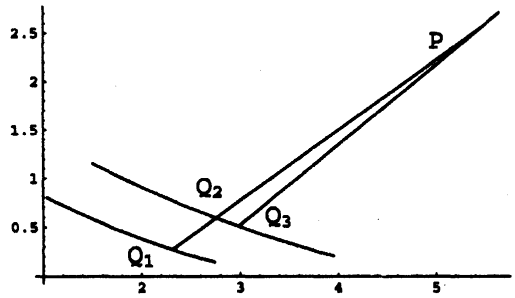
the map ψ from $\mathbf{C} \setminus \bar{\mathbf{D}}$ to $\mathbf{C} \setminus S$ is a homeomorphism.





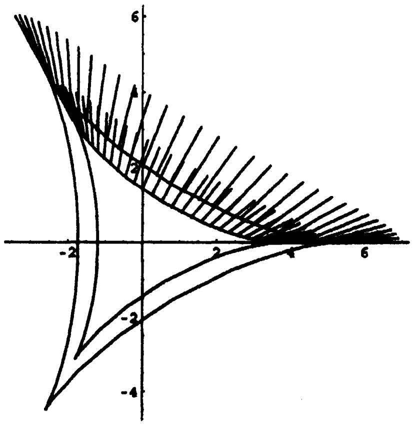
The image of a radial line $\{re^{i\theta} : r > 1\}$ under the map ψ is also a half-line. Let h_λ be a function from $\mathbf{C} \setminus S$ to $\mathbf{C} \setminus \lambda S$ defined by $h_\lambda(z) = \lambda z$ with $\lambda \geq 1$. The composition $h_\lambda \circ \psi$ is a map from $\mathbf{C} \setminus \bar{\mathbf{D}}$ onto $\mathbf{C} \setminus \lambda S$. Then the image of a radial line under the map $h_\lambda \circ \psi$ is a half-line which is called a λ-radial line.

We assume that $c > 1$. For any point z in $\mathbf{C} \setminus cS$ we set $(h_c \circ \psi)^{-1}(z) = re^{i\alpha}$, and $(h_c \circ \psi)^{-1}(g_c(z)) = r_1 e^{i\beta}$. Then we will prove that $\quad r_1 > \sqrt{c}r$.



We evaluate the length $\quad | PQ_1 | = c(r_1 + \frac{1}{r_1} - 2)$, where $\quad P = g_c(z)$. Clearly,

$$| PQ_1 | = | PQ_2 | + | Q_1Q_2 | \, .$$



The following four properties hold:
(1) $|PQ_2| \geq |PQ_3| = c^k(r^k + 1/r^k - 2)$.
(2) $|Q_1Q_2| \geq c^k - c$.
(3) If $c > 1$, $r > 1$ and $k \geq 2$, then $c^k(r^k+1/r^k-2)+c^k-c > c(\sqrt{cr}+1/(\sqrt{cr})-2)$.
(4) $c(r_1 + \frac{1}{r_1} - 2) > c(\sqrt{cr} + \frac{1}{\sqrt{cr}} - 2)$.
End of the proof of (B). □

We note that $\{f_c^n(C)\}$ converges uniformly to ∞.
End of the proof of Proposition 3.1. □

Lastly we prove that $K(f_c) = K(g_c)$ is a Cantor set. This is proved from the following three results.
(1) Theorem. (Fornaess and Sibony[2001]). Let f be a regular polynomial endomorphism of $\mathbf{C^k}$. Assume that $K(f) \cap C(f) = \phi$. Then
(i) The map f is strictly expanding on $K(f)$.
(ii) Repelling periodic points are dense in $K(f)$.
(iii) $K(f) = supp\ \mu$.
(2) When $c > 1$, any periodic points of $f_c(x, y)$ lies on the plane $\{x = \bar{y}\}$ and belongs to the set $K(g_c)$.
(3) When $c > 1$, there is a positive integer n such that g_c^n is uniformly expanding on $K(g_c)$.
Then we have the following theorem.

Theorem *Assume that* $c > 1$. *Then*
(i) $K(f_c) = supp\ \mu \subset \{x = \bar{y}\}$.
(ii) $K(f_c) = K(g_c)$ *is a Cantor set.*

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