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Microfunctions and a transfer operator for complex dynamical systems.

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§1. Functions with regular singularities

Let us begin with the following situation. Let $R(z) = z^2 + c$ be a quadratic polynomial on the Riemann sphere. We assume that the complex dynamical system defined by this quadratic polynomial is postcritically finite, i.e., the forward orbit $\{f^n(0) \mid n = 0, 1, 2, \ldots\}$ of the critical point $0$ of $R(z)$ is a finite set. For the sake of simplicity, we denote by $F$ the Fatou set of $R(z)$ and by $J$ the Julia set $\mathbb{C} \setminus F$. In order to illustrate the situation, we consider especially the case $c = i$. Then, the critical point is $0$, and its forward orbit is $\{0, i, -1, -i\}$ and $R(0)$ and $R(i)$ form a periodic cycle of period $2$. $R(z)$ has two fixed points, which we denote by $\alpha$ and $\beta$ as in the picture. These two fixed points are so-called the $\alpha$-fixed point and the $\beta$-fixed point.
As $R(\beta)$ is postcritically finite, there exists an external ray, say $\gamma_c$, landing at the critical value $c$. We give an orientation to this curve as $\gamma_c \to c$. Note that this external ray is spiralling near $c$.

Let $\Omega_c = \mathbb{C} \setminus (\gamma_c \cup f(\gamma_c))$ be a domain in the complex plane. This domain $\Omega_c$ has smooth boundaries along the external ray $\gamma_c$. We consider an abstract "neighborhood" $\Omega^\pm$ of $\Omega_c$ doubly sheeted near $\gamma_c$.

The domain $\Omega_c$ has two smooth curves on the boundary. We add two curves $\gamma_c^+$ and $\gamma_c^-$ to this domain to each sides of the external ray $\gamma_c$, and we denote this set by $\overline{\Omega_c}$.

$\overline{\Omega_c}$ is an open set containing $\Omega_c$. For domain $\Omega$, we denote by $\mathcal{O}(\Omega)$ the set of holomorphic functions on $\Omega$.

Let $f: \overline{\Omega_c} \to \mathbb{C}$ be a holomorphic function on $\Omega_c$ which can be extended holomorphically to some "neighborhood" $\Omega^+$ of $\overline{\Omega_c}$. Such a function $f$ is said to be equivalent to $g: \overline{\Omega_c} \to \mathbb{C}$ which is a holomorphic function on $\Omega_c$ and extendable to some "neighborhood" $\Omega^-$ of $\overline{\Omega_c}$ holomorphically, if there exists a "neighborhood" $\Omega^\pm$ of $\Omega_c$ such that $f$ and $g$ coincide on $\Omega^\pm$. This equivalence relation defines a concept of germ. Note that $f: \Omega_c \to \mathbb{C}$ itself gives a representative of its germ since analytic continuation is unique if it exists. We call such function $f$ a (general)
premicrofunction along \( z \). A (general) microfunction at \( z \) along \( z \) is defined by an equivalence class of germs of (general) pre-microfunctions \( f : \Omega_c \to C \) at \( z \) modulo germs of holomorphic functions at \( c \).

More precisely, (general) pre-microfunctions \( f : \Omega_c \to C \) and \( g : \Omega_c \to C \) define the same microfunction at \( c \) if there exists an open neighborhood \( U \) of \( c \) in the complex plane \( C \) and a holomorphic function \( h : U \to C \) such that \( f(z) - g(z) = h(z) \) holds for \( z \in U \cap \Omega_c \).

The above definition of (general) microfunction is so general that the singularities of such functions at \( z \) are too much complicated. So, we restrict our singularities to "regular singularities" defined as follows.

**Definition 1.1.** A pre-microfunction \( f : \Omega_c \to C \) is said to have a regular singularity at \( c \) if there exist positive numbers \( \varepsilon \) and \( K \) such that inequality

\[
|f(z)| < K |z - c|^{-1+\varepsilon}
\]

holds near \( c \).

**Definition 1.2.** A pre-microfunction \( f : \Omega_c \to C \) is said to have a regular singularity at \( \infty \) if there exist positive numbers \( \varepsilon \) and \( K \) such that inequality

\[
|f(z)| < K |z|^{-1-\varepsilon}
\]

holds near the infinity.

We denote by \( \mathcal{M}_c \) the set of pre-microfunctions along \( z \) with regular singularities both at \( c \) and \( \infty \). More precisely, we denote \( \mathcal{M}_c \) instead of \( \mathcal{M}_c \) when there are more than one external rays landing at \( c \). The space of equivalence classes of germs of pre-microfunctions with regular singularities at \( z \) modulo the space of germs of holomorphic functions \( O(z) \) at \( z \), will be denoted by \( \mathcal{M}_c \).

Let \( P(\mathcal{R}) \) denote the postcritical set. For each point
\( p \in \mathbb{P}(\mathbb{R}) \), the space of pre-microfunctions along its external rays with regular singularities at both points is defined in a similar manner and will be denoted by \( \mathcal{M}_p \) for simplicity, and by \( \mathcal{M}_R \) when it is necessary to indicate the external ray.

For \( p \in \mathbb{F} \) with multiple external rays, say \( \mathcal{R}_1, \ldots, \mathcal{R}_r \), landing at \( p \), we define the space \( \mathcal{M}_p \) by the direct sum

\[
\mathcal{M}_p = \bigoplus_{k=1}^{r} \mathcal{M}_{\mathcal{R}_k}
\]

where the sum is taken as a formal sum, since each component belongs to different spaces. However, each element of \( \mathcal{M}_p \) defines a function holomorphic in the intersection of the domains of definitions and the decomposition of a holomorphic function

\[
f : \mathbb{C} \setminus \left( \bigcup_{k=1}^{r} \mathcal{R}_k \cup \{p\} \right) \to \mathbb{C}
\]

defined by an element of \( \mathcal{M}_p \) into components \( f_k \) in \( \mathcal{M}_{\mathcal{R}_k} \),

\[
f_k : \mathbb{C} \setminus \left( \mathcal{R}_k \cup \{p\} \right) \to \mathbb{C}
\]

is unique since we are considering the pre-microfunctions with regular singularities at the infinity. We denote

\[
\mathcal{M}_+ = \bigoplus_{p \in \mathbb{P}(\mathbb{R})} \mathcal{M}_p
\]

\[
\mathcal{M}_0 = \mathcal{M}_{\mathcal{R}_0}^+ \bigoplus \mathcal{M}_{\mathcal{R}_0}^-
\]

\[
\mathcal{M}_- = \bigoplus_{k=1}^{\infty} \bigoplus_{p \in \mathbb{P}^R(\mathcal{R}_k)} \mathcal{M}_p
\]

and

\[
\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_0 \oplus \mathcal{M}_-.
\]

Here, the origin \( 0 \) is the critical point of our quadratic map \( \mathbb{R}(z) \) and there are two external rays landing at \( 0 \), which are pre-images of the external ray \( \mathcal{R}_0 \). \( \mathcal{R}_0^+ \) and \( \mathcal{R}_0^- \) denote the external angles \( \frac{1}{12} \) and \( \frac{7}{12} \) respectively. Note that \( \mathcal{R}_0 \) is the external angle \( \frac{1}{6} \), since it is mapped to period two cycle of external rays with angles \( \frac{1}{3} \) and \( \frac{2}{3} \). Here, the infinite direct sum is only in a formal sense.
§ 2 Difference operator and an exact sequence

Let $O(\infty)$ denote the space of holomorphic functions in a neighborhood of the external ray $\infty$. An element of $O(\infty)$ is represented by a continuous function $f: \infty \to \mathbb{C}$ which can be extended to some neighborhood of $\infty$ holomorphically. The space of holomorphic functions along the external ray $\infty$ with regular singularities at both $\infty$ and the infinity is defined by the following.

$$O_0(\infty) = \{ f \in O(\infty) \mid \exists \varepsilon > 0, \exists k > 0, \exists \text{ nbd of } \infty \text{ s.t.}$$

$$\lim_{z \to \infty} |f(z) - k| z^{-1-\varepsilon} \text{ near } \infty$$

Now, we define a difference operator along an external ray.

**Definition 2.1** Difference operator $\Delta_c: M_c \to O_0(\infty)$ is defined by the difference of boundary values along $\Gamma_c$

$$\Delta_c(f) = f(z) - f((z - c) e^{-2\pi i} + c).$$

Here $z \in \Delta_c$ is considered a point in the boundary of $\Delta_c$ of the clockwise side, and $(z - c) e^{-2\pi i} + c$ represents the same point but considered as a point in the boundary of $\Delta_c$ of the counterclockwise side.

For each $p \in J$ and its external ray $\Gamma_p$, difference operator $\Delta_p: M_p \to O_0(\infty)$ is defined in a similar way. We denote $\Delta_p$ instead of $\Delta_p$ if there are more than two external rays landing at $p$ and we need to indicate it.

**Remark** Difference operator can be defined for functions holomorphic along $\infty$ in a doubly sheeted domains. The domain of definition of such a holomorphic function need not be connected.

Let us fix a double sheeted neighborhood $\Delta_c$ of our domain $\Delta_c = \mathbb{C} \setminus (\infty \cup \infty)$, and let $\Delta_c$ denote the neighborhood of $\infty$ where $\Delta_c$ is double sheeted.
Theorem 2.2 The following sequence is exact:

\[ 0 \rightarrow O(C \setminus \{c\}) \rightarrow O(D) \quad \Delta_c \rightarrow O(S_c) \rightarrow 0. \]

Proof. We gave an orientation to the external ray \( S_c \) defining an order to the points in \( S_c \) so that \( \infty < p < c \). Take points \( S_0, S_1, \ldots, S_d \in S_c \) for \( d \in \mathbb{Z} \) ordered along \( S_c \) as

\[ \infty < \ldots < S_0 < S_1 < S_2 < \ldots < \infty. \]

and

\[ \lim_{k \to +\infty} S_k = \lim_{k \to -\infty} R_k = \infty, \]

\[ \lim_{k \to +\infty} S_k = \lim_{k \to -\infty} R_k = c. \]

Then open arcs \( \overline{S_j S_{j+1}} \) (\( j \in \mathbb{Z} \)) form an open covering of the external ray \( S_c \). The space \( O(C \setminus \{c\}) \) of holomorphic functions on \( \mathbb{C} \setminus \{c\} \) can be injectively embedded in the space of holomorphic functions on \( S_c \) and the values of such a function on the two sheets of \( S_c \) coincide on the overlapped sector \( S_c \), hence the difference of these values vanishes. So, we need only to prove the ontoeness of the difference operator \( \Delta_c \). For \( \varphi \in O(S_c) \), we want to construct a holomorphic function in \( O(C \setminus \{c\}) \). Note that such a function is not unique since the kernel of \( \Delta_c \) contains \( O(C \setminus \{c\}) \).

Let

\[ F_j(z) = \frac{1}{2\pi i} \int_{S_{j+1}}^{S_j} \frac{\varphi(t)}{t - z} \, dt \]

for \( j \in \mathbb{Z} \). Such integration is called a Cousin's integral along the arc \( \overline{S_j S_{j+1}} \). Note that this arc includes the arc \( \overline{S_j S_{j+1}} \) in its interior. The function \( F_j(z) \) is holomorphic in \( \mathbb{C} \setminus \bigcup_{j \in \mathbb{Z}} \{S_j, S_{j+1}\} \).

By deforming the path of integration of the Cousin's integral we see that \( F_j(z) \) can be holomorphically extended beyond the arc from both sides into the other sides, except
at \( R_{g+1} \) and \( S_{g+1} \). Next let us take a family of annuli in a \( \mathcal{Y} \) separating \( c \) and \( \infty \) with smooth boundaries as follows. We take annulus \( B_{g} \) for each \( j \in \mathbb{Z} \) so that the intersection of \( B_{g} \) with the external ray \( y_{c} \) is the arc \( R_{g} \), and \( S_{g} \) belong to the outer and inner boundary of \( B_{g} \) respectively. Furthermore, for \( j, k \in \mathbb{Z} \), \( B_{j} \cap B_{k} \) is empty if \( |j - k| > 1 \) hold, and for each \( j \in \mathbb{Z} \), \( B_{j} \cap B_{j+1} \) is an annulus. We impose that

\[
\bigcup_{j \in \mathbb{Z}} B_{j} = \mathcal{Y} \setminus \{c\}
\]

For each \( j \), we denote by \( \tilde{B}_{j} \) a covering of \( B_{j} \) such that \( \tilde{B}_{j} \) covers twice on the sector \( S_{c} \cap B_{j} \). An function \( F_{j}(z) \) defined by Cousin's integration can be extended holomorphically to \( \tilde{B}_{j} \). It is further extendable to a wider domain \( \tilde{B}_{g} \cup \tilde{B}_{g} \cup \tilde{B}_{g+1} \). Hence \( F_{j}(z) \) is bounded in \( \tilde{B}_{j} \). As is easily verified by considering the integration, we have

\[
F_{j}(z) = F_{j}(z - c) e^{-2\pi i} + c)
\]

for \( z \in S_{c} \cap \tilde{B}_{j} \).

For \( j, k \in \mathbb{Z} \) with \( B_{j} \cap B_{k} = \emptyset \), define a holomorphic function

\[
H_{jk} : B_{j} \cap B_{k} \to \mathbb{C}
\]

by

\[
H_{jk}(z) = F_{j}(z) - F_{k}(z).
\]

\( F_{j}(z) \) and \( F_{k}(z) \) are holomorphic on \( \tilde{B}_{j} \cap \tilde{B}_{k} \). But, as we have

\[
F_{j}(z - c) e^{-2\pi i} + c) - F_{k}(z - c) e^{-2\pi i} + c)
\]

\[= F_{j}(z) - F_{k}(z)\]
along $\gamma_c$, $H_{\beta_k}((z - c)e^{2\pi i} + c) = H_{\beta_k}(z)$ holds on $S_c \cap B_{2r} \cap B_0$, so that $H_{\beta_k}(z)$ is well defined and holomorphic on the annulus $B_{2r} \cap B_0$. This family of holomorphic functions $\{H_{\beta_k}\}$ forms a "Cousin data" i.e. for $i, j, k \in \mathbb{B}$,

$$H_j + H_{j+1} + H_{j+2} = 0 \text{ on } B_j \cap B_{j+1} \cap B_k.$$

As we assumed $B_j \cap B_{j+1} = \emptyset$ if $|y - x| > 1$, this above fact is easily verified.

For each $j \in \mathbb{Z}$, take a loop $\gamma_j$ in $B_j \cap B_{j+1}$, making a clockwise turn once and define $h_j(z)$ and $\rho_j(z)$ by

$$h_j(z) = \frac{1}{2\pi i} \int_{\gamma_j} \frac{H_{i+1}(t)}{z - t} \, dt$$

defined and holomorphic in $\bigcup_{k=-\infty} u B_k \cup \mathbf{f}_c$ (outside of the annulus), and

$$\rho_j(z) = \frac{1}{2\pi i} \int_{\gamma_j} \frac{H_{i+1}(t)}{z - t} \, dt$$

declare the holomorphic in $\bigcup_{k=0} u B_k \cup \mathbf{f}_c$ (inside of the annulus).

By deforming the integration path we see that they are well defined and we have

$$h_j(z) = \rho_j(z) - \rho_{j+1}(z) \quad \text{for } z \in B_j \cap B_{j+1}.$$

By Runge's theorem, $\rho_j : \bigcup_{k=-\infty} u B_k \cup \mathbf{f}_c \to \mathbb{C}$ can be approximated by polynomials in the sense of uniform convergence on compact sets, and $h_j : \bigcup_{k=0} u B_k \cup \mathbf{f}_c$ can be approximated by rational functions with poles only at $c$.

For each $j > 0$, find a rational function $g_j : \mathbb{C} \setminus \{f_c\} \to \mathbb{C}$ such that

$$|g_j(z) - h_j(z)| < \frac{1}{2^j}$$

for $z \in \bigcup_{k=-\infty} u B_k \cup \mathbf{f}_c$.

And for each $j < 0$, find a polynomial $g_j : \mathbb{C} \to \mathbb{C}$ such that

$$|g_j(z) - h_j(z)| < \frac{1}{2^j}$$

for $z \in \bigcup_{k=0} u B_k \cup \mathbf{f}_c$.

Note that these functions $g_j$ are all holomorphic in $\mathbb{C} \setminus \{f_c\}$. 

Let \( \overline{F}_d = h_d - g_d : \bigcup_{i=0}^{\infty} B_{2^i} U 2^{i+1} \rightarrow C \)

for \( d \geq 0 \), and \( \overline{F}_{d+1} = h_{d+1} - g_{d+1} : \bigcup_{i=0}^{\infty} B_{2^i} U 2^{i+1} \rightarrow C \)

for \( d < 0 \). Then we have

\[
|\overline{F}_d| < \frac{1}{2^d} \quad (d \geq 0)
\]

and

\[
|\overline{F}_{d+1}| < \frac{1}{2^{d+1}} \quad (d < 0).
\]

We still have

\[
H_d(z) = \overline{F}_d(z) - \overline{F}_{d+1}(z) \quad \text{for } z \in B_d \cap B_{d+1}.
\]

Now, we set

\[
H_d(z) = -\sum_{i=0}^{d} \overline{F}_i(z) - \sum_{i=d+1}^{\infty} \overline{F}_i(z).
\]

For \( i \leq d \), \( \overline{F}_i(z) \) is holomorphic in \( \bigcup_{i=0}^{\infty} B_{2^i} U 2^{i+1} \), hence they are all holomorphic in the smallest disk \( \bigcup_{i=d}^{\infty} B_{2^i} U 2^{i+1} \) and that we have the estimate of the supremum of the functions, the sum of \( \overline{F}_i \)'s is uniformly convergent on \( B_d \).

Similarly, the sum of \( \overline{F}_i \)'s converge uniformly convergent on \( B_{d+1} \), too. Hence \( H_d(z) \) is holomorphic in \( B_d \).

In the overlapping annulus \( B_d \cap B_{d+1} \), we have

\[
H_{d+1} - H_d = -\sum_{i=d+1}^{\infty} \overline{F}_i + \overline{F}_{d+1} = H_{d+1}.
\]

Finally, in \( B_d \), let \( G_d(z) = H_d(z) + F_d(z) \). These functions \( G_d \)'s on \( B_d \) defines a holomorphic function

\[
G_d : B_d \rightarrow C
\]

define on the overlapped neighborhood \( B_d \) of \( B_c \). We can verify that these functions coincide and \( G \) is well defined by an immediate calculations as follows.

In \( B_d \cap B_{d+1} \),

\[
G_{d+1}(z) = H_{d+1}(z) + F_{d+1}(z) = H_d(z) + H_{d+1}(z) + F_{d+1}(z)
\]

\[
= H_d(z) + F_d(z) + F_{d+1}(z) = H_d(z) + F_d(z) = G_d(z).
\]
Thus, we conclude that $G \in \mathcal{O}(\mathbb{C})$ and

$$\Delta_c G = \Phi$$

holds. This completes the proof of the our Theorem 2.2.

We remark that such a function $G$ satisfying $\Delta_c G = \Phi$ is not unique since $\ker \Delta_c = \mathcal{O}(\mathbb{C}) \setminus \{0\}$.

§3. Cousin's integral operator and decomposition of pre-microfunctions.

In the previous section, we discussed the surjectivity of the difference operator $\Delta_c$. In this section, we restrict the space of (general) pre-microfunctions to the space of pre-microfunctions with regular singularities, and consider an inverse operator of $\Delta_c$, which we call a Cousin's integral operator.

**Definition 3.1**

$I_c : \mathcal{O}_0(\mathbb{C}_c) \rightarrow \mathcal{M}_c$ is defined by

$$I_c[\Phi](z) = \frac{1}{2\pi i} \int_{\Gamma_c} \frac{\Phi(t)}{t-z} \, dt$$

for $\Phi \in \mathcal{O}_0(\mathbb{D}_c)$.

Here, we use notation $I_c[\Phi]$ as $I_c : \mathcal{O}_0(\mathbb{C}_c) \rightarrow \mathcal{M}_c$ is an operator and we want to emphasize it, i.e. the argument of the operator is a function and not its value.

**Definition 3.2**

Let $f$ be a bi-valued function defined in a neighborhood of $\mathbb{C}_c$, both of the two branches are holomorphic and the difference $\Delta_c f$ of $f$ has regular singularities at $c$ and at $\infty$, i.e. $\Delta_c f \in \mathcal{O}_0(\mathbb{C}_c)$. The $\mathcal{M}_c$ component of $f$ is defined as

$$[f]_c = [f]_{\mathbb{C}_c} = I_c[\Delta_c f]$$

This mapping $[\ ]_c$ is a projection map onto $\mathcal{M}_c$. 
We have the following identities.

**Theorem 3.3**
\[ T_c \circ \Delta_c = \text{id} \quad \text{on} \quad M_c, \]
\[ \Delta_c \circ T_c = \text{id} \quad \text{on} \quad \mathcal{O}_c(\mathcal{V}_c). \]

**Proof.** These identities are easily verified.

For each point \( p \in \mathcal{I} \) (and an external ray \( Y_p \) landing at \( p \)), projection \( \mathcal{V} \cap \mathcal{I}_p \) is similarly defined.

Let \( \mathcal{O}_c(\mathcal{V}_c) \) denote the space of holomorphic functions \( f: \mathcal{V}_c \rightarrow \mathbb{C} \) such that \( f \) can be extended holomorphically to some double-sheeted neighborhood \( \mathcal{V}_c \) and satisfies \( \Delta f \in \mathcal{O}_c(\mathcal{V}_c) \). Function \( f \in \mathcal{O}_c(\mathcal{V}_c) \) is holomorphic in \( \mathcal{V}_c = \mathcal{C} \setminus \{ \mathcal{V}_c \cup \mathcal{I}_c \} \) and has singularities at \( c \) and at the infinity together with its difference along \( \mathcal{V}_c \).

Let \( \mathcal{H}_c \) denote the space of hyperfunctions supported at \( c \), i.e., \( \psi \in \mathcal{H}_c \), if and only if \( \psi \) is holomorphic in \( \mathcal{C} \setminus \{ \mathcal{V}_c \cup \mathcal{I}_c \} \). The space of entire functions is denoted by \( \mathcal{O}(\mathcal{C}) \). Let us define the operators that extract singularities of \( f \).

**Definition 3.4.** Operator \( \Gamma_c : \mathcal{O}_c(\mathcal{V}_c) \rightarrow \mathcal{H}_c \) is defined by
\[
\Gamma_c[f](\zeta) = \frac{1}{2\pi i} \int_{\mathcal{C} \setminus \{ \mathcal{V}_c \cup \mathcal{I}_c \}} \frac{f(t)}{t-\zeta} \, dt
\]
\[
- \frac{1}{2\pi i} \int_{\mathcal{V}_c} \frac{\Delta f(t)}{t-\zeta} \, dt
\]
where \( \zeta \in \mathcal{C} \setminus \{ \mathcal{V}_c \cup \mathcal{I}_c \} \), \( \varepsilon > 0 \) is chosen sufficiently small so that the \( \varepsilon \)-ball around \( c \) does not contain \( \zeta \), and \( \mathcal{V}_c \cap \mathcal{I}_c \) is the intersection point of \( \mathcal{V}_c \) and the circle \( |z-c| = \varepsilon \). The orientation of the path of integration along the circle is the counter-clockwise with respect to \( \zeta \).
As \( \Delta f \) has a regular singularity at \( c \), this defines a holomorphic function on \( \left( \mathbb{C} \setminus \{0\} \right) \setminus \{c\} \). That is, \( \Gamma_c[f] \in \mathcal{H}_c \).

**Definition 3.5.** Operator \( \Gamma_{\infty} : \mathcal{O}_0(\Omega_c) \to \mathcal{O}(\mathbb{C}) \) is defined by

\[
\Gamma_{\infty}[f](z) = \frac{1}{2\pi i} \int_{|t-c|=\infty} \frac{f(t)}{t-z} \, dt + \frac{1}{2\pi i} \int_{R_0} \frac{G_c(f)(t)}{t-z} \, dt
\]

where \( w > 0 \) is taken sufficiently large for each \( z \), so that the circle of integration path surrounds \( z \), and \( R_0 \in \mathbb{R} \) is the intersection point of \( R_0 \) and the big circle. The orientation is taken as the counterclockwise with respect to \( z \).

As \( \Delta f \) has a regular singularity at \( \infty \), this defines an entire function. Hence \( \Gamma_{\infty}[f] \in \mathcal{O}(\mathbb{C}) \).

Just for the sake of consistency of notation, we define

\[ \Gamma_c : \mathcal{O}_0(\Omega_c) \to \mathcal{H}_c \]

by \( \Gamma_c[f] = [f]_c \). We have the following decomposition.

**Theorem 3.6.** \( \mathcal{O}_0(\Omega_c) = \mathcal{H}_c \oplus \mathcal{M}_c \oplus \mathcal{O}(\mathbb{C}) \)

and \( \Gamma_c : \mathcal{O}_0(\Omega_c) \to \mathcal{H}_c \), \( \Gamma_m : \mathcal{O}_0(\Omega_c) \to \mathcal{M}_c \)

\( \Gamma_{\infty} : \mathcal{O}_0(\Omega_c) \to \mathcal{O}(\mathbb{C}) \)

gives the projections to components.

**Proof.** Clearly, the kernel of the difference operator \( \Delta_c \) is \( \mathcal{O}(\mathbb{C} \setminus \{c\}) \) and \( \mathcal{O}(\mathbb{C} \setminus \{c\}) = \mathcal{H}_c \oplus \mathcal{O}(\mathbb{C}) \).

Note that these operators can be defined if \( f \) is defined and holomorphic in a double covered neighborhood of \( \Omega_c \). In this case, \( \Gamma_c + \Gamma_m + \Gamma_{\infty} \) defines a projection to \( \mathcal{O}_0(\Omega_c) = \mathcal{H}_c \oplus \mathcal{M}_c \oplus \mathcal{O}(\mathbb{C}) \) if \( \Delta_c f \in \mathcal{O}_0(\Omega_c) \).
§ 4 Space of pre-microfunctions and a transfer operator

Let us go back to our complex dynamical system $R(z)$. Let $J$ denote the Julia set of $R(z)$ and let $F$ denote the Fatou set of $R(z)$. We suppose $R(z)$ is postcritically finite, especially the case of $c = i$. We denote by $O(J)$ the space of germ of continuous functions $f : J \to \mathbb{C}$ which are holomorphic in some neighborhood of $J$. The space of holomorphic functions $f : F \to \mathbb{C}$ of the Fatou set satisfying $f(0) = 0$ will be denoted by $O_0(F)$. The postcritical set of $R(z)$ is denoted by $P(R)$. The space of pre-microfunctions at $P(R)$ is defined by

\[ M_{P(R)} = \bigoplus_{p \in P(R)} N_p, \]

and

\[ M = \bigoplus_{p \in P(R)} N_p \oplus \bigoplus_{k=0}^{\infty} M_{p_k}, \]

denotes the space of formal sum of pre-microfunctions at the graded orbit of the critical point 0.

Let $f \in O(J) \oplus M_{P(R)} \oplus O_0(F)$ and $p \in P(R)$ with $R_p$ its external ray. Then the $N_p$-component $[f]_p$ of $f$ is given by a projection

\[ [f]_p(z) = \frac{1}{2\pi i} \int_{R_p} \frac{A_{f}(t)}{t-z} \, dt. \]

Let us consider the most simple postcritically finite case (except $c = -2$ case) of $R(z) = z^2 + i$. Fixed points of $R$ are denoted by $d$ and $B$. The preimage of the external ray $R$ consists of two external rays, say $R_0^+$ and $R_0^-$, of the critical point 0, with external angles $\frac{1}{2}$ and $\frac{7}{12}$ respectively. These external rays are oriented as $\infty \to 0$. Let $D_0$ denote the upper connected component of $\mathbb{C} \setminus (\mathbb{C} \cup \mathbb{C}^\infty)$ which contains the critical value $c = i$. The $\alpha$-fixed point belongs to this domain. We denote the other connected point by $T_B$. It contains the $B$-fixed point. The quadratic map $R$ is of degree two. The critical value $c$ is a branch point. We denote the two branches of $R^{-1}$ by $Q_0$ and $Q_B$ defined in $R_c = \mathbb{C} \setminus \{ \infty, 0, i \}$. 
Let \( \varphi_{\alpha}, \varphi_{\beta} : C \setminus \left( \gamma_{c} \cup \gamma_{p} \right) \rightarrow U_{\alpha}, U_{\beta} \) with \( \varphi_{\alpha}(z) = -\sqrt{z-c} \), \( \varphi_{\beta}(z) = \sqrt{z-c} \), where the branch of the square root is chosen by assigning \( \varphi_{p}(c+1) = 1 \).

If we regard \( \varphi_{\alpha} \) and \( \varphi_{\beta} \) as holomorphic functions on \( \Omega_{c} \), we can naturally consider holomorphic functions \( (\varphi_{\alpha}^{s})^{\overline{s}} \) and \( (\varphi_{\beta}^{s})^{\overline{s}} \) for \( 0 < s < 2 \). They can be extended to a double-sheeted neighborhood \( \Omega_{2i} \), holomorphically. We define a holomorphic function

\[
\gamma_{s}(z) = \frac{1}{(2 \varphi_{\beta}(z))^{s}}
\]
defined in \( \Omega_{c} \).

For \( z = c + r e^{i\theta} \in \Omega_{c} \), we have

\[
(\Delta_{c} \gamma_{s})(z) = \gamma_{s}(z) - \gamma_{s}(z - c) e^{-2i\theta} + c)
= \frac{1 - e^{2s\theta}}{(2 \sqrt{r} e^{i\theta})^{s}}
\]

Hence \( |\Delta_{c} \gamma_{s}| \leq \text{const.} \cdot r^{-\frac{s}{2}} \)

which implies \( \Delta_{c} \gamma_{s} \) has regular singularities at \( c \) and \( \infty \) if \( 0 < s < 2 \). Therefore \( \gamma_{s} \in M_{c} \).

Now, take a function \( f \in C(c) \oplus M_{E}(R) \oplus C_{0}(F) \).

Here, we abuse the formal sum of function in different spaces and the sum as a function defined in the common domain of definition. So, \( f \) is defined and holomorphic in \( F \setminus \left( \cup_{p \in P(R)} \gamma_{p} \right) \). For an external ray \( \gamma_{p} \), we denote by
The path of integration coming from $\infty$ to $p$ along $\gamma_p$ taking the clockwise side value of the integrand function and going back from $p$ to $\infty$ along $\gamma_p$ taking the counterclockwise side value of the integrand function. That is,

$$\int_{\gamma_p} f(z) \, dz = \int_{\gamma_p} f(z) \, dz - \int_{\gamma_p} f((z-p)e^{2\pi i} + p) \, dz$$

$$= \int_{\gamma_p} \partial_{\gamma_p} f(z) \, dz.$$

By $\gamma_f$ we represent an integration path along the boundary of the Fatou set, passing near the Julia set and taking values of the function on the Fatou set. And finally by $\gamma_J$ we represent an integration path running around the Julia set in the counterclockwise direction.

Let $\mathcal{H}_+ = \mathcal{O}(\mathbb{C}) \oplus \mathcal{M}(\mathbb{R}) \oplus \mathcal{C}(\mathbb{R})$.

Definition 4.1 Transfer operator $L_s : \mathcal{H}_+ \rightarrow \mathcal{H}_+$ is defined for $0 < s < 2$ and for $t \in \mathcal{H}_+$ by

$$(L_s f)(z) = \sum_{y \in \mathcal{R}(z)} \gamma_s(R(y)) f(y).$$

We can rewrite the transfer operator in an integral operator form as

$$(L_s f)(z) = \frac{1}{2\pi i} \int_{\gamma_f + \gamma_R + \gamma_J} \frac{\gamma_s(R(t)) R(t) f(t)}{R(t) - z} \, dt$$

and as

$$(L_s f)(z) = \gamma_s(z) (f \circ Q_\omega(z) + f \circ Q_\theta(z)).$$

Definition 4.2 Push forward operator $R_x$ is defined by

$$R_x f = f \circ Q_\omega + f \circ Q_\theta.$$
For point $\gamma \in \Gamma$, we denote by $\chi_\gamma(z) = \frac{1}{z-\gamma}$ the unit pole at $\gamma$. If $\gamma \in \Gamma$, then $\chi_\gamma \in \mathcal{O}(\mathbb{F})$. If $\gamma \in \Gamma_p$, then $\chi_\gamma \in \mathcal{O}(\mathbb{F}_p)$. Note that $\chi_\gamma(\gamma) = -\chi_\gamma(\gamma)$.

**Proposition 4.3** For $\gamma \in \Gamma \cup \cup_{p \in \mathbb{N}} \Gamma_p$,

$$L_\gamma \chi_\gamma = R(\gamma) \chi_\gamma (R(\gamma)) X_{\chi_\gamma} + R(\gamma) \left[ \chi_\gamma \cdot X_{\chi_\gamma} \right]_c.$$  

**Proof.** By a direct computation, we have

$$R_\gamma \chi_\gamma(z) = \chi_\gamma(z) \left( \chi_\gamma \circ Q_0(z) + \chi_\gamma \circ Q_1(z) \right) = \chi_\gamma(z) \sum_{\delta \in \Gamma \cup \Gamma_p} (-1)^\delta \chi_\gamma(\gamma)$$

$$= \chi_\gamma(z) \sum_{\delta \in \Gamma \cup \Gamma_p} \frac{1}{z - \delta - \gamma}.$$  

By the residue formula, we have

$$0 = \frac{1}{2\pi i} \int_{\gamma_j} \frac{R(\gamma)}{(z-R(\gamma))(y-\gamma)} \, dz = \sum_{\delta \in \Gamma \cup \Gamma_p} \frac{1}{y - \delta - \gamma} + \frac{R(\gamma)}{z - R(\gamma)}.$$  

Hence

$$(L_\gamma \chi_\gamma)(z) = \chi_\gamma(z) \frac{R(\gamma)}{z - R(\gamma)} = \chi_\gamma(z) R(\gamma) X_{\chi_\gamma}(z)$$

$$= \chi_\gamma(R(\gamma)) R(\gamma) X_{\chi_\gamma}(z) + \left[ R(\gamma) \cdot \chi_\gamma \cdot X_{\chi_\gamma} \right]_c.$$  

Here $\left[ R(\gamma) \cdot \chi_\gamma \cdot X_{\chi_\gamma} \right]_c(z) = (\chi_\gamma(z) - \chi_\gamma(R(\gamma)) R(\gamma)) X_{\chi_\gamma}(z)$ is holomorphic near $R(\gamma)$ so that it belongs to $\mathcal{MC}$. The first term $\chi_\gamma(R(\gamma)) R(\gamma) X_{\chi_\gamma}(z)$ is a multiple of unit pole at $R(\gamma)$.

Unit poles $\chi_\gamma \gamma \in \Gamma \Gamma_p$ form a basis of function space $\mathcal{O}(\mathbb{F})$ and the family of unit poles $\chi_\gamma \gamma \gamma \in \Gamma_p$ form a basis of space of pre-microfunctions $\mathcal{M}$, $\mathcal{M}_p$. This splitting of the transfer operator $L_\gamma$ gives a decomposition of the operator into components of the direct sum decomposition of function spaces.

**Theorem 4.4** For $\gamma \in \Gamma \Gamma_p$ and $g \in \mathcal{M}_p$ we have the following decomposition

$$L_\gamma g = \left[ g \circ Q_0 \cdot \chi_\gamma \right]_c(R(\gamma)) + \left[ \chi_\gamma \cdot [g \circ Q_p]_c(R(\gamma)) \right]_c$$

where $Q_p = Q_0$ or $Q_p$ according to $p \in \Gamma_0$ or $\Gamma_p$. 

As \( g \in M_p \), \( g(z) \) can be expressed in an integration of Cauchy type for \( z \in \Omega_p = \mathbb{C} \setminus (\delta_p \cup i \delta_p) \) as
\[
\mathcal{L}_S g(z) = \frac{1}{2\pi i} \int_{\partial P} (A_p g)(t) \frac{dt}{t-z} = \frac{1}{2\pi i} \int_{\partial P} \mathcal{L}_S g(t)(-\mathcal{L}_S^\ast (z)) dt.
\]
Hence, we have
\[
\mathcal{L}_S g(z) = \frac{-1}{2\pi i} \int_{\partial P} (\Delta_p g)(\eta) (\mathcal{L}_S \mathcal{K}_t)(\eta)(\mathcal{L}_S \mathcal{K}_P)(\eta) d\eta.
\]
\[
= \frac{1}{2\pi i} \int_{\partial P} (\Delta_p g)(\eta) (\mathcal{K}_t(\delta_p)(\eta)(\mathcal{K}_P(\eta)) \mathcal{K}_R(\eta)) d\eta - \frac{1}{2\pi i} \int_{\partial P} (\Delta_p g)(\eta) (\mathcal{K}_t(\delta_p)(\eta)(\mathcal{K}_R(\eta)) d\eta.
\]
\[
= \frac{1}{2\pi i} \int_{\partial P} (\Delta_R \mathcal{K}_P)(\eta) (\mathcal{K}_t(\delta_p)(\eta)(\mathcal{K}_R(\eta)) d\eta + (\mathcal{K}_t(\delta_p)(\eta)(\mathcal{K}_R(\eta)) d\eta.
\]
Note that in the above calculations, \( \eta \in \delta_p \) is a variable along \( \delta_p \) and we changed variables by \( s = R(\eta) \) and \( d\xi = R^\ast(\eta) d\eta \).

This theorem shows that the transfer operator \( \mathcal{L}_S \) sends \( M_p \) into \( M_{R(p)} \oplus M_c \) for \( p \in J \). In the case of postcritically finite complex dynamical system case, the space of pre-microfunctions for postcritical set is invariant for \( \mathcal{L}_S \), i.e.

**Proposition.** \( \mathcal{L}_S (M_p) \subset M_p \).

§5. Decomposition of the transfer operator

Our space of pre-microfunctions \( H_p \) has a direct sum decomposition
\[
H_p = (\mathcal{O}(C) \oplus M_p) \oplus \mathcal{O}(F)
\]
and the transfer operator \( \mathcal{L}_S \) maps this space into itself. We denote the components of \( \mathcal{L}_S \) in a matrix form as
\[
\mathcal{L}_S = \begin{pmatrix}
\mathcal{L}_{ST} & \mathcal{L}_{SM} & \mathcal{L}_{SF} \\
\mathcal{L}_{MT} & \mathcal{L}_{MN} & \mathcal{L}_{MF} \\
\mathcal{L}_{PT} & \mathcal{L}_{PM} & \mathcal{L}_{PF}
\end{pmatrix}.
\]
For $f_n \in V_2(I)$ we have the following proposition.

**Proposition 5.1**
\[
\mathcal{L}_n f_n = \mathcal{L}_n f_n = \mathcal{L}_n f_n = \mathcal{L}_n f_n = \mathcal{L}_n f_n.
\]

with
\[
\mathcal{L}_n f_n = \mathcal{L}_n f_n = \mathcal{L}_n f_n = \mathcal{L}_n f_n = \mathcal{L}_n f_n.
\]

\[
\mathcal{L}_n f_n = \mathcal{L}_n f_n = \mathcal{L}_n f_n = \mathcal{L}_n f_n = \mathcal{L}_n f_n.
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\]

\[
\mathcal{L}_n f_n = \mathcal{L}_n f_n = \mathcal{L}_n f_n = \mathcal{L}_n f_n = \mathcal{L}_n f_n.
\]

**Proof.** As $0 < s < 2$, we have
\[
\lim_{t \to 0} \frac{1}{2\pi i} \oint_{C_p} \frac{\psi_k(R(t)) \cdot R(t) \cdot \theta(t)}{R(t) - z} \, dt = 0,
\]
where $z = c$ and $C_p$ is the integration path $C_p$ is the circle of radius $e around the origin point. Since the singularity at the origin is of order $1 - s$. In the next calculation, integration paths are as explained in the previous section. We have
\[
(\mathcal{L}_n f_n)(z) = \frac{1}{2\pi i} \oint_{C_p} \frac{\psi_k(R(t)) \cdot R(t) \cdot \theta(t)}{R(t) - z} \, dt = \frac{1}{2\pi i} \oint_{C_p} \frac{\psi_k(R(t)) \cdot R(t) \cdot \theta(t)}{R(t) - z} \, dt
\]
\[
= \psi_k(z) \sum \limits_{y \in \gamma_c} \overline{f_n(y)} - \frac{1}{2\pi i} \oint_{C_p} \frac{\psi_k(R(t)) \cdot R(t) \cdot \theta(t)}{R(t) - z} \, dt
\]
\[
= \psi_k(z) \sum \limits_{y \in \gamma_c} \overline{f_n(y)} + \frac{1}{2\pi i} \oint_{C_p} \frac{\psi_k(R(t)) \cdot R(t) \cdot \theta(t)}{R(t) - z} \, dt
\]
\[
= \psi_k(z) \cdot R_n f_n(z) - \frac{1}{2\pi i} \oint_{C_p} \frac{\psi_k(R(t)) \cdot R(t) \cdot \theta(t)}{R(t) - z} \, dt
\]

Here we made a change of variables $x = R(t)$ and $dt = R(t) \, dx.$ $Q^+: \gamma_c \to \gamma_c^+$ and $Q^- : \gamma_c \to \gamma_c^-$ denotes the inverse branches of $R$ along $\gamma_c$. $R_n f_n = f_n \circ Q^+ + f_n \circ Q^-$ holds along $\gamma_c$ and $R_n f_n = f_n \circ Q^-$. We continue the calculation.

\[
(\mathcal{L}_n f_n)(z) = \psi_k(z) \cdot R_n f_n(z) - \frac{1}{2\pi i} \oint_{C_p} \frac{\psi_k(R(t)) \cdot R_n f_n}{R(t) - z} \, dt
\]

This completes the first line. Note that $\mathcal{L}_n f_n = \psi_k \cdot R_n f_n$ has singularities along $\gamma_c$. Next, compute the component $\mathcal{L}_n f_n$ as follows.
\( (L_{MJ} f_J)(z) = \frac{1}{2\pi i} \int_{\gamma_M} \frac{\gamma_s(R(t)) R(t) f_J(t)}{R(t) - z} \, dt \)

\[ = \left[ \gamma_s \cdot R_x f_J \right] c \]

and
\[ (L_{FJ} f_J)(z) = \frac{1}{2\pi i} \int_{\gamma_F} \frac{\gamma_s(R(t)) R(t) f_J(t)}{R(t) - z} \, dt \]

\[ = \frac{1}{2\pi i} \int_{|\sigma| = |c|} \frac{\gamma_s(\sigma)(R_x f_J)(\sigma)}{\sigma - z} \, d\sigma \]

\[ = 0 \]

Since
\[ | L_{FJ} f_J | \leq \frac{2\pi p}{2\pi} \frac{|2z|^{1-s} | f_x(0) + f(0) \rho |}{|z - c| - p} \]

\[ \to 0 \quad \text{(as } p \to 0) \]

For the second column of the operator matrix, we have the following.

**Proposition 5.2.** For \( f_M \in M_p \subset M_+(p \neq 0) \),

\[ L_{JM} f_M = 0 \]

\[ L_{MM} f_M = \left[ \gamma_s \cdot R_x f_M \right] c + \left[ \gamma_s \cdot f_M \cdot Q \rho \right] R(p) \]

\[ L_{FM} f_M = 0 \]

**Remark.** \( L_{F} f_M \in M_c \oplus M_{R(p)} \) if \( f_M \in M_p \)

**Proof.** As \( f_M \in M_p \), there exists a positive number \( \varepsilon \) and a positive constant \( K \) such that

\[ |f_M(z)| < K |z|^{-\varepsilon} \]

holds near the infinity. Hence we have

\[ |L_{JM} f_M(z)| \leq \frac{1}{2\pi} \int_{\gamma_J} \frac{|\gamma_s(R(t)) R(t) f_M(t)|}{|R(t) - z|} \, dt \]

\[ \leq \frac{2\pi |\gamma|}{2\pi} \frac{|2z|^{1-s} | f_x(0) + f(0) \rho |}{|z - c| - p} \cdot |z|^{-\varepsilon} \]

\[ \leq \text{Const.} \cdot |z|^{-s-\varepsilon} \to 0 \quad (|t| \to \infty) \]

and \( L_{JM} f_M = 0 \).

Next, we show that \( L_{FM} = 0 \) in the following.
\[(L_{FM} f_M)(z) = \frac{1}{2\pi i} \int_{|z| = \rho} \frac{\gamma_s(R(\tau)) \frac{f_M(\tau)}{R(\tau) - z}}{R(\tau) - z} d\tau\]
\[= \frac{1}{2\pi i} \left( \int_{|z| = \rho} + \sum_{|\tau| = \rho} \right) \frac{\gamma_s(R(\tau)) \frac{f_M(\tau)}{R(\tau) - z}}{R(\tau) - z} d\tau.
\]

This goes to zero as \(\rho \to 0\).

For \(\gamma_s\) belongs to \(M_0\) so that the second term vanishes and the first term vanishes since \(f_M\) belongs to \(M_0\). Note that in this computation, \(z\) is taken from the Fatou set and the integration path \(\gamma_F\) runs near the Julia set. Note that this argument cannot be applied if \(p = 0\) since the integrand might have a singularity at \(p\). which is not regular. We need regularity of the singular points to have this kind of integral vanish. Finally,
\[(L_{MN} f_M)(z) = \frac{1}{2\pi i} \int_{|z| = \rho} \frac{\gamma_s(R(\tau)) \frac{R(\tau)}{R(\tau) - z}}{R(\tau) - z} d\tau + \frac{1}{2\pi i} \int_{|\tau| = \rho} \frac{\gamma_s(R(\tau)) \frac{f_M(\tau)}{R(\tau) - z}}{R(\tau) - z} d\tau\]
\[= \mathcal{I} [\gamma_s \cdot R_y f_M] + \mathcal{I}_{R_0} [\gamma_s \cdot (4_p f_M) \cdot Q] \]
\[= [\gamma_s \cdot R_y f_M]_C + [\gamma_s \cdot f_M \cdot Q]_{R(\rho)}. \]
This completes the proof of Proposition 5.2.

Remark. If \(f_M \in M_0\), then the integrand may have a non-regular singularity since we have a product of two regular singularities at \(p = 0\). Hence,
\[L_{FM} f_M = [\gamma_s \cdot R_y f_M]_F\]
\[L_{MN} f_M = [\gamma_s \cdot R_y f_M]_C\]
\[L_{JM} f_M = [\gamma_s \cdot R_y f_M]_J.\]

For \(f_F \in \mathcal{O}_0(F)\), we have the following

**Proposition 5.3**
\[L_{JM} f_F = 0\]
\[L_{MF} f_F = [\gamma_s \cdot R_y f_F]_C\]
\[L_{FF} f_F = \gamma_s \cdot R_y f_F - [\gamma_s \cdot R_y f_F]_C\]
Proof. As \( f_F \in U_0(F) \), we have an estimate \( |f_F(t)| < k |t|^{-1} \) near \( \infty \). Hence

\[
\left| (L_{TF} f_F)(z) \right| = \frac{2\pi |z|}{2\pi} \left| \frac{12\pi^{-5} |2\pi| k |t|^{-1}}{|t^2| + \frac{2}{|t|}} \right| \to 0 \quad \text{(as} \ t \to \infty \text{)}
\]

Therefore we have \( L_{TF} f_F = 0 \). For the second component,

\[
L_{MF} f_F = \mathcal{I}_c [ \zeta_c \cdot R_y f_F ] = [ \zeta_c \cdot R_y f_F ]_c
\]

And the third component is computed similarly.

\[
L_{Eff} f_F(z) = \frac{1}{2\pi i} \int_{\gamma_F} \frac{R(c)(R(t)) f_F(t)}{R(t) - z} dt
\]

\[
= \frac{1}{2\pi i} \int_{\gamma_F} \frac{\zeta(c)(R_y f_F)(\sigma)}{\sigma - z} d\sigma
\]

\[
= \zeta_c(z) (R_y f_F)(z) - [ \zeta_c \cdot R_y f_F ]_c
\]

Putting the above propositions together, we have the decomposition of our transfer operator into components.

\[
\begin{bmatrix}
L_{TT} & L_{TM} & L_{TF} \\
L_{MT} & L_{MM} & L_{MF} \\
L_{FT} & L_{FM} & L_{FF}
\end{bmatrix}
\begin{bmatrix}
f_T \\
f_M \\
f_F
\end{bmatrix}
= \begin{bmatrix}
[ \zeta_T \cdot R_y f_T ]_c & [ \zeta_T \cdot R_y f_M ]_c & 0 \\
[ \zeta_M \cdot R_y f_T ]_c & [ \zeta_M \cdot R_y f_T ]_c & [ \zeta_M \cdot R_y f_F ]_c \\
[ \zeta_T \cdot R_y f_T ]_c & [ \zeta_F \cdot R_y f_M ]_c & [ \zeta_T \cdot R_y f_F ]_c
\end{bmatrix}
\begin{bmatrix}
f_T \\
f_M \\
f_F
\end{bmatrix}
+ [ \zeta_T \cdot f_M ] \Theta_{R_y f_F}(t)
\]

Note that if \( f_T \) or \( f_F \) are not identically zero, then \( [ \zeta_T \cdot R_y f_T ]_c \neq 0 \) or \( [ \zeta_M \cdot R_y f_F ]_c \neq 0 \). Hence, there is no eigenfunction in subspace \( \mathcal{O}(C) \oplus U_0(F) \).

§ 6. Invariant subspace of the transfer operator

Our transfer operator \( \mathcal{T}_c : \mathcal{N} \rightarrow \mathcal{N} \) maps the space of pre-minifunctions supported on the forward orbit of the critical point. For \( \mathcal{O}_0 \in \mathcal{N}_c \), \( \mathcal{O}_0(z) \) can be written in a form of Cauchy integral.

\[
\mathcal{O}_0(z) = \frac{1}{2\pi i} \int_{\gamma_c} \frac{(\Delta c \mathcal{O}_0)(\xi) \Lambda_c(\xi) d\xi}{\Lambda_c(\xi)} = \frac{1}{2\pi i} \int_{\gamma_c} \frac{(\Delta c \mathcal{O}_0)(\xi) \Lambda_c(\xi) d\xi}{\Lambda_c(\xi)}
\]

In the following, we denote as \( \mathcal{V}_{\xi,c}(z) = \mathcal{V}_c (R_c(z)) \cdot \mathcal{V}_c (R_{c1}(z)) \cdots \mathcal{V}_c (R_{cN}(z)) \)
Where $R_k(\varepsilon) = R^0 R_k(\varepsilon)$ denote the $k$-th iteration of $R(\varepsilon)$. Note that $\gamma_k \circ R_k \in \mathcal{H}_{R^0 k+1(\varepsilon)}$ and that $\gamma_k \circ R_k$ are regular on $\gamma_k$. For each $k = 1, 2, \ldots$, we consider a pre-microfunction $\gamma_k$ in $\mathcal{H}_{R_k(\varepsilon)}$ expressed in terms of a pre-microfunction $\gamma_k$ in $\mathcal{H}_{\varepsilon}$. For $\gamma_k \in \mathcal{H}_{\varepsilon}$, let

$$G_k(\beta) = \gamma_k(3) \cdot \gamma_k(3)$$

and define $R_k \circ \mathcal{H}_{R_k(\varepsilon)}$ by

$$R_k(\varphi) = \frac{-1}{2\pi i} \int_{\gamma_k} (\Delta_c G_k(\varphi, t)) R_k''(t) Y_{R_k''(t)}(\varphi) \, dt.$$ 

Let $Q_k = (R_k|_{\gamma_k})^{-1} : \gamma_k \rightarrow \gamma_k$ be the inverse branch of $R_k$. By a change of variables $\sigma = R_k'(t), d\sigma = R_k''(t) \, dt$ and $t = Q_k(\sigma)$, we have

$$R_k(\varphi) = \frac{-1}{2\pi i} \int_{Q_k} \left( \Delta_c G_k(\varphi, Q_k(\sigma)) \right)(\sigma) Y_{Q_k}(\varphi) \, d\sigma.$$ 

This implies $R_k = [G_k \circ Q_k] \circ R_k$ and $R_k \in \mathcal{H}_{R_k(\varepsilon)}$. We have

$$\Delta_c G_k = (\Delta_c R_k') \circ R_k$$

along $\gamma_k$. So, this correspondence induces an isomorphism $\mathcal{H}_{R_k(\varepsilon)} \cong \mathcal{H}_{\varepsilon}$. As $\mathcal{H}_{\varepsilon} \in \mathcal{H}_{R_k(\varepsilon)}$, we have $\mathcal{H}_{\varepsilon} \in \mathcal{H}_{R_k(\varepsilon)} \oplus \mathcal{H}_{\varepsilon}$.

More precisely, we have the following explicit formula.

**Proposition 6.1** If $R_k = [G_k \circ Q_k] \circ R_k \in \mathcal{H}_{R_k(\varepsilon)}$ with $G$ as above, we have the following decomposition.

$$L_s R_k = [G_k \circ Q_k] \circ R_k + \gamma_k \cdot [G_k \circ Q_k] \circ R_k.$$ 

**Proof.** This is immediately verified by applying Theorem 4.4. By an immediate calculation, we can obtain the proof as follows. First, component of $L_s R_k$ is given by

$$[L_s R_k]_{R_k(\varepsilon)} = L_s \left[ \frac{-1}{2\pi i} \int_{Q_k} \left( \Delta_c G_k(\varphi, Q_k(\sigma)) \right)(\sigma) Y_{Q_k}(\varphi) \, d\sigma \right]_{R_k(\varepsilon)}$$

$$= \left[ \int_{Q_k} \frac{-1}{2\pi i} \int_{R_k''(t)} \left( \Delta_c G_k(\varphi, t) \right) \gamma_k(3) \, d\sigma \right]_{R_k(\varepsilon)}.$$
\[ \frac{1}{2\pi i} \int \left( \Delta_{R_{\phi(c)}} \left[ G_{\phi(c)} \circ Q_{\phi(c)} \right] \right)(\sigma) \left[ L_5 \chi_\sigma \right]_{R_{\phi(c)}} d\sigma \]

\[ = \frac{1}{2\pi i} \int \left( \Delta_{R_{\phi(c)}} \left[ G_{\phi(c)} \circ Q_{\phi(c)} \right] \right)(\sigma) \frac{1}{2\pi i} \int \left[ R(\sigma) R(\tau) \chi_{R(\tau)} \right]_{R_{\phi(c)}} d\tau d\sigma \]

\[ = \frac{1}{2\pi i} \int \left( \Delta_{R_{\phi(c)}} \left[ G_{\phi(c)} \circ Q_{\phi(c)} \right] \right)(\sigma) \left[ \chi_{R(\sigma)} \chi_{R(\tau)} \right]_{R_{\phi(c)}} d\sigma d\tau \]

\[ = \left[ \chi_{\phi(c)} \left[ G_{\phi(c)} \circ Q_{\phi(c)} \right] \right]_{R_{\phi(c)}} \]

where we made a change of variables \( \rho = R(\sigma), \quad d\rho = R(\sigma) d\sigma \).

Similarly, the second component is computed as follows.

\[ \left[ L_5 \chi_{R(c)} \right]_c = \frac{1}{2\pi i} \int \left( \Delta_{R_{\phi(c)}} \left[ G_{\phi(c)} \circ Q_{\phi(c)} \right] \right)(\sigma) \left[ L_5 \chi_\sigma \right]_{R_{\phi(c)}} d\sigma \]

\[ = \frac{1}{2\pi i} \int \left( \Delta_{R_{\phi(c)}} \left[ G_{\phi(c)} \circ Q_{\phi(c)} \right] \right)(\sigma) \left[ R(\sigma) \chi_{R(\tau)} \right]_{R_{\phi(c)}} d\sigma d\tau \]

\[ = \frac{1}{2\pi i} \int \left( \Delta_{R_{\phi(c)}} \left[ G_{\phi(c)} \circ Q_{\phi(c)} \right] \right)(\sigma) \left[ \chi_{R(\sigma)} \chi_{R(\tau)} \right]_{R_{\phi(c)}} d\sigma d\tau \]

\[ = \left[ \chi_{\phi(c)} \left[ G_{\phi(c)} \circ Q_{\phi(c)} \right] \right]_{R_{\phi(c)}} \]

8.7 Eigenvalue problem

In this section, we consider the eigenvalue problem for our transfer operator \( L_5 \) restricted to an invariant subspace of pre-microfunctions \( M_{\phi(c)} \) defined in section 4 as

\[ M_{\phi(c)} = \sum_{k=0}^{\infty} M_{R_{\phi(c)}} \]

Here, the sum is considered as formal sum. In the case of post-critically finite maps, the post critical set \( P(c) \) is a finite set and the sum is finite. In this case

\[ M_{\phi(c)} = \bigoplus_{P \in P(c)} M_P \]
In order to analyze the eigenvalue problem of $L_s$, we consider a formal sum of pre-microfunctions.

$$
\hat{\phi} = \sum_{k=0}^{\infty} \hat{\phi}_k \quad \text{with} \quad \hat{\phi}_k \in M_{\mathbb{P}^k}(c)
$$

**Proposition 7.1.** If $\hat{\phi}$ is an eigenfunction of $L_s$ satisfying

$$\lambda L_s \hat{\phi} = \hat{\phi} \quad \text{and} \quad P(R) \text{ is infinite},$$

then

$$\lambda \left[ L_s \hat{\phi}_k \right]_{R_{k+1}(c)} = \hat{\phi}_k \quad \text{and} \quad \lambda \sum_{k=0}^{\infty} [L_s \hat{\phi}_k]_c = \hat{\phi}_0.$$

**Proof.** By a straightforward calculation, we have

$$L_s \hat{\phi}_k = \left[ L_s \hat{\phi}_k \right]_{R_{k+1}(c)} + \left[ L_s \hat{\phi}_k \right]_c.$$

**Theorem 7.2.** The eigenvalue problem $\lambda L_s \hat{\phi} = \hat{\phi}$ of our transfer operator $L_s : M_{\mathbb{P}^k}(c) \rightarrow M_{\mathbb{P}^k}(c)$ reduces to an "eigenvalue" problem of an integral operator

$$T_\lambda : \mathcal{C}_0(\mathbb{P}^k(c)) \rightarrow \mathcal{C}_0(\mathbb{P}^k(c))$$

defined by

$$(T_\lambda \varphi)(u) = (A \varphi \chi_{\mathbb{P}})(u) \cdot \frac{1}{2\pi} \int_{\mathbb{P}^k} H_s(u, t; \lambda) \varphi(t) \, dt,$$

where

$$H_s(u, t; \lambda) = -\sum_{k=0}^{\infty} \lambda^k \gamma_{\mathbb{P}^k}(t) R_{k+1}(t) \chi_{\mathbb{P}^k}(u),$$

with $\lambda T_\lambda \hat{\phi}_0 = \hat{\phi}_0$, $\hat{\phi}_0 = A \varphi \chi_{\mathbb{P}}$.

**Proof.** As $R_{k+1} = \lambda \left[ L_s \hat{\phi}_k \right]_{R_{k+1}(c)} = \lambda \left[ \gamma_{\mathbb{P}^k} \chi_{\mathbb{P}^k} \circ Q_{k+1} \right]_{R_{k+1}(c)}$

$$= \lambda \left[ g_{k+1} \circ Q_{k+1} \cdot \gamma_{\mathbb{P}^k} \chi_{\mathbb{P}^k} \circ Q_{k+1} \cdot \gamma_{\mathbb{P}^k} \chi_{\mathbb{P}^k} \circ Q_{k+1} \right]_{R_{k+1}(c)}$$

and $Q_{k+1} = \left[ \gamma_{\mathbb{P}^k} \chi_{\mathbb{P}^k} \circ Q_{k+1} \right]_{R_{k+1}(c)} = \left[ g_{k+1} \circ Q_{k+1} \cdot \gamma_{\mathbb{P}^k} \chi_{\mathbb{P}^k} \circ Q_{k+1} \right]_{R_{k+1}(c)}$

we have $g_{k+1} = \lambda^k \gamma_{\mathbb{P}^k}$ for $k \geq 0$.

Hence $\hat{\phi}_k = \lambda^k \hat{\phi}_0$, which implies

$$\hat{\phi}_k = \left[ g_{k+1} \circ Q_{k+1} \right]_{R_{k+1}(c)} = \left[ \lambda^k \hat{\phi}_0 \circ Q_{k+1} \cdot \gamma_{\mathbb{P}^k} \chi_{\mathbb{P}^k} \circ Q_{k+1} \right]_{R_{k+1}(c)}.$$
\[
\lambda \sum_{k=0}^{\infty} \left( \frac{1}{2\pi i} \int_{\gamma_c} \Delta R_n(t) \gamma_S(t) \gamma_R(t) R_n(t) \chi_{R_n(t)}(u) \frac{dt}{t} \right) \chi_{R_n(t)}(u)
\]

This yields an integral equation for \( \lambda \in M_c \):

\[
(\Delta R_0) = \lambda (\Delta \gamma_S)(u) \frac{1}{2\pi i} \int_{\gamma_c} H_S(u,t; \lambda)(\Delta R_0)(t) dt
\]

more briefly,

\[
\lambda \sum_{k=0}^{\infty} \left( \frac{1}{2\pi i} \int_{\gamma_c} H_S(u,t; \lambda)(\Delta R_0)(t) dt \right) \chi_{R_n(t)}(u)
\]

by setting

\[
H_S(u,t; \lambda) = \frac{1}{2\pi i} \int_{\gamma_c} H_S(u,t; \lambda)(\Delta R_0)(t) dt
\]

\[\S8. \text{ Dual spaces and Cauchy's transformations.}\]

Let \( O(T) \) denote the space of germs of holomorphic functions along the Julia set \( J = J(T) \). Each element of \( O(T) \) has a representative \( f : J \to C \) which can be extended to a holomorphic function in a neighborhood of \( J \). As \( J \) is a perfect set, the analytic continuation is uniquely determined by \( f \).

Topology in \( O(T) \) is given by the uniform convergence on \( J \).

Linear functional \( G^T : O(T) \to C \) is said to be holomorphic if for every holomorphic family \( f \in O(T) \), \( G^T[f] : \Lambda \to C \) is holomorphic. We require the continuity of \( G^T \) with respect to the sup norms on \( O(T) \). The space of continuous holomorphic linear functionals \( G^T : O(T) \to C \) will be denoted by \( O^T \).

As in the previous sections, \( O_0(F) \) denotes the space of holomorphic functions in the Fatou set \( F \) vanishing at the infinity.

If \( p \in F \) then \( \chi_p = \frac{1}{|z-p|} \) belongs to \( O^T \). For holomorphic linear functional \( G^T \in O^T \), define a holomorphic function \( \bar{G}^T \in O_0(F) \) by \( \bar{G}^T(z) = G^T[1/|z|^2] \). Then, family
of holomorphic functions \( F \rightarrow \mathcal{O}(J) \) defined by \( z \mapsto -z \)
for a holomorphic family, \( g^J : F \rightarrow \mathcal{O}(J) \) is a holomorphic function, since the functional \( G^J \) is holomorphic. By the continuity of \( G^J \), we see immediately \( g^J(0) = 0 \) and hence \( g^J \in \mathcal{O}_0(F) \).
This correspondence between \( \mathcal{O}^*(J) \) and \( \mathcal{O}_0(F) \) is called the Cauchy transformation.

**Proposition 8.1.** For \( f_J \in \mathcal{O}(J) \), \( G^J[f_J] \) can be expressed in an integration form

\[
G^J[f_J] = \frac{1}{2\pi i} \int_{X_F} f_J(\tau) g^J(\tau) d\tau,
\]
where the integration path \( X_F \) goes around the Julia set in the clockwise direction.

**Proof.** As \( f_J \) is holomorphic near \( J \), for \( z \) in a neighborhood of \( J \),

\[
f_J(z) = \frac{1}{2\pi i} \int_{X_J} f_J(\tau) \lambda_z(\tau) d\tau = -\frac{1}{2\pi i} \int_{X_J} f_J(\tau) \lambda_z(\tau) d\tau,
\]
where \( X_J \) runs around the Julia set in the counterclockwise direction. The right-hand side of this equality gives an expression of \( f_J(z) \) as a "linear combination" of unit poles.
We have

\[
G^J[f_J] = \frac{-1}{2\pi i} \int_{X_J} f_J(\tau) G^J[\lambda_z] d\tau = \frac{-1}{2\pi i} \int_{X_J} f_J(\tau) g^J(\tau) d\tau
\]
\[
= \frac{1}{2\pi i} \int_{X_F} f_J(\tau) g^J(\tau) d\tau.
\]

In the following, we shall denote such pairings of functions as

\[
\langle g^J, f_J \rangle_F = \frac{1}{2\pi i} \int_{X_F} f_J(\tau) g^J(\tau) d\tau.
\]

**Proposition 8.2** \( \mathcal{O}^*(J) \cong \mathcal{O}_0(F) \)

**Proof.** The Cauchy transformation defines a complex linear map from \( \mathcal{O}^*(J) \) to \( \mathcal{O}_0(F) \), and the pairing along \( X_F \) defines a complex linear map from \( \mathcal{O}_0(F) \) to \( \mathcal{O}^*(J) \). These two transformations are mutually inverse.
Let \( \mathcal{O}_0^*(F) \) denote the space of holomorphic linear and continuous functional \( G^F : \mathcal{O}_0(F) \to \mathbb{C} \).

**Proposition 8.3**

\( \mathcal{O}(J) \subset \mathcal{O}_0^*(F) \).

**Proof.** If \( z \in J \) then \( \Omega_z \in \mathcal{O}_0(F) \). For \( G^F \in \mathcal{O}_0^*(F) \), let

\[
g^F(z) = G^F[\Omega_z].
\]

Then \( g^F : J \to \mathbb{C} \) is a continuous function.

If \( g^F \in \mathcal{O}(J) \), then for \( f_F \in \mathcal{O}_0(F) \) with

\[
f_F(z) = \frac{i}{2\pi i} \int f_F(t) \Omega_z(t) dt = \frac{i}{2\pi i} \int f_F(t) \Omega_z(t) dt,
\]

we have

\[
G^F[f_F] = \frac{i}{2\pi i} \int f_F(t) G^F[\Omega_z] dt = \frac{i}{2\pi i} \int f_F(t) g^F(t) dt,
\]

where the integration path \( \gamma_F \) is given by \( g^F \) which is holomorphic in a neighborhood of \( J \). This pairing will be denoted as \( \langle g^F, f_F \rangle_J \).

We define the third pairings \( \langle \cdot, \cdot \rangle_M \) and \( \langle \cdot, \cdot \rangle_{\mathcal{M}} \) related to the external rays and pre-microfunctions. Let \( M \) denote the space of pre-microfunctions and let \( \gamma_M \) denote the "sum" of external rays supporting the pre-microfunctions. We use symbol \( M \) to indicate the object is related to the pre-microfunction component. When we apply operations \( \Delta_M, \mathcal{I}_M \) etc., we take the "sum" of the objects over external rays. The dual space \( M^* \) is the space of holomorphic linear and continuous functionals \( \mathcal{O}^* : M \to \mathbb{C} \).

For \( \gamma_M \), we denote by \( \gamma_M \) the integration path passing along the external ray both sides of \( \gamma_M \) coming from the infinity to \( p \) in the negative side of \( \gamma_M \) and coming back from \( p \) to the infinity on the positive side of \( \gamma_M \). If \( g^M \in \mathcal{O}_0(\mathbb{D}_M) \), that is, \( g^M \) is a holomorphic function in a neighborhood of \( \gamma_M \) with regular singularities at the infinity and each landing points.
For \( f_M \in M \), we can rewrite it in the following form:
\[
\tilde{f}_M(z) = \frac{1}{2\pi i} \int_{\gamma_M} (\Delta_M f_M)(\tau) \chi_M(\tau) \, d\tau = \frac{1}{2\pi i} \int_{\gamma_M} (\Delta_M f_M)(\tau) \chi_M(\tau) \, d\tau \]

For \( g^M \in C(\beta_M) \) define a homomorphic functional \( G^M : M \to \mathbb{C} \) by
\[
G^M[f_M] = \langle g^M, \Delta_M f_M \rangle_M = \langle g^M, f_M \rangle_M.
\]

\( G^M[f_M] \) is defined if \( \Delta_M f_M \cdot g^M \in L_1(\beta_M) \). Note that
\[
G^M[-\chi] = \langle g^M, -\chi \rangle_M = g^M(z)
\]

by Cauchy's integration formula. Note that if \( g^M \in C(\beta_M) \)
and \( g^M = T_M g^M_J \), then for \( z \in C \setminus \beta_M \)
\[
\hat{g}^M(z) = \langle g^M, \chi \rangle_M = \langle g^M, \chi \rangle_M
\]
fails. If \( f_M \in M \), then
\[
G^M[f_M] = G^M \left[ \frac{1}{2\pi i} \int_{\gamma_M} (\Delta_M f_M)(\tau) (-\chi) \, d\tau \right] = \frac{1}{2\pi i} \int_{\gamma_M} (\Delta_M f_M)(\tau) G^M[-\chi] \, d\tau
\]
\[
= \frac{1}{2\pi i} \int_{\gamma_M} (\Delta_M f_M)(\tau) g^M(\tau) \, d\tau = \langle g^M, \Delta_M f_M \rangle_M.
\]

We have a splitting of pre-microfunctions,
\[
f = f_J \oplus f_M \oplus f_F \in C(\beta) \oplus M \oplus C_0(F)
\]
and a splitting of its dual space
\[
G = G_J \oplus G^M \oplus G^F \in C^*(\beta) \oplus M^* \oplus C_0^*(F).
\]

with Cauchy's transforms given by
\[
G^J[-\chi] = \hat{g}^J(z), \quad z \in F, \quad g^J \in C_0(F)
\]
\[
G^M[-\chi] = \hat{g}^M(z), \quad z \in \beta, \quad g^M \in C(\beta)
\]
\[
G^M[\chi] = \hat{g}^M(z), \quad z \in C \setminus \beta, \quad g^M \in \mathcal{M}
\]
\[
G^F[-\chi] = \hat{g}^F(z), \quad z \in F, \quad g^F \in C(\beta)
\]

The pairing of \( f \) on \( G \) is defined by
\[
\]
\[
= \langle g^J, f_J \rangle_M + \langle g^M, \Delta_M f_M \rangle_M + \langle g^F, f_F \rangle_M.
\]

Projections of \( G = C(\beta) \oplus M \oplus C_0(F) \) to components are denoted by
\[
f \mapsto [f]_J, \quad f \mapsto [f]_M, \quad f \mapsto [f]_F
\] respectively.
These projections are given by
\[
\left[ f \right]_J (z) = \frac{1}{2\pi i} \int_{\delta J} f(t) \chi_2(t) \, dt \quad (z \in \mathcal{J}),
\]
\[
\left[ f \right]_M (z) = \frac{1}{2\pi i} \int_{\delta M} (\Delta_M f)(t) \chi_2(t) \, dt \quad (z \in \mathcal{C} \setminus \delta M),
\]
\[
\left[ f \right]_F (z) = \frac{1}{2\pi i} \int_{\delta F} f(t) \chi_2(t) \, dt \quad (z \in \mathcal{F}).
\]

And the projections of \( \mathcal{N} = \mathcal{O}^*(\mathcal{J}) \oplus \mathcal{M}^* \oplus \mathcal{O}^*(\mathcal{F}) \) are denoted as
\[
\begin{align*}
\left[ J^* \right] : \mathcal{N}^* &\rightarrow \mathcal{O}^*(\mathcal{J}) \subset \mathcal{O}^*(\mathcal{I}) \\
\left[ M^* \right] : \mathcal{N}^* &\rightarrow \mathcal{O}^*(\mathcal{M}) \subset \mathcal{M}^* \\
\left[ F^* \right] : \mathcal{N}^* &\rightarrow \mathcal{O}^*(\mathcal{I}) \subset \mathcal{O}^*(\mathcal{F}).
\end{align*}
\]

Let \( L^*_\mathcal{B} \) denote the dual of our transfer operator \( L^*_\mathcal{B} \). We abuse notations and confuse functionals and its Cauchy's transforms. \( L^*_\mathcal{B} : \mathcal{O}^*(\mathcal{F}) \oplus \mathcal{O}^*(\mathcal{M}) \oplus \mathcal{O}^*(\mathcal{J}) \otimes \mathcal{K} \) is decomposed as
\[
L^*_\mathcal{B} = \begin{pmatrix} L^*_\mathcal{J} & L^*_\mathcal{M} & L^*_\mathcal{F} \\ L^*_\mathcal{I} & L^*_\mathcal{M} & L^*_\mathcal{F} \\ L^*_\mathcal{I} & L^*_\mathcal{M} & L^*_\mathcal{F} \end{pmatrix}.
\]

In the rest of this section, we compute these components more explicitly.

**Proposition 8.4.**
\[
\begin{align*}
L^*_\mathcal{J} g^J &= \chi_5 \circ R \cdot R^* \cdot g^J \circ R - [\chi_5 \circ R \cdot R^* \cdot g^J \circ R]_0 \\
L^*_\mathcal{M} g^J &= \Delta_0 [\chi_5 \circ R \cdot R^* \cdot g^J \circ R] \\
L^*_\mathcal{F} g^J &= 0
\end{align*}
\]

**Proof.** For \( g^J \in \mathcal{O}^*(\mathcal{F}) \), we compute \( L^*_\mathcal{J} g^J (z) \) for \( z \in \mathcal{F} \).
\[
(L^*_\mathcal{J} g^J)(z) = \left[ (L^*_\mathcal{J} g^J)[z] \right]^T = \left[ G^T [-L^*_5 \chi_5] \right]^T
\]
\[
= \left[ G^T [-\chi_5 (R(3)) \cdot R^*(3) \chi_{R(3)} - [\chi_5 \cdot R^*(3) \cdot \chi_{R(3)}]^T \right]
\]
\[
= \left[ G^T [-\chi_5 (R(3)) \cdot R^*(3) \chi_{R(3)}] \right]^T \quad \text{(since \( \chi_5 \in \mathcal{M}_c \))}
\]
\[
= \left[ \chi_5 (R(3)) \cdot R^*(3) \right] G^T [-\chi_{R(3)}] = \left[ \chi_5 (R(3)) \cdot R^*(3) \cdot G^T [-\chi_{R(3)}] \right]_F
\]
\[
= \chi_5 (R(3)) \cdot R^*(3) \cdot g^T (R(3)) - [\chi_5 \circ R \cdot R^* \cdot g^J \circ R]_0 (z).
\]
Next, for $z \in \mathcal{Z}_1$, $L^*_F g^F \in C(\mathcal{Z}_1)$ is computed as follows.

$$(L^*_F g^F)(z) = \left[ (L^*_F g^F) [-\chi_3] \right]^F$$

$$= \left[ g^F \left[ -\gamma_z R(3) \cdot R'(3) \cdot \chi_{R(3)} \right] - \left[ \gamma_z \cdot R'(3) \cdot \chi_{R(3)} \right] \right]^F$$

$$= \left[ \gamma_z R(3) \cdot R'(3) \cdot g^F \left[ -\chi_{R(3)} \right] \right]^F = \left[ \gamma_z R(3) \cdot R'(3) \cdot g^F(R(3)) \right]^F$$

$$= \Delta_0 \left[ \gamma_z \cdot R \cdot R' \cdot g^F \cdot R \right](z) = \Delta_0 \gamma_z \cdot R \cdot R' \cdot g^F \cdot R.$$ 

For $z \in J$, we have

$$(L^*_F g^F)(z) = \left[ (L^*_F g^F) [-\chi_3] \right]^F$$

$$= \left[ g^F \left[ -\gamma_z (R(3)) \cdot R'(3) \cdot \chi_{R(3)} \right] - \left[ \gamma_z \cdot R'(3) \cdot \chi_{R(3)} \right] \right]^F$$

$$= 0.$$ 

The last equality holds since $\gamma_z (R(3)) \cdot R'(3) \cdot \chi_{R(3)} \in C_0(F)$ and $\left[ \gamma_z \cdot R'(3) \cdot \chi_{R(3)} \right] \in M.$

**Proposition 8.5**

$L^*_F \gamma^F = 0$

$L^*_F g^F = \Delta_0 \left[ \gamma_z \cdot R \cdot R' \cdot g^F \cdot R \right]$,

$L^*_F \gamma^F = \gamma_z \cdot R \cdot R' \cdot g^F \cdot R - \left[ \gamma_z \cdot R \cdot R' \cdot g^F \cdot R \right]_0.$

**Proof.** For $z \in F$, we have $-\chi_3 \in C(J)$. For $g^F \in C(J)$,

$$(L^*_F g^F)(z) = \left[ (L^*_F g^F) [-\chi_3] \right]^F = \left[ g^F \left[ -\gamma_z \chi_3 \right] \right]^F = 0.$$ 

Here, the last equality holds since $\gamma_z \cdot R(3) \cdot R'(3) \cdot \chi_{R(3)} \in C(\mathcal{Z}_1)$,

$\left[ \gamma_z \cdot R(3) \cdot \chi_{R(3)} \right] \in M$ and $g^F \left[ -\chi_{R(3)} \right] = 0.$

For $z \in J$, $-\chi_3$ belongs to $C_0(F).$ Hence

$$(L^*_F g^F)(z) = \left[ (L^*_F g^F) [-\chi_3] \right]^F = \left[ g^F \left[ -\gamma_z \chi_3 \right] \right]^F = \left[ \gamma_z \cdot R(3) \cdot R'(3) \cdot \chi_{R(3)} \right]^F = \gamma_z \cdot R(3) \cdot R'(3) \cdot g^F \cdot R(3) - \left[ \gamma_z \cdot R \cdot R' \cdot g^F \cdot R \right]_0(z).$$ 

We used the fact $-\gamma_z (R(3)) \cdot R'(3) \cdot \chi_{R(3)} \in C_0(F)$ and $\left[ \gamma_z \cdot R'(3) \cdot \chi_{R(3)} \right] \in M.$
To compute \( \delta_{\omega} g^F \), we take \( \omega \in \gamma_M \subset F \). Then,
\[
(\delta_{\omega} g^F)(\gamma) = \left[ (L^* \gamma^F) [-\gamma_5] \right]^T
\]
\[
= G^F [- \gamma_5 (R(\gamma))^T \chi_{R(\gamma)}] \quad \left[ \gamma_5 \cdot R(\gamma) \chi_{R(\gamma)} \right]_C
\]
\[
= \left[ \gamma_5 (R(\gamma))^T \chi_{R(\gamma)} \right] \quad \left[ \gamma_5 (R(\gamma))^T \chi_{R(\gamma)} \right]_C
\]
\[
= \Delta_0 \left[ \gamma_5 \cdot R \cdot R' \cdot g^F R \right](\gamma).
\]
In the above calculations, we used the fact \( \left[ \gamma_5 \cdot R(\gamma) \chi_{R(\gamma)} \right]_C \in M_0 \). During the computations, \( \omega \) is regarded as constant and the final result gives the formula as a function of \( \omega \).

**Proposition 8.6.** \( \delta_{\omega} g^M = \left[ \gamma_5 \cdot R \cdot R' \cdot (I_M g^M) \cdot R \right]_F \)

\[
\delta_{\omega} g^M = \left\{ \left[ \gamma_5 \cdot R \cdot R' \cdot g^M (\gamma) \right]_C \right\} \quad \left[ \gamma_5 \cdot R \cdot R' \cdot g^M (\gamma) \right]_C
\]

\[
= \left[ \gamma_5 \cdot R \cdot R' \cdot g^M (\gamma) \right]_C \quad \left[ \gamma_5 \cdot R \cdot R' \cdot g^M (\gamma) \right]_C
\]

\[
= \Delta_0 \left[ \gamma_5 \cdot R \cdot R' \cdot g^M R \right](\gamma).
\]

**Proof.** For \( g^M \in M_0 \), let \( g^M(\gamma) = g^M [-\gamma_5] \), \( \gamma \in M_0 \), \( g^M = C(M) \).

For \( \omega \in F \), \( \omega \in C \setminus \gamma_M \), \( g^M \in C(M) \),
\[
(\delta_{\omega} g^M)(\gamma) = \left[ (L^* \gamma^M) [-\gamma_5] \right]^T
\]
\[
= \left[ G^M \left[ \gamma_5 (R(\gamma))^T \chi_{R(\gamma)} \right] \right]^T
\]
\[
= \left[ G^M \left[ \gamma_5 (R(\gamma))^T \chi_{R(\gamma)} \right] \right]^T
\]
\[
= \left[ \gamma_5 (R(\gamma))^T \chi_{R(\gamma)} \right] \quad \left[ \gamma_5 (R(\gamma))^T \chi_{R(\gamma)} \right] \quad \left[ \gamma_5 (R(\gamma))^T \chi_{R(\gamma)} \right]_C
\]
\[
= \left[ \gamma_5 (R(\gamma))^T \chi_{R(\gamma)} \right] \quad \left[ \gamma_5 (R(\gamma))^T \chi_{R(\gamma)} \right]_C
\]
\[
= \Delta_0 \left[ \gamma_5 \cdot R \cdot R' \cdot g^M R \right](\gamma).
\]

Here, the first equality holds since \( \gamma \in F_0 \setminus (C \setminus \gamma_M) \) and a

Note: The rest of the text is repeated, indicating a cycle in the document.
\[(L^\ast_{FM} G^M)(3) = \left[ (L^\ast G^M)[ \chi_3 ] \right]^F \]

\[= \left[ G^M [ \chi_3 (R(3)), R(3), \chi_{R(3)} ] + \left[ \chi_3 \cdot R(3), \chi_{R(3)} \right] \right]_F \]

\[= \left[ \chi_3 (R(3)), R(3), \left( \chi_3, g^M \right)(R(3)) \right]_F + \left[ \frac{1}{2\pi i} \int_{\gamma_3} g^M(t) : R(3) \cdot \chi_{R(3)}(t) \cdot \frac{dt}{t-R(3)} \right] \]

The second term of the above line is computed as follows.

\[= \frac{1}{2\pi i} \int_{\gamma_3} \frac{dt}{t-R(3)} \cdot \left[ \chi_3 \cdot \left( \chi_3, g^M \right)(R(3)) \right]_F \]

\[= \frac{1}{2\pi i} \int_{\gamma_3} \frac{dt}{t-R(3)} \cdot \left( \chi_3, g^M \right)(R(3)) \cdot \frac{dt}{t-R(3)} \]

\[= 0. \]

The last equality holds since \( R(3) \) is of degree one and the denominator \( (t-R(3)) \) is of degree three with respect to the variable of integration and the integration path \( \gamma_3 \) turns around the Julia set along a circle of infinitely large radius. Hence we have

\[(L^\ast_{FM} G^M)(3) = \left[ \chi_3 (R(3)), R(3), \left( \chi_3, g^M \right)(R(3)) \right]_F. \]

Finally, let us compute \( L^\ast_{MM} G^M \in \Omega^\ast(M) \).

For \( x \in Y_p \) with \( p \in \mathbb{P}(R) \), the component \( (L^\ast_{MM} G^M)_x \in \Omega^\ast(x) \) is computed as follows.

\[(L^\ast_{MM} G^M)(3) = \left[ (L^\ast G^M)[ -\chi_3 ] \right]_{M_p} = \left[ G^M \cdot [ -L^\ast \chi_3 ] \right]_{M_p} \]

\[= \left[ \chi_3 (R(3)), R(3), \left[ -\chi_{R(3)} \right] \right]_{M_p} + \left[ \chi_3 \cdot R(3), \chi_{R(3)} \right]_{M_p} \]

\[= \left[ \chi_3 (R(3)), R(3), g^M(3) \right]_{M_p} + \left[ \frac{1}{2\pi i} \int_{\gamma_3} g^M(t) : \chi_3 \cdot \left( \chi_3, g^M \right)(R(3)) \cdot \frac{dt}{t-R(3)} \right]_{M_p} \]

\[= \left[ \chi_3 \cdot R \cdot \left[ -\chi_{R(3)} \right] \right]_{M_p} + \left[ \chi_3 \cdot R \cdot \left[ -\chi_{R(3)} \right] \right]_{M_p} \]

In the case of \( p = 0 \), i.e., for \( x \in Y_0 \), we have

\[(L^\ast_{MM} G^M)(3) = \left[ (L^\ast G^M)[ -\chi_3 ] \right]_{M_0} = \left[ G^M \cdot [ -L^\ast \chi_3 ] \right]_{M_0} \]

\[= \left[ G^M \cdot [ -\chi_3 \cdot R(3), \chi_{R(3)} ] \right]_{M_0} = \left[ < g^M, \chi_3 \cdot R(3), \chi_{R(3)} > \right]_{M_0}. \]
\[ = \left[ \frac{1}{2\pi i} \int_{\gamma_5} g^M_R(\tau) \cdot \chi^R_5(\tau) \cdot \frac{d\tau}{\tau - R(\xi)} \right]^{M_0} \]
\[ = \left[ \frac{1}{2\pi i} \int_{\gamma_6} g^M_R(\tau) \cdot \chi^R_6(\tau) \cdot R(\xi) \cdot \chi^R_5(\tau) \cdot d\tau \right]^{M_0} \]
\[ = R(\xi) \cdot \left( \Delta \left[ g^M_6, \chi^R_6, \chi^R_5 \right] \right)(3). \]

§ 9. Example

In this section, we compute the operator \( L^*_{MM} \) more precisely for \( R(\xi) = 2\tau + i \) case. In this case, the critical value \( C = i \) is preperiodic and the postcritical set \( P(R) = \{ i, i-1, -i \} \) consists of three points.

Let us compute \( L^*_{MM} g^M \) for \( g^M \in \mathcal{A}(\xi) \), where \( G^M_c \in \mathcal{M}_c \) and \( g^M_c(\xi) = G^M_c[-\chi^R_5] \) for \( \xi \in \xi_c \).

For \( \xi \in \gamma_5 \) with \( p \neq 0 \), \( p \in P(R) \)
\[ (L^*_{MM} G^M_c)(3) = \left[ \left( L^*_{\xi_5} G^M_c \right)[-\chi^R_5] \right] \cdot \left[ G^M_c[-\chi^R_5] \right] \]
\[ = \left[ G^M_c[-\chi^R_5(\xi_5) \cdot R(\xi_5) \cdot \chi^R_5(\xi_5)] \right] \cdot \left[ G^M_c[-\chi^R_5] \right] \]

Here, as \( p \neq 0 \), \( R(\xi) \in \xi_c \), \( \chi^R_5(\xi_5) \in \mathcal{M}_c \). And as \( \left[ \chi^R_5(\xi_5) \cdot R(\xi_5) \cdot \chi^R_5(\xi_5) \right] \) belongs to \( \mathcal{M}_c \), we have
\[ (L^*_{MM} G^M_c)(3) = \left[ G^M_c[-\chi^R_5(\xi_5) \cdot R(\xi_5) \cdot \chi^R_5(\xi_5)] \right] \cdot \left[ G^M_c[-\chi^R_5] \right] \]
\[ = \left[ \frac{1}{2\pi i} \int_{\gamma_5} G^M_c(\tau) \cdot \chi^R_5(\tau) \cdot R(\xi) \cdot \chi^R_5(\tau) \cdot d\tau \right] \cdot \left[ G^M_c[-\chi^R_5] \right] \]
\[ = \left[ R(\xi) \frac{1}{2\pi i} \int_{\gamma_5} G^M_c(\tau) \cdot \chi^R_5(\tau) \cdot \chi^R_5(\tau) \cdot d\tau \right] \cdot \left[ G^M_c[-\chi^R_5] \right] \]
\[ = \left[ R(\xi) \int_{\gamma_5} \left[ g^M_c \cdot \Delta \chi^R_5 \right] \cdot \left[ R(\xi) \right] \right] \cdot \left[ G^M_c[-\chi^R_5] \right] \]
\[ = (R' \cdot \chi^R_5 \left[ g^M_c \cdot \Delta \chi^R_5 \right] \cdot \left[ R(\xi) \right] \).

For \( \xi \in \gamma_6 \),
\[ (L^*_{MM} G^M_c)(3) = \left[ \left( L^*_{\xi_6} G^M_c \right)[-\chi^R_6] \right] \cdot \left[ G^M_c[-\chi^R_6] \right] \]
\[ = \left[ G^M_c[-\chi^R_6(\xi_6) \cdot \chi^R_5(\xi_6)] \right] \cdot \left[ G^M_c[-\chi^R_6] \right] \]
\[ = \left[ R(\xi) \frac{1}{2\pi i} \int_{\gamma_6} G^M_c(\tau) \cdot \chi^R_6(\tau) \cdot \chi^R_5(\tau) \cdot d\tau \right] \cdot \left[ G^M_c[-\chi^R_6] \right] \]
\[ = \left[ R(\xi) \int_{\gamma_6} \left[ g^M_c \cdot \Delta \chi^R_6 \right] \cdot \left[ R(\xi) \right] \right] \cdot \left[ G^M_c[-\chi^R_6] \right]. \]
\[
\left[ R(\beta) \frac{-1}{2\pi i} \int_{\hat{c}} g^M_c(\tau) \psi_\delta(\tau) \frac{d\tau}{\tau - R(\beta)} \right]^{M_0} \\
= [ R(\beta) \{ \psi_\delta \psi^M_\delta \ Circ \ R(\beta) \}]^{M_0}
\]

where \( \{ \psi_\delta \psi^M_\delta \} \) denotes the regular part of \( \psi_\delta \psi^M_\delta \) along \( \delta \).

\[\int_{\hat{c}} \psi_\delta \psi^M_\delta \left( \tau \right) \psi_\delta (\tau) \psi^M_\delta (\tau) \frac{d\tau}{\tau - \xi} \quad \text{for} \ \xi \in \hat{c}, \]

We have a decomposition

\[\psi_\delta \psi^M_\delta = \left[ \psi_\delta \psi^M_\delta \right] + \left[ \psi_\delta \psi^M_\delta \right] \]

with \( \left[ \psi_\delta \psi^M_\delta \right] \in M_c \) and \( \left[ \psi_\delta \psi^M_\delta \right] \in C(\hat{c}). \)

810. Complex conformal measures.

Let \( G \in \Omega^N \oplus M^* \oplus \Omega^N (F) \). Let \( A \subset C \) be an open set with smooth boundary \( \partial A \) (oriented by the counter-clockwise direction). The characteristic function \( \chi_A(z) \) of \( A \) is expressed as

\[\chi_A(z) = \frac{-1}{2\pi i} \int_{\partial A} \chi_\eta(\zeta) \frac{d\eta}{\zeta - \eta}\]

So, we can rewrite

\[\chi_A = \frac{-1}{2\pi i} \int_{\partial A} \chi_\eta \frac{d\eta}{\zeta - \eta}\]

Hence,

\[G[\chi_A] = \frac{-1}{2\pi i} \int_{\partial A} G[\chi_\eta] \frac{d\eta}{\zeta - \eta}\]

defines a set function. If \( G = g^J + g^M + g^F \), then

\[G[\chi_\eta] = g^J + g^M + g^F\]

with \( g^J \in \Omega^N (F) \), \( g^M \in M \) \( g^F \in \Omega^N (F) \), and

\[G[\chi_A] = \frac{-1}{2\pi i} \int_{\partial A} \left( g^{J(\eta)} + g^{M(\eta)} + g^{F(\eta)} \right) \frac{d\eta}{\zeta - \eta}\]

defines an additive set function. Suppose \( \lambda \) be a characteristic value of our transfer operator \( \mathcal{L} \) and let \( f \in \mathcal{X} = \Omega^N \oplus M \oplus \Omega^N (F) \) be an eigenfunction...
of \( \mathcal{L}_f \) for singular value \( \lambda \), i.e., \( \lambda \mathcal{L}_f f = f \). And let \( G \in \mathcal{H} = (C^0(I) \oplus M^\ast \oplus G^*(E)) \) be the co-eigenfunctional of \( \mathcal{L}_f \) for \( \lambda \), i.e., \( \lambda \mathcal{L}_f G = G \), with \( G(z) = G \mathcal{L}_f^{-1} X_k \), \( z \in C^0(I) \oplus M \oplus G^*(E) \).

Define a set function \( \mu \mathcal{A} \) by

\[
\mu \mathcal{A}(A) = \frac{1}{2\pi i} \int_{\partial A} s(t) g(t) \, dt.
\]

Then, we have

\[
\mu \mathcal{A}(R(A)) = \frac{1}{2\pi i} \int_{\partial R^{-1}(A)} s(z) g(z) \, dz
\]

\[
= \frac{1}{2\pi i} \int_{\partial R^{-1}(A)} s(z) \lambda (\mathcal{L}_f g)(z) \, dz
\]

\[
= \frac{1}{2\pi i} \int_{\partial R^{-1}(A)} \lambda f(z) R(z) \mathcal{L}_f(R(z)) g(R(z)) \, dz.
\]

Then by a change of variables \( \zeta = R(z) \) with \( d\zeta = R'(z) \, dz \), we have

\[
\mu \mathcal{A}(R(A)) = \frac{1}{2\pi i} \int_{\partial A} \lambda \mathcal{L}_f \mathcal{A}(\zeta) \cdot g(\zeta) \sum_{z \in \mathcal{L}_f^{-1}(A)} f(z) \, d\zeta.
\]

\[
= \mu \mathcal{A}(A),
\]

where we used \( \mathcal{L}_f = \mathcal{L}_f R \cdot R \) and \( \mathcal{L}_f g = R^{-1}(\mathcal{L}_f^{-1} g) \cdot R \).

Our set function \( \mu \mathcal{A} \) is backward invariant under \( R \).

Finally, we consider the pull-back of the set function defined by the co-eigenfunction \( g \). Suppose \( \mathcal{L}_f g = g \), then, for \( A \) with \( R^{-1} : A \rightarrow R(A) \) injective, we have

\[
\mu \mathcal{A}(R(A)) = \frac{1}{2\pi i} \int_{\partial R(A)} g(\zeta) \, d\zeta = \frac{1}{2\pi i} \int_{\partial R(A)} g(R(z)) R'(z) \, dz
\]

\[
= \frac{1}{2\pi i} \int_{\partial A} \mathcal{L}_f (R(\zeta)^{-1} \cdot R'(\zeta) \cdot \mathcal{L}_f R(\zeta) \cdot g(R(z))) \, d\zeta.
\]

This shows a kind of complex conformal property of the set function \( \mu \mathcal{A} \) for co-invariant function \( g \).