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Microfunctions and a transfer operator
for complex dynamical systems.

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§1. Functions with regular singularities

Let us begin with the following situation. Let
\[ R(z) = z^2 + c \]
be a quadratic polynomial on the Riemann sphere. We assume that the complex dynamical system defined by this quadratic polynomial is postcritically finite, i.e., the forward orbit \( f^m(c); \ n = 1, 2, \ldots \) of the critical point 0 of \( R(z) \) is a finite set. For the sake of simplicity, we denote by \( F \) the Fatou set of \( R(z) \) and by \( J \) the Julia set \( \mathbb{C} \setminus F \). In order to illustrate the situation, we consider especially the case \( c = i \). Then the critical point is 0 and its forward orbit is \( 0, i, -1, -i \) and \( R(c) \). We denote by \( \alpha \) and \( \beta \) the two fixed points, which are so-called the \( \alpha \)-fixed point and the \( \beta \)-fixed point.
As \( R(\mathbb{C}) \) is postcritically finite, there exists an external ray, say \( \gamma_c \), landing at the critical value \( c \). We give an orientation to this curve as \( \infty \to c \).

Note that this external ray is spiralling near \( c \).

Let \( \Omega_c = \mathbb{C} \setminus (\gamma_c \cup f(\gamma_c)) \) be a domain in the complex plane. This domain \( \Omega_c \) has smooth boundaries along the external ray \( \gamma_c \).

We consider an abstract "neighborhood" \( \bar{\Omega}_c \) of \( \Omega_c \) doubly sheeted near \( \gamma_c \).

The domain \( \bar{\Omega}_c \) has two smooth curves on the boundary. We add two curves \( \gamma_c^+ \) and \( \gamma_c^- \) to this domain to each side of the external ray \( \gamma_c \), and we denote this set by \( \bar{\Omega}_c \).

\( \bar{\Omega}_c \) is an open set containing \( \Omega_c \). For domain \( \Omega \), we denote by \( \mathcal{O}(\Omega) \) the set of holomorphic functions on \( \Omega \).

Let \( f: \Omega_c \to \mathbb{C} \) be a holomorphic function on \( \Omega_c \) which can be extended holomorphically to some "neighborhood" \( \bar{\Omega}_c \) of \( \bar{\Omega}_c \). Such a function \( f \) is said to be equivalent to \( g: \Omega_c \to \mathbb{C} \) which is a holomorphic function on \( \Omega_c \) and extendable to some "neighborhood" \( \bar{\Omega}_c \) of \( \bar{\Omega}_c \) holomorphically, if there exists a "neighborhood" \( \bar{\Omega}_c' \) of \( \bar{\Omega}_c \) such that \( f \) and \( g \) coincide on \( \bar{\Omega}_c' \). This equivalence relation defines a concept of germ. Note that \( f: \Omega_c \to \mathbb{C} \) itself gives a representative of its germ since analytic continuation is unique if it exists. We call such function \( f \) a (general)
premicrofunction along \( \infty \). A (general) microfunction at \( c \) along \( \infty \) is defined by an equivalence class of germs of (general) pre-microfunctions \( f : \Omega_c \to \mathbb{C} \) at \( c \) modulo germs of holomorphic functions at \( c \).

More precisely, (general) pre-microfunctions \( f : \Omega_c \to \mathbb{C} \) and \( g : \Omega_c \to \mathbb{C} \) define the same microfunction at \( c \) if there exists an open neighborhood \( U \) of \( c \) in the complex plane \( \mathbb{C} \) and a holomorphic function \( h : U \to \mathbb{C} \) such that \( f(z) - g(z) = h(z) \) holds for \( z \in U \cap \Omega_c \).

The above definition of (general) microfunction is so general that the singularities of such functions at \( c \) are too much complicated. So, we restrict our singularities to "regular singularities" defined as follows.

**Definition 1.1.** Pre-microfunction \( f : \Omega_c \to \mathbb{C} \) is said to have a regular singularity at \( c \) if there exist positive numbers \( \varepsilon \) and \( \kappa \) such that inequality

\[
|f(z)| < \kappa |z - c|^{-1+\varepsilon}
\]

holds near \( c \).

**Definition 1.2.** Pre-microfunction \( f : \Omega_c \to \mathbb{C} \) is said to have a regular singularity at \( \infty \) if there exist positive numbers \( \varepsilon \) and \( \kappa \) such that inequality

\[
|f(z)| < \kappa |z|^{-1-\varepsilon}
\]

holds near the infinity.

We denote by \( M_c \) the set of pre-microfunctions along \( \infty \) with regular singularities both at \( c \) and \( \infty \). More precisely, we denote \( M_c \) instead of \( M_c \) when there are more than one external rays landing at \( c \). The space of equivalence classes of germs of pre-microfunctions with regular singularities at \( c \) modulo the space of germs of holomorphic functions \( O(\infty) \) at \( c \), will be denoted by \( M_c \).

Let \( P(R) \) denote the postcritical set. For each point
p ∈ P(R), the space of pre-microfunctions along its external rays with regular singularities at both points is defined in a similar manner and will be denoted by \( M_p \). For simplicity, and by \( M_p \) when it is necessary to indicate the external ray.

For \( p \in \mathcal{J} \) with multiple external rays, say \( \gamma_1, \ldots, \gamma_r \), landing at \( p \), we define the space \( M_p \) by the direct sum

\[
M_p = \bigoplus_{k=1}^{r} M_{\gamma_k}^k
\]

where the sum is taken as a formal sum, since each component belongs to different spaces. However, each element of \( M_p \) defines a function holomorphic in the intersection of the domains of definitions and the decomposition of a holomorphic function

\[
f: \mathbb{C} \setminus (\bigcup_{k=1}^{r} U_{\gamma_k^i}) \to \mathbb{C}
\]

defined by an element \( f_k \) of \( M_p \) into components \( f_k \) in \( M_{\gamma_k}^k \),

\[
f_k: \mathbb{C} \setminus (\gamma_k \cup \{p\}) \to \mathbb{C}
\]

is unique since we are considering the pre-microfunctions with regular singularities at the infinity. We denote

\[
M_+ = \bigoplus_{p \in \mathcal{P}(R)} M_p
\]

\[
M_0 = M_{\gamma_0^+} \oplus M_{\gamma_0^-}
\]

\[
M_- = \bigoplus_{k=1}^{\infty} \bigoplus_{p \in \mathcal{P}(R_{\gamma_k})} M_p
\]

and

\[
M = M_+ \oplus M_0 \oplus M_-
\]

Here, the origin 0 is the critical point of our quadratic map \( R(z) \) and there are two external rays landing at 0, which are pre-images of the external ray \( \Gamma_0 \). \( \Gamma_0^+ \) and \( \Gamma_0^- \) denotes the external angles \( \frac{1}{12} \) and \( \frac{7}{12} \), respectively. Note that \( \Gamma_0 \) is the external angle \( \frac{7}{12} \), since it is mapped to period two cycle of external rays with angles \( \frac{7}{12} \) and \( \frac{7}{12} \).

Here, the infinite direct sum is only in a formal sense.
§2 Difference operator and an exact sequence.

Let $O(\delta_c)$ denote the space of holomorphic functions in a neighborhood of the external ray $\delta_c$. An element of $O(\delta_c)$ is represented by a continuous function $f: \delta_c \to \mathbb{C}$ which can be extended to some neighborhood of $\delta_c$ holomorphically. The space of holomorphic functions along the external ray $\delta_c$ with regular singularities at both $c$ and the infinity is defined by the following.

$$O_0(\delta_c) = \{ f \in O(\delta_c) \mid \exists \varepsilon > 0, \exists k > 0, \exists \text{nbd of } \delta_c \text{ s.t.} \}
\begin{align*}
&|f(z)| < k|z-c|^{-1+\varepsilon} \text{ near } c \\
&|f(z)| < k|z|^{-\varepsilon} \text{ near } \infty
\end{align*}$$

Now, we define a difference operator along an external ray.

**Definition 2.1** Difference operator $\Delta_c : M_c \to O_0(\delta_c)$ is defined by the difference of boundary values along $\delta_c$

$$\Delta_c f(z) = f(z) - f((z-c)e^{-2\pi i} + c).$$

Here $z \in \delta_c$ is considered as a point in the boundary of $\Omega_c$ of the clockwise side, and $(z-c)e^{-2\pi i} + c$ represents the same point but considered as a point in the boundary of $\Omega_c$ of the counterclockwise side.

For each $p \in J$ and its external ray $\delta_p$, difference operator $\Delta_p : M_p \to O_0(\delta_p)$ is defined in a similar way.

We denote $\Delta_p$ instead of $\Delta_p$ if there are more than two external rays landing at $p$ and we need to indicate it.

**Remark** Difference operator can be defined for functions holomorphic along $\delta_c$ in a doubly sheeted domains. The domain of definition of such a holomorphic function need not be connected.

Let us fix a double sheeted neighborhood $\Omega_c$ of our domain $\Omega_c = \mathbb{C} \setminus (\delta_c \cup \delta_c^*)$, and let $\delta_c'$ denote the neighborhood of $\delta_c$ where $\Omega_c$ is double sheeted.
Theorem 2.2  The following sequence is exact.

\[ 0 \to \mathcal{O}(C \setminus \{c\}) \to \mathcal{O}(S_c) \xrightarrow{d_c} \mathcal{O}(S_c) \to 0. \]

Proof. We gave an orientation to the external ray \( S_c \)

defining an order to the points in \( S_c \) so that \( \infty < p < c \).

Take points \( S_j, S_{j+1} \in S_c \) for \( j \in \mathbb{Z} \) ordered along \( S_c \) as

\[ \infty < \ldots < S_{j+1} < S_j < S_{j+2} < \ldots < c \]

and

\[ \lim_{k \to -\infty} S_{j+1} = \lim_{k \to -\infty} S_{j} = \infty, \]

\[ \lim_{k \to +\infty} S_{j+1} = \lim_{k \to +\infty} S_{j} = c. \]

Then open arcs \( \overline{S_j S_{j+1}} \) form an open covering of the external ray \( S_c \). The space \( \mathcal{O}(C \setminus \{c\}) \) of holomorphic functions on \( \mathbb{C} \setminus \{c\} \) can be injectively embedded in the space of holomorphic functions on \( S_c \) and the values of such a function on the two sheets of \( S_c \) coincide on the overlapped sector \( S_c \), hence the difference of these values vanishes. So, we need only to prove the ontoess of the difference operator \( d_c \). For \( \varphi \in \mathcal{O}(S_c) \), we want to construct a holomorphic function in \( \mathcal{O}(C \setminus \{c\}) \). Note that such a function is not unique since the kernel of \( d_c \) contains \( \mathcal{O}(C \setminus \{c\}) \).

Let

\[ F_j(z) = \frac{1}{2\pi i} \int_{S_{j+1}}^{S_j} \frac{\varphi(t)}{t-z} \, dt \]

for \( j \in \mathbb{Z} \). Such integration is called a Cousin's integral

along the arc \( \overline{S_{j+1} S_j} \). Note that this arc includes the arc \( \overline{S_j S_{j+1}} \) in its interior. The function \( F_j(z) \)

is holomorphic in \( C \setminus (\overline{S_{j+1} S_j} \cup \overline{S_j S_{j+1}}) \).

By deforming the path of integration of the Cousin's integral we see that \( F_j(z) \) can

be holomorphically extended beyond the arc from both sides into the other sides, except...
at $r_{j+1}$ and $r_{j+1}$. Next let us take a family of annuli in a lexicographically ordered way with smooth boundaries as follows. We take annulus $B_j$ for each $j \in \mathbb{Z}$ so that the intersection of $B_j$ with the external ray $r_{\infty}$ is the arc $t_{j+1}r_j$ and $r_j^{-1}r_{j+1}$ belong to the outer and inner boundary of $B_j$ respectively. Furthermore, we for $j, k \in \mathbb{Z}$, $B_j \cap B_k$ is empty if $|j-k| > 1$, hold, and for each $j \in \mathbb{Z}$, $B_j \cap B_{j+1}$ is an annulus. We impose that

$$\bigcup_{j \in \mathbb{Z}} B_j = \mathbb{C} \setminus \{c\}$$

For each $j$, we denote by $\tilde{B}_j$ a covering of $B_j$ such that $\tilde{B}_j$ covers twice on the sector $S_c \cap \tilde{B}_j$. An function $F_j(z)$ defined by Cousin's integration can be extended holomorphically to $\tilde{B}_j$. It is further extendable to a wider domain $\tilde{B}_{j-1} \cup \tilde{B}_j \cup \tilde{B}_{j+1}$. Hence $F_j(z)$ is bounded in $\tilde{B}_j$. As in easily verified by considering the integration, we have

$$F_j(z) - F_j ((z-c)e^{-2\pi i} + c) = \Phi(z)$$

for $z \in S_c \cap \tilde{B}_j$.

For $j, k \in \mathbb{Z}$ with $B_j \cap B_k = \emptyset$, define a holomorphic function

$$H_{jk} : \tilde{B}_j \cap \tilde{B}_k \rightarrow \mathbb{C}$$

by

$$H_{jk}(z) = F_j(z) - F_k(z).$$

$F_j(z)$ and $F_k(z)$ are holomorphic on $\tilde{B}_j \cap \tilde{B}_k$. But, as we have

$$F_j((z-c)e^{-2\pi i} + c) - F_k((z-c)e^{-2\pi i} + c)$$

$$= F_j(z) - F_k(z)$$
along $S_c$, \( H_{j,k} ((z-c) e^{2\pi i c} + c) = H_{j,k}(z) \) holds on
\( S_c \cap B_{j+1} \setminus B_{j} \), so that \( H_{j,k}(z) \) is well defined and
holomorphic on the annulus \( B_{j+1} \setminus B_{j} \). This family
of holomorphic functions \( \{ H_{j,k} \} \) forms a "Cousin data",
\( \text{i.e. for } i,j \in \mathbb{Z}, \)
\[
H_{j+1} + H_{j} + H_{j-1} = 0 \quad \text{on } B_{j} \cap B_{j+1} \cap B_{j-1}.
\]

As we assumed \( B_{j+1} \setminus B_{j} = \emptyset \) if \( |\overline{z} - z| > 1 \), this above fact is
easily verified.

For each \( j \in \mathbb{Z} \), take a loop \( \gamma_j \) in \( B_{j} \cap B_{j+1} \), making
a clockwise turn once and define \( \delta_j(z) \) and \( \kappa_j(z) \) by
\[
\delta_j(z) = \frac{1}{2\pi i} \int_{\gamma_j} \frac{H_{j+1}(t)}{t-z} \, dt
\]
defined and holomorphic in \( U \cup B_{-\infty} \cup B_{+\infty} \) (outside of
the annulus), and
\[
\kappa_j(z) = \frac{1}{2\pi i} \int_{\gamma_j} \frac{H_{j+1}(t)}{t-z} \, dt
\]
define are holomorphic in \( \bigcup_{k=\infty} B_{k} \cup \mathfrak{F} \cup \Gamma \) (inside of
the annulus).

By defining the integration path we see that they are
well defined and we have
\[
H_{j,j+1}(z) = \delta_j(z) - \kappa_{j+1}(z) \quad (z \in B_{j} \cap B_{j+1}).
\]

By Runge's theorem, \( \delta_j : \bigcup_{k=\infty} B_{k} \cup \mathfrak{F} \cup \Gamma \to \mathbb{C} \)
can be approximated by polynomials in the sense
of uniform convergence on compact sets, and
\( \delta_j : \bigcup_{k=\infty} B_{k} \cup \mathfrak{F} \cup \Gamma \) can be approximated by rational functions
with poles only at \( \infty \).

For each \( j \geq 0 \), find a rational function \( \varphi_j : \mathfrak{R} \to \mathbb{C} \)
such that
\[
|\varphi_j(z) - \delta_j(z)| < \frac{1}{2^{2j}} \quad \text{for } z \in \bigcup_{k=\infty} B_{k} \cup \mathfrak{F} \cup \Gamma.
\]

And for each \( j < 0 \), find a polynomial
\( \varphi_j : \mathbb{C} \to \mathbb{C} \) such that
\[
|\varphi_j(z) - \delta_j(z)| < \frac{1}{2^{2j}} \quad \text{for } z \in \bigcup_{k=\infty} B_{k} \cup \mathfrak{F} \cup \Gamma.
\]

Note that these functions \( \varphi_j \) are all holomorphic
in \( \mathbb{C} \setminus \mathfrak{F} \).
Let \( \widetilde{h}_d = h_d - g_d : \bigcup_{i=0}^{\infty} B_{d_i} \cup c \to \mathbb{C} \) for \( d \geq 0 \), and \( \widetilde{h}_{d+1} = h_{d+1} - g_{d+1} : \bigcup_{i=0}^{\infty} B_{d_i} \cup c \to \mathbb{C} \) for \( d < 0 \). Then we have
\[
|\widetilde{h}_d| < \frac{1}{2^d} \quad (d \geq 0)
\]
and
\[
|\widetilde{h}_{d+1}| < \frac{1}{2^{d+1}} \quad (d < 0).
\]
We still have
\[
H_{d+1}(z) = \widetilde{h}_d(z) - \widetilde{h}_{d+1}(z) \quad \text{for } z \in B_d \cap B_{d+1}.
\]
Now, we set
\[
H_d(z) = -\sum_{i=0}^{d} \widetilde{h}_i(z) - \sum_{i=d+1}^{\infty} \widetilde{h}_i(z).
\]
For \( i \leq d \), \( \widetilde{h}_i(z) \) is holomorphic in \( \bigcup_{i=0}^{\infty} B_{d_i} \cup c \), hence they are all holomorphic in the \( d \)-smallest disk \( \bigcup_{i=0}^{\infty} B_{d_i} \cup c \), and that we have the estimate of the supremum of the functions, the sum of \( \widetilde{h}_i \)'s is uniformly convergent on \( B_d \).
Similarly, the sum of \( \widetilde{h}_i \)'s converge uniformly convergent on \( B_{d+1} \), too. Hence \( H_d(z) \) is holomorphic in \( B_d \).
In the overlapping annulus \( B_d \cap B_{d+1} \), we have
\[
H_{d+1} - H_d = -\sum_{i=0}^{d} \widetilde{h}_i - \sum_{i=d+1}^{\infty} \widetilde{h}_i + \sum_{i=0}^{d} \widetilde{h}_i = \widetilde{h}_d - \widetilde{h}_{d+1} = H_{d+1}.
\]
Finally, in \( B_d \), let \( G_d(z) = H_d(z) + F_d(z) \). These functions \( \{G_d\} \) on \( B_d \) defines a holomorphic function
\[
G : D_c \to \mathbb{C}
\]
define on the overlapped neighborhood \( D_c \) of \( D_c \). We can verify that these functions coincide and \( G \) is well defined by an immediate calculations as follows.
In \( B_d \cap B_{d+1} \),
\[
G_{d+1}(z) = H_{d+1}(z) + F_{d+1}(z) = H_d(z) + H_{d+1}(z) + F_{d+1}(z)
= H_d(z) + F_d(z) - F_{d+1}(z) + F_{d+1}(z) = H_d(z) + F_d(z) = G_d(z).
\]
Thus, we conclude that \( G \in \mathcal{O}(\mathbb{C}^c) \) and
\[
\Delta_c G = \varphi
\]
holds. This completes the proof of our Theorem 2.2.

We remark that such a function \( G \) satisfying \( \Delta_c G = \varphi \)
is not unique since \( \text{Ker} \Delta_c = \mathcal{O}(\mathbb{C} \setminus \mathbb{C}^c) \).

§3. Cousin's integral operator and decomposition of pro-microfunctions.

In the previous section, we discussed the surjectivity of the difference operator \( \Delta_c \). In this section, we restrict the space of (general) pro-microfunctions to the space of pro-microfunctions with regular singularities, and consider an inverse operator of \( \Delta_c \), which we call a Cousin's integral operator.

**Definition 3.1** \( I_c : \mathcal{O}_c(\mathbb{C}^c) \to M_c \) is defined by
\[
I_c[\varphi](z) = \frac{1}{2\pi i} \oint_{\mathbb{C}^c} \frac{\varphi(t)}{t-z} \, dt
\]
for \( \varphi \in \mathcal{O}_c(\mathbb{C}^c) \).

Here, we use notation \( I_c[\varphi] \) as \( I_c : \mathcal{O}_c(\mathbb{C}^c) \to M_c \) is an operator and we want to emphasize it, i.e. the argument of the operator is a function and not its value.

**Definition 3.2** Let \( f \) be bi-valued function defined in a neighborhood of \( \mathbb{C}^c \), both of the two branches are holomorphic and the difference \( \Delta_c f \) of \( f \) has regular singularities at \( c \) and at \( \infty \), i.e. \( \Delta_c f \in \mathcal{O}_c(\mathbb{C}^c) \). The \( M_c \) component of \( f \) is defined as
\[
[f]_c = [f]_{\mathbb{C}^c} = I_c[\Delta_c f].
\]
This mapping \([f]_c \) is a projection map onto \( M_c \).
We have the following identities.

**Theorem 3.3** \( I_c \circ D_c = \text{id} \) on \( \mathcal{M}_c \),
\[ \Delta_c \circ I_c = \text{id} \] on \( \mathcal{O}_c(\mathcal{V}_c) \).

**Proof.** These identities are easily verified.

For each point \( p \in \mathcal{F} \) (and an external ray \( \Gamma_p \) landing at \( p \)), projection \( \mathcal{F} \to \mathcal{F}_p \) is similarly defined.

Let \( \mathcal{O}_c(\mathcal{V}_c) \) denote the space of holomorphic functions \( f : \mathcal{V}_c \to \mathbb{C} \) such that \( f \) can be extended holomorphically to some double-sheeted neighborhood \( \mathcal{V}_c \) and satisfies \( \Delta f \in \mathcal{O}_c(\mathcal{V}_c) \). Function \( f \in \mathcal{O}_c(\mathcal{V}_c) \) is holomorphic in \( \mathcal{V}_c = \mathbb{C} \setminus (\mathcal{V}_c \cup \{\infty\}) \) and has singularities at \( c \) and at the infinity together with its difference along \( \mathcal{V}_c \).

Let \( \mathcal{V}_c \) denote the space of hyperfunctions supported at \( c \), i.e., \( \mathcal{V}_c \in \mathcal{K}_c \) if and only if \( f \) is holomorphic in \( (\mathbb{C} \cup \{0\}) \setminus \{c\} \). The space of entire functions is denoted by \( \mathcal{O}(\mathbb{C}) \). Let us define the operators that extract singularities of \( f \).

**Definition 3.4.** Operator \( \Gamma_c : \mathcal{O}_c(\mathcal{V}_c) \to \mathcal{V}_c \) is defined by
\[
\Gamma_c[f](z) = \frac{1}{2\pi i} \int_{|z-c| = \varepsilon} \frac{f(t)}{t-z} \, dt
- \frac{1}{2\pi i} \int_{\mathcal{V}_c} \frac{(\Delta f)(t)}{t-z} \, dt
\]
where \( z \in \mathbb{C} \setminus \{c\} \), \( \varepsilon > 0 \) is chosen sufficiently small so that the \( \varepsilon \)-ball around \( c \) does not contain \( z \), and \( \mathcal{V}_c \in \mathcal{K}_c \) is the intersection point of \( \mathcal{V}_c \) and the circle \( |z-c| = \varepsilon \). The orientation of the path of integration along the circle is the counter-clockwise with respect to \( z \).
As $\Delta f$ has a regular singularity at $c$, this defines a holomorphic function on $(\mathbb{C}^* \setminus \{0\}) \setminus \{c\}$. That is, $\Gamma_c[f] \in \mathcal{H}_c$.

**Definition 3.5.** Operator $\Gamma_{\infty} : \mathcal{O}_0(\mathbb{C}) \to \mathcal{O}(\mathbb{C})$ is defined by

$$\Gamma_{\infty}[f](z) = \frac{1}{2\pi i} \oint_{|c| = \infty} \frac{f(t)}{t - z} \, dt + \frac{1}{2\pi i} \int_{R_0} \frac{f(t)}{t - z} \, dt$$

when $w > 0$ is taken sufficiently large for each $z$, so that the circle of integration path surrounds $z$, and $R_0 \in \mathbb{R}$ is the intersection point of $R_c$ and the big circle. The orientation is taken as the counterclockwise with respect to $z$.

As $\Delta f$ has a regular singularity at $\infty$, this defines an entire function. Hence $\Gamma_{\infty}[f] \in \mathcal{O}(\mathbb{C})$.

Just for the sake of notation, we define

$$\Gamma_M : \mathcal{O}_0(\mathbb{C}) \to \mathcal{M}_c$$

by $\Gamma_M[f] = [f]_c$. We have the following decomposition.

**Theorem 3.6.** $\mathcal{O}_0(\mathbb{C}) = \mathcal{H}_c \oplus \mathcal{M}_c \oplus \mathcal{O}(\mathbb{C})$

and $\Gamma_c : \mathcal{O}_0(\mathbb{C}) \to \mathcal{H}_c$, $\Gamma_M : \mathcal{O}_0(\mathbb{C}) \to \mathcal{M}_c$, $\Gamma_{\infty} : \mathcal{O}_0(\mathbb{C}) \to \mathcal{O}(\mathbb{C})$ gives the projections to components.

**Proof.** Clearly, the kernel of the difference operator $\Delta_c$ is $\mathcal{O}(\mathbb{C} \setminus \{c\})$ and $\mathcal{O}(\mathbb{C} \setminus \{c\}) = \mathcal{H}_c \oplus \mathcal{O}(\mathbb{C})$.

Note that these operators can be defined if $f$ is defined and holomorphic in a double wounded neighborhood of $\mathbb{C}_c$. In this case, $\Gamma_c + \Gamma_M + \Gamma_{\infty}$ defines a projection to $\mathcal{O}_0(\mathbb{C}) = \mathcal{H}_c \oplus \mathcal{M}_c \oplus \mathcal{O}(\mathbb{C})$ if $\Delta_c f \in \mathcal{O}_0(\mathbb{C})$. 
§4 Space of pre-microfunctions and a transfer operator

Let us go back to our complex dynamical system $R(\mathcal{E})$. Let $J$ denote the Julia set of $R(\mathcal{E})$ and let $F$ denote the Fatou set of $R(\mathcal{E})$. We suppose $R(\mathcal{E})$ is postcritically finite, especially the case of $c = -i$. We denote by $\mathcal{O}(J)$ the space of germs of continuous functions $f: J \to \mathbb{C}$ which are holomorphic in some neighborhood of $J$. The space of holomorphic functions $f: F \to \mathbb{C}$ of the Fatou set satisfying $f(c) = 0$ will be denoted by $\mathcal{O}_0(F)$. The postcritical set of $R(\mathcal{E})$ is denoted by $\mathcal{P}(R)$. The space of pre-microfunctions at $\mathcal{P}(R)$ is defined by

$$M_{\mathcal{P}(R)} = \bigoplus_{P \in \mathcal{P}(R)} M_P,$$

and

$$M = M_{\mathcal{P}(R)} \bigoplus \bigoplus_{k=0}^{\infty} M_{P_k},$$

denotes the space of formal sum of pre-microfunctions at the graph orbit of the critical point $0$.

Let $f \in \mathcal{O}(J) \oplus M_{\mathcal{P}(R)} \oplus \mathcal{O}_0(F)$ and $P \in \mathcal{P}(R)$ with $P_o$ its external ray. Then the $M_P$-component $[f]_P$ of $f$ is given by a projection

$$[f]_P(z) = \frac{1}{2\pi i} \int_{P_o} \frac{A_P f(z)}{z - z} \, dz.$$

Let us consider the most simple postcritically finite case (except $c = -2$ case) of $R(\mathcal{E}) = \mathbb{C}^2$. Fixed points of $R$ are denoted by $\alpha$ and $\beta$. The preimage of the external ray $R$ consists of two external rays, say $R^+$ and $R^-$, of the critical point $0$, with external angles $\frac{\pi}{12}$ and $\frac{\pi}{12}$ respectively. These external rays are oriented as $\infty \to 0$. Let $U_0$ denote the upper connected component of $\mathbb{C} \setminus (\infty \cup \{0\})$ which contains the critical value $c = -i$. The $\alpha$-fixed point belongs to this domain. We denote the other connected points by $T_P$. It contains the $\beta$-fixed point. The quadratic map $R$ is of degree two. The critical value $c$ is a branch point. We denote the two branches of $R^{-1}$ by $Q_\alpha$ and $Q_\beta$ defined in $\mathbb{C} = \mathbb{C} \setminus (\infty \cup \{0\})$. 
$Q_\alpha : \mathbb{C} \setminus (\gamma_c U \cup \gamma) \rightarrow U_\alpha$
$Q_\beta : \mathbb{C} \setminus (\gamma_c U \cup \gamma) \rightarrow U_\beta$

with $Q_\alpha(z) = -\sqrt{z-c}$, $Q_\beta(z) = \sqrt{z-c}$, where the branch of the square root is chosen by assigning $Q_\beta(c+1) = 1$. If we regard $Q_\alpha$ and $Q_\beta$ as holomorphic functions on $\Omega_c$, we can naturally consider holomorphic functions $(Q_\alpha)^s$ and $(Q_\beta)^s$ for $0 < s < 2$. They can be extended to a double-sheeted neighborhood $52i$ holomorphically. We define a holomorphic function

$$\gamma_s(z) = \frac{1}{(2Q_\beta(z))^s}$$

defined in $\Omega_c$.

For $z = c + r e^{i\theta} \in \Omega_c$, we have

$$(\Delta_c \gamma_s)(z) = \gamma_s(z) - \gamma_s((z-c) e^{-2\pi i} + c)$$

$$= \frac{1 - e^{5\pi i}}{(2 \sqrt{r} e^{\frac{5\pi i}})^s}$$

Hence $|\Delta_c \gamma_s| = \text{const.} r^{-\frac{s}{2}}$.

which implies $\Delta_c \gamma_s$ has regular singularities at $c$ and $\infty$ if $0 < s < 2$. Therefore $\gamma_s \in M_c$.

Now, take a function $f \in \mathcal{O}(\Omega) \oplus \mathcal{N}_{E(R)} \oplus \mathcal{C}(F)$. Here, we abuse the formal sum of functions in different spaces and the sum as a function defined in the common domain of definition. So, $f$ is defined and holomorphic in $F \setminus (\bigcup_{p} \gamma_p)$. For an external ray $\gamma_p$, we denote by
\[ \gamma_p \text{ the path of integration coming from } \infty \text{ to } p \text{ along } \gamma_p \text{ taking the clockwise side value of the integrand function and going back from } p \text{ to } \infty \text{ along } \gamma_p \text{ taking the counterclockwise side value of the integrand function. That is,} \\
\int_{\gamma_p} f(z) \, dz = \int_{\gamma_p} f(z) \, dz - \int_{\gamma_p} f((z-p)e^{2\pi i} + p) \, dz = \int_{\gamma_p} \partial_p f(z) \, dz. \]

By \( \gamma_F \) we represent an integration path along the boundary of the Fatou set, passing near the Julia set and taking values of the function on the Fatou set. And finally by \( \gamma_J \) we represent an integration path running around the Julia set in the counterclockwise direction.

Let \( \mathcal{N} = \mathcal{U}(C) \oplus \mathcal{N}(R) \oplus \mathcal{U}_+(F). \)

**Definition 4.1** Transfer operator \( \mathcal{L}_S: \mathcal{N} \rightarrow \mathcal{N} \) is defined for \( 0 < s < 2 \) and for \( f \in \mathcal{N} \) as:

\[ (\mathcal{L}_S f)(z) = \sum_{\gamma \in \mathcal{R}_n} \gamma_S(R(\gamma z)) f(\gamma). \]

We can rewrite the transfer operator in an integral operator form as:

\[ (\mathcal{L}_S f)(z) = \frac{1}{2\pi i} \int_{\gamma_F + \gamma_R + \gamma_J} \frac{\psi(R(t)) f(t)}{R(t) - z} \, dt. \]

and as:

\[ (\mathcal{L}_S f)(z) = \gamma_S(z) (f \circ Q_\infty(z) + f \circ Q_\infty(z)). \]

**Definition 4.2** Push forward operator \( R \) is defined by

\[ R_x f = f \circ Q_\infty + f \circ Q_\infty. \]
For point \( \gamma \in \mathbb{C} \), we denote by \( X_\gamma(z) = \frac{1}{z - \gamma} \) the unit pole at \( \gamma \). If \( \gamma \in \mathbb{J} \) then \( X_\gamma \in \mathcal{O}_\mathbb{C}(\mathbb{C}) \). If \( \gamma \in \mathbb{R}_p \) then \( X_\gamma \in \mathcal{O}_\mathbb{C}(\mathbb{R}_p) \). Note that \( X_\gamma^{-1}(z) = X_{\gamma}(-z) \).

Proposition 4.3 For \( \gamma \in \mathbb{J} \cup \mathbb{R}_p \),

\[
\mathcal{L}_\gamma X_\gamma = R(\gamma) X_\gamma(\mathbb{R}_p) X_{\gamma}(\mathbb{R}_p) + R(\gamma) \left[ \gamma_\gamma \mathcal{X}_{\gamma} \right] \mathcal{C}.
\]

Proof. By a direct computation, we have

\[
R(\gamma)(z) = \gamma_\gamma(\gamma) (X_\gamma \circ \mathcal{O}_\mathbb{C}(\mathbb{C})) \gamma_\gamma(\gamma) = \gamma_\gamma(\gamma) \sum_{\gamma \in \mathbb{R}_p} \frac{1}{\gamma - \gamma}.
\]

By the residue formula, we have

\[
o = \frac{1}{2\pi i} \int_{\gamma} \frac{R(\gamma)}{(z - R(\gamma))(y - \gamma)} \, dy = \sum_{\gamma \in \mathbb{R}_p} \frac{1}{y - \gamma} + \frac{R(\gamma)}{z - R(\gamma)}.
\]

Hence

\[
\left( \mathcal{L}_\gamma X_\gamma \right)(z) = \gamma_\gamma(\gamma) \frac{R(\gamma)}{z - R(\gamma)} = \gamma_\gamma(\gamma) R(\gamma) X_{\gamma}(\mathbb{R}_p)(z)
\]

\[
= \gamma_\gamma(\gamma) R(\gamma) X_{\gamma}(\mathbb{R}_p)(z) + \left[ R(\gamma) \gamma_\gamma \mathcal{X}_{\gamma} \right] \mathcal{C}.
\]

Here \( \left[ R(\gamma) \gamma_\gamma \mathcal{X}_{\gamma} \right] \mathcal{C} = (\gamma_\gamma(\gamma) - \gamma_\gamma(\gamma)) R(\gamma) X_{\gamma}(\mathbb{R}_p)(z) \) is holomorphic near \( \mathbb{R}_p \) so that it belongs to \( \mathcal{M}_\mathbb{C} \). The first term \( \gamma_\gamma(\gamma) R(\gamma) X_{\gamma}(\mathbb{R}_p)(z) \) is a multiple of unit pole at \( R(\gamma) \).

Unit poles \( X_\gamma \gamma \in \mathbb{J} \) form a basis of function space \( \mathcal{O}(\mathbb{C}) \) and the family of unit poles \( \mathcal{X}_\gamma \gamma \in \mathbb{R}_p \) form a basis of space of pre-inner functions \( \mathcal{M}_\mathbb{C} \). This splitting of the transfer operator \( \mathcal{L}_\gamma \) gives a decomposition of the operator into components of the direct sum decomposition of function spaces.

Theorem 4.4 For \( \mathbf{p} \in \mathbb{J} \) and \( \mathbf{g} \in \mathcal{M}_\mathbb{C} \) we have the following decomposition

\[
\mathcal{L}_\gamma g = \left[ \mathbf{g} \circ \mathcal{O}_\mathbb{C} \gamma_\gamma \right] \mathcal{R}_\mathbb{C} + \left[ \gamma_\gamma \mathcal{X}_\gamma \mathbf{g} \circ \mathcal{O}_\mathbb{C} \right] \mathcal{R}_\mathbb{C} \mathcal{C}
\]

where \( \mathcal{O}_\mathbb{C} = \mathcal{O}_\mathbb{C} \) or \( \mathcal{O}_\mathbb{R} \) according to \( \mathbf{p} \in \mathcal{T}_\mathbb{C} \) or \( \mathcal{T}_\mathbb{R} \).
Proof. As \( q \in M_p \), \( g(z) \) can be expressed in an integration of Cauchy type for \( z \in \Omega_p = \mathbb{C} \setminus (\delta_p \cup \Gamma_p) \) as

\[
    g(z) = \frac{1}{2\pi i} \int_{\delta_p} (\delta q)(t) \frac{dt}{t-z} = \frac{1}{2\pi i} \int_{\Gamma_p} [\delta q](t) (-\gamma^*_q(t)) dt.
\]

Hence, we have

\[
    \mathcal{L}_s g)(z) = -\frac{1}{2\pi i} \int_{\delta_p} (\delta q)(\gamma) (\nu_s X_q)(2) d\gamma
    \]

\[
    = -\frac{1}{2\pi i} \int_{\delta_p} \left( \int_{\mathbb{C}} (2 \gamma) \gamma \cdot \left( \gamma_s \cdot R(\gamma) \right) X_{R(\gamma)}\right) d\gamma
    \]

\[
    = -\int_{\delta_p} \left( \int_{\mathbb{C}} \gamma_s \cdot R(\gamma) \cdot X_{R(\gamma)}\right) d\gamma
    \]

\[
    = -\int_{\delta_p} \left( \int_{\mathbb{C}} \gamma_s \cdot R(\gamma) \cdot X_{R(\gamma)}\right) d\gamma
    \]

\[
    = \left[ \int_{\delta_p} \gamma_s \right] R(\gamma) + \left[ \gamma_s \cdot \left[ q \circ \delta_q \right] R(\gamma) \right] \mathcal{L}_s.
\]

Note that in the above calculations, \( \eta \in \delta_p \) is a variable along \( \delta_p \) and we changed variables by \( \delta = R(\eta) \) and \( d\delta = R(\eta) d\eta \).

This theorem shows that the transfer operator \( \mathcal{L}_s \) sends \( M_p \) into \( M_{R(\delta)} \oplus M_c \), for \( p \in \Delta \). In the case of postcritically finite complex dynamical system case, the space of pre-microfunctions for postcritical set is invariant for \( \mathcal{L}_s \), i.e.

Proposition. \( \mathcal{L}_s(M_+) \subseteq M_+ \).

§5. Decomposition of the transfer operator

Our space of pre-microfunctions \( H_+ \) has a direct sum decomposition

\[
    H_+ = \mathcal{L}(C) \oplus M_+ \oplus C'(F)
\]

and the transfer operator \( \mathcal{L}_s \) maps this space into itself.

We denote the components of \( \mathcal{L}_s \) in a matrix form as

\[
    \mathcal{L}_s = \begin{pmatrix}
    L_{T J} & L_{T M} & L_{T F} \\
    L_{M J} & L_{M M} & L_{M F} \\
    L_{P J} & L_{P M} & L_{P F}
    \end{pmatrix}
\]
For \( f_j \in \Omega(\mathbb{C}) \) we have the following proposition.

**Proposition 5.1**

\[
L_j f_j = \gamma_j \cdot R_j f_j = L_{tJ} f_j + \lambda_{tJ} f_j + \lambda_{rJ} f_j
\]

\[
L_{tJ} f_j = \gamma_j \cdot R_j f_j - \left[ \gamma_j \cdot R_j f_j \right]_c,
\]

\[
\lambda_{tJ} f_j = \left[ \gamma_j \cdot R_j f_j \right]_c,
\]

\[
\lambda_{rJ} f_j = 0.
\]

**Proof.** As \( 0 < s < 2 \), we have

\[
\lim_{\delta \to 0} \frac{1}{2\pi i} \oint_{C_\delta} \frac{\gamma_k(R(t)) \cdot R(t) f_k(t)}{R(t) - z} \, dt = 0,
\]

where \( C_\delta \) is the circle of radius \( \delta \) around the origin \( 0 \), since the singularity at the origin is of order \( 1 - s \). In the next calculation, integration paths are as explained in the previous section. We have

\[
(L_{tJ} f_j)(\delta) = \frac{1}{2\pi i} \oint_{C_\delta} \frac{\gamma_k(R(t)) \cdot R(t) f_k(t)}{R(t) - z} \, dt = 0.
\]

\[
= \frac{1}{2\pi i} \oint_{C_\delta} \frac{\gamma_k(R(t)) \cdot R(t) f_k(t)}{R(t) - z} \, dt = 0.
\]

\[
= \gamma_k(R(t)) \cdot R(t) f_k(t) = \left[ \gamma_k(R(t)) \cdot R(t) f_k(t) \right]_c
\]

Here we made a change of variables \( \sigma \), \( \sigma = R(t) \), and \( d\sigma = R(t) \, dt \). \( Q^+ : \gamma_j \to \gamma_j^+ \) and \( Q^- : \gamma_j \to \gamma_j^- \) denotes two inverse branches of \( R \) along \( \gamma_j \). \( R_j f_j = f_j \cdot Q_j^+ + f_j \cdot Q_j^- \) defines along \( \gamma_j \) and \( R_j f_j = f_j \cdot Q_j^+ + f_j \cdot Q_j^- \) elsewhere. We continue the calculation.

\[
(L_{tJ} f_j)(\delta) = \gamma_k(R(t)) \cdot R(t) f_k(t) - \left[ \gamma_k(R(t)) \cdot R(t) f_k(t) \right]_c
\]

This completes the first line. Note that \( L_{tJ} f_j = \gamma_j \cdot R_j f_j \) has singularities along \( \gamma_j \) only. Next, compute the component \( \lambda_{tJ} f_j \) as follows.
\[ (L_{Mf} f_J)(z) = \frac{1}{2\pi i} \int_{C_M} \frac{\psi_s(R(t)R(t) f_J(t))}{R(t) - z} \, dt \]

and
\[ (L_{Mf} f_J)(z) = \frac{1}{2\pi i} \int_{C_F} \frac{\psi_s(R(t)R(t) f_J(t))}{R(t) - z} \, dt \]

\[ = \frac{1}{2\pi i} \int_{C_{ClO}} \frac{\psi_s(\sigma)(R_E f_J(\sigma))}{\sigma - z} \, d\sigma \]

\[ = 0 \]

Since
\[ \left| L_{Mf} f_J \right| \leq \frac{2\pi p}{2\pi} \frac{|2p|^{-5}}{|12 - c| - \rho} \rightarrow 0 \quad \text{(as } p \to 0) \]

For the second column of the operator matrix, we have the following.

**Proposition 5.2.** For \( f_M \in M_p \subset M_+ \quad (p \neq 0) \),

\[ L_{Mf} f_M = 0 \]

\[ L_{Mf} f_M = [\psi_s \cdot R_E f_M]_c + [\psi_s \cdot f_M^{\circ} \varphi_p]_{R(p)} \]

\[ L_{Mf} f_M = 0 \]

**Remark:** \( L_{s} f_M \in M_c \oplus M_{R(p)} \) if \( f_M \in M_p \)

**Proof.** As \( f_M \in M_p \), there exists a positive number \( \varepsilon \) and a positive constant \( K \) such that

\[ \left| f_M(z) \right| < K |z|^{-\varepsilon} \]

holds near the infinity. Hence we have

\[ \left| L_{Mf} f_M(z) \right| \leq \frac{1}{2\pi} \int_{C_{Mf}} \frac{|\psi_s(R(t)R(t) f_M(t))|}{|R(t) - z|} \, dt \]

\[ \leq \frac{2\pi |C_{Mf}|}{2\pi} \frac{1}{2^5 |(z)^2| + \frac{c}{3}} |z|^{-3} \]

\[ \leq \text{const.} \quad |z|^{-5-\varepsilon} \quad \to 0 \quad (|z| \to \infty) \]

and

\[ L_{Mf} f_M = 0 \]

Next, we show that \( L_{Mf} = 0 \) in the following.
\( (L_{FM} f_M)(z) = \frac{1}{2\pi i} \int_{\Gamma_0 - z} \frac{\gamma_s(R(\tau)) R(\tau) f_M(\tau)}{R(\tau) - z} \, d\tau \)

\[
= \frac{1}{2\pi i} \left( \int_{\Gamma_0 - z} + \int_{\Gamma_1 = \rho} \int_{|\tau| = \rho} \right) \frac{\gamma_s(R(\tau)) R(\tau) f_M(\tau)}{R(\tau) - z} \, d\tau.
\]

This goes to zero as \( p \to 0 \).

For, \( \gamma_s \) belongs to \( M_0 \) so that the second term vanishes and the first term vanishes since \( f_M \) belongs to \( M_0 \). Note that in this computation, \( z \) is taken from the Fatou set and the integration path \( \Gamma_F \) runs near the Julia set. Note that this argument cannot be applied if \( p = 0 \) since the integrand might have a singularity at \( p \) which is not regular. We need regularity of the singular points to have this kind of integral vanish. Finally,

\[
(L_{MN} f_M)(z) = \frac{1}{2\pi i} \int_{\Gamma_0 - z} \frac{\gamma_s(R(\tau) R(\tau) f_M(\tau)}{R(\tau) - z} \, d\tau
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma_0 - z} \frac{\gamma_s(R(\tau) R(\tau) f_M(\tau)}{R(\tau) - z} \, d\tau + \frac{1}{2\pi i} \int_{\Gamma_1 = \rho} \frac{\gamma_s(R(\tau) R(\tau) f_M(\tau)}{R(\tau) - z} \, d\tau
\]

\[
= I_{\gamma_s \cdot R_X f_M} + [\gamma_s \cdot R_X f_M]_{R(p)}.
\]

This completes the proof of Proposition 5.2.

**Remark.** If \( f_M \in M_0 \), then the integrand may have a non-regular singularity since we have a product of two regular singularities at \( p = 0 \). Hence,

\[
L_{FM} f_M = [\gamma_s \cdot R_X f_M]_{E}
\]

\[
L_{MN} f_M = [\gamma_s \cdot R_X f_M]_{C}
\]

\[
L_{JM} f_M = [\gamma_s \cdot R_X f_M]_{J}.
\]

For \( f_F \in \mathcal{O}_S(F) \), we have the following

**Proposition 5.3**

\[
L_{TF} f_F = 0
\]

\[
L_{MT} f_F = [\gamma_s \cdot R_X f_F]_{C}
\]

\[
L_{TF} f_F = \gamma_s \cdot R_X f_F - [\gamma_s \cdot R_X f_F]_{C}
\]
Proof} As $f_F \in \mathcal{O}_b(F)$, we have an estimate $|f_F(t)| < k |t|^{-1}$ near $\infty$. Hence
$$
|\mathcal{L}_F f_F(t)| \leq \frac{2\pi |t|}{2\pi} \frac{12 \pi |t|^{-5} \cdot 2 \pi |t|^{-1}}{1 + \frac{2\pi |t|^{-1}}{\pi}} \to 0 \quad (as \quad t \to \infty)
$$
Therefore, we have $\mathcal{L}_F f_F = 0$. For the second component,
$$
\mathcal{L}_M f_F = I_c \{ \mathcal{Y}_c \cdot R_y f_F \} = [\mathcal{Y}_c \cdot R_y f_F]_c.
$$
And the third component is computed similarly.
$$
(\mathcal{L}_F f_F)(t) = \frac{1}{2\pi i} \int_{\gamma_F} \frac{\mathcal{Y}_c(R(t)) \cdot R_y f_F(t)}{R(t) - z} \, dz
$$
$$
= \frac{1}{2\pi i} \int_{\gamma_F} \frac{\mathcal{Y}_c(\sigma) \cdot (R_y f_F)(\sigma)}{\sigma - z} \, d\sigma
$$
$$
= \mathcal{Y}_c(\sigma)(R_y f_F)(\sigma) - [\mathcal{Y}_c \cdot R_y f_F]_c.
$$
Putting the above propositions together, we have the decomposition of our transfer operator into components.

Note that if $f_T$ or $f_F$ are not identically zero, then $[\mathcal{Y}_c \cdot R_y f_F]_c \neq 0$ or $[\mathcal{Y}_c \cdot R_y f_F]_c \neq 0$. Hence, there is no eigen-function in subspace $\mathcal{O}(\mathcal{O}) \oplus \mathcal{O}_b(F)$.

§6. Invariant subspace of the transfer operator

Our transfer operator $\mathcal{T}_c : M_c \to M_c$ maps the space of pre-minimun functions supported on the forward orbit of the critical point. For $h_0 \in M_c$, $\mathcal{T}_c h_0$ can be written in a form of Cauchy integral.
$$
\mathcal{T}_c h_0(t) = \frac{1}{2\pi i} \int_{\gamma_c} (\Lambda_c h_0)(\sigma) \mathcal{Y}_c(\sigma) \, d\sigma = \frac{1}{2\pi i} \int_{\gamma_c} (\Lambda_c h_0)(\sigma) \mathcal{Y}_c(\sigma) \, d\sigma.
$$
In the following, we denote as $\mathcal{Y}_c \cdot h(t) = \mathcal{Y}_c(R(t)) \cdot h(R(t))$.
where $R_k^t(\omega) = R^t \circ R_k^0(\omega)$ denote the $k$-th iteration of $R^t(\omega)$.

Note that $\gamma_k \circ R_k^t \in M_{R_k^t(\gamma_k)}$, and that $\gamma_k \circ R_k^t$ are regular on $\gamma_k(\omega)$.

For each $k = 1, 2, \ldots$, we consider a pre-microfunction $g_k$ on $\phi_{R_k^t(\gamma_k)}$ expressed in terms of a pre-microfunction $g_{k-1}$ on $\nu_{R_k^t(\gamma_k)}$.

For $\phi_k \in \phi_{R_k^t(\gamma_k)}$, let

$$g_k(\phi_k) = g_{k-1}(\phi_k) \cdot \gamma_k(\phi_k)$$

and define $h_k \in M_{R_k^t(\gamma_k)}$ by

$$h_k(\phi_k) = \frac{-1}{2\pi i} \int_{\gamma_k} \left( \Delta_{R_k^t(\gamma_k)}(\phi_k) R_k^t(\gamma_k) \right) \gamma_k(\phi_k) d\gamma_k.$$

Let $Q_k = (R_k|\gamma_k)^{-1} : \gamma_k(\phi_k) \to \gamma_k$ be the inverse branch of $R_k$. By a change of variables $\sigma = R_k^t(t) = R_k(t) dt$ and $t = Q_k(\sigma)$, we have

$$h_k(\phi_k) = \frac{-1}{2\pi i} \int_{\gamma_k} \left( \Delta_{R_k^t(\gamma_k)}(\phi_k) \left[ G_k \circ Q_k \right] \right)(\sigma) \gamma_k(\phi_k) d\sigma.$$

This implies $h_k = [G_k \circ Q_k]_{R_k^t(\gamma_k)}$ and $h_k \in M_{R_k^t(\gamma_k)}$.

We have

$$\Delta_{R_k^t(\gamma_k)} = \left( \Delta_{R_k^t(\gamma_k)}(\phi_k) \right) \circ R_k^t \circ Q_k.$$

So, this correspondence induces an isomorphism $M_{R_k^t(\gamma_k)} \cong M_{\gamma_k(\nu_{R_k^t(\gamma_k)})}$. As $h_k \in M_{R_k^t(\gamma_k)}$, we have $\nu_{h_k} \in M_{\gamma_k(\nu_{R_k^t(\gamma_k)})} \oplus M_{\nu_{R_k^t(\gamma_k)}}$.

More precisely, we have the following explicit formula.

**Proposition 6.1** If $h_k = [G_k \circ Q_k]_{R_k^t(\gamma_k)} \in M_{R_k^t(\gamma_k)}$ with $G$ as above, we have the following decomposition.

$$\nu_{h_k} = \left[ G_k \circ Q_k \cdot \gamma_k \right]_{R_k^t(\gamma_k)} + \left[ \gamma_k \cdot G_k \circ Q_k \right]_{R_k^t(\gamma_k)}.$$

**Proof.** This is immediately verified by applying Theorem 4.4. By an immediate calculation, we can obtain the proof as follows. First component of $\nu_{h_k}$ is given by

$$\left[ G_k \circ Q_k \cdot \gamma_k \right]_{R_k^t(\gamma_k)} = \left[ \frac{-1}{2\pi i} \int_{\gamma_k} \left( \Delta_{R_k^t(\gamma_k)}(\phi_k) \left[ G_k \circ Q_k \right] \right)(\sigma) \gamma_k(\phi_k) d\sigma \right]_{R_k^t(\gamma_k)}.$$

$$= \left[ \frac{-1}{2\pi i} \int_{\gamma_k} \left( \Delta_{R_k^t(\gamma_k)}(\phi_k) \left[ G_k \circ Q_k \right] \right)(\sigma) \nu_{\phi_k}(\sigma) d\sigma \right]_{R_k^t(\gamma_k)}.$$
\[
= \frac{1}{2\pi i} \int_{\gamma_{R_0}(c)} (\Delta_{R_0(c)} (G_{x \circ Q_{k_0}}))(\sigma) \left[ L_{x} \chi_{R_0}(c) \right] d\sigma
\]

\[
= \frac{1}{2\pi i} \int_{\gamma_{R_0}(c)} (\Delta_{R_0(c)} (G_{x \circ Q_{k_0}}))(\sigma) \chi_{R_0}(c) R(\sigma) \chi_{R_0}(c) d\sigma
\]

\[
= \frac{1}{2\pi i} \int_{\gamma_{R_0}(c)} (\Delta_{R_0(c)} (G_{x \circ Q_{k_0}}))(\sigma) \chi_{R_0}(c) \chi_{R_0}(c) \chi_{R_0}(c) d\sigma
\]

\[
= [ \chi_{x \circ Q_{k_0}} ] R_{0}(c) \chi_{R_0}(c)
\]

where we made change of variables \( \rho = R(\sigma) \), \( d\rho = R'(\sigma) d\sigma \). Similarly, the second component is computed as follows.

\[
[L_{x} \chi_{R_0}(c)] = \frac{1}{2\pi i} \int_{\gamma_{R_0}(c)} (\Delta_{R_0(c)} (G_{x \circ Q_{k_0}}))(\sigma) \left[ L_{x} \chi_{R_0}(c) \right] d\sigma
\]

\[
= \frac{1}{2\pi i} \int_{\gamma_{R_0}(c)} (\Delta_{R_0(c)} (G_{x \circ Q_{k_0}}))(\sigma) R(\sigma) \chi_{R_0}(c) \chi_{R_0}(c) d\sigma
\]

\[
= \frac{1}{2\pi i} \int_{\gamma_{R_0}(c)} (\Delta_{R_0(c)} (G_{x \circ Q_{k_0}}))(\sigma) \chi_{R_0}(c) \chi_{R_0}(c) \chi_{R_0}(c) d\sigma
\]

\[
= [ \chi_{x \circ Q_{k_0}} ] R_{0}(c) \chi_{R_0}(c)
\]

\[
= [ \chi_{x \circ Q_{k_0}} ] R_{0}(c) \chi_{R_0}(c)
\]

8.7 Eigenvalue problem

In this section, we consider the eigenvalue problem for our transfer operator \( L_{x} \) restricted to an invariant subspace of pre-microfunctions \( M_{P}(r) \) defined in section 4 as

\[
M_{P}(r) = \sum_{k=0}^{\infty} M_{R_{k}(c)}
\]

Here, the sum is considered an formal sum. In the case of postcritically finite maps, the post critical set \( P(r) \) is a finite set and the sum is finite. In this case

\[
M_{P}(r) = \bigoplus_{P \in P(r)} M_{P}
\]
In order to analyze the eigenvalue problem of $L_5$, we consider a formal sum of pre-microfunctions.

\[ \hbar = \sum_{k=0}^{\infty} \hbar_k \quad \text{with} \quad \hbar_k \in \mathcal{M}_5(k) \]

**Proposition 7.1.** If $h$ is an eigenfunction of $L_5$ satisfying

\[ \lambda L_5 h = h \quad \text{and} \quad P(R) \text{ is infinite}, \]

then

\[ \lambda \left[ \mathcal{L}_5 \hbar_k \right]_{R_{5t_1}(c)} = h_{k+1} \quad \text{and} \quad \lambda \sum_{k=0}^{\infty} [L_5 \hbar_k]_c = h_0. \]

**Proof.** By a straightforward calculation, we have

\[ L_5 \hbar_k = [L_5 \hbar_k]_{R_{5t_1}(c)} + \mathcal{L}_5 \hbar_k I_c. \]

**Theorem 7.2.** The eigenvalue problem \( \lambda L_5 h = h \) of our transfer operator \( L_5 : \mathcal{M}_5(0) \rightarrow \mathcal{M}_5(R) \) reduces to an "eigenvalue" problem of an integral operator

\[ T_5 : \mathcal{C}_0 (Y_c) \rightarrow \mathcal{C}_0 (Y_c) \]

defined by

\[ (T_5 \phi)(u) = (\Delta_c \gamma_s \psi_5 \psi_5 u) \cdot \frac{1}{2\pi i} \int_{Y_c} H_5 (u, \cdot; \lambda) \psi(t) \, dt \]

where

\[ H_5 (u, t; \lambda) = - \sum_{k=0}^{\infty} \lambda^k \gamma_s \psi_5 \psi_5 (t) R_{5t_1}^+ (t) \psi R_{5t_1} (u), \]

with \( \lambda T_5 h_0 = h_0 \quad \text{and} \quad h_0 = \Delta_c h_0 \).

**Proof.** As \( R_{5t_1} = \lambda \left[ L_5 \hbar_k \right]_{R_{5t_1}(c)} = \lambda \left[ \mathcal{L}_5 \hbar_k \psi_5 \psi_5 \right]_{R_{5t_1}(c)} \]

\[ = \lambda \left[ \gamma_s \psi_5 \psi_5 \cdot \gamma_s \psi_5 \psi_5 \cdot \gamma_s \psi_5 \psi_5 \cdot \gamma_s \psi_5 \psi_5 \cdot \gamma_s \psi_5 \psi_5 \right]_{R_{5t_1}(c)}, \]

and

\[ h_{5t_1} = [G_{5t_1} \circ Q_{5t_1}]_{R_{5t_1}(c)} = [g_{5t_1} \circ Q_{5t_1} \cdot \gamma_s \psi_5 \psi_5 \cdot \gamma_s \psi_5 \psi_5]_{R_{5t_1}(c)}, \]

we have

\[ g_{5t_1} = \lambda \gamma_s \psi_5 \psi_5 \quad \text{for} \quad k \geq 0. \]

Hence

\[ \hbar_k = \lambda^k h_0, \quad \text{which implies} \]

\[ \hbar_k = [G_{5t_1} \circ Q_{5t_1}]_{R_{5t_1}(c)} = [\lambda^k h_0 \circ Q_{5t_1} \cdot \gamma_s \psi_5 \psi_5 \cdot \gamma_s \psi_5 \psi_5]_{R_{5t_1}(c)} \]
and $h_0 = \lambda \sum_{k=0}^{\infty} \left[ \mathcal{P}_k \rho_k \right]_c = \lambda \sum_{k=0}^{\infty} \left[ \mathcal{P}_k \left[ G_{2k} \circ Q_{2k+1} \right] R_{2k+1}(c) \right]_c$

$= \lambda \sum_{k=0}^{\infty} \left[ \mathcal{P}_k \rho_k \left[ \mathcal{P}_k \circ Q_{2k+1} \right] \right]_c$

$= \lambda \sum_{k=0}^{\infty} \left[ \mathcal{P}_k \rho_k \left[ \mathcal{P}_k \circ Q_{2k+1} \right] \left[ \Delta_{R_{2k+1}}(c) \right]_c \left[ \mathcal{P}_k \circ Q_{2k+1} \right] \right]_c$

$= \lambda \sum_{k=0}^{\infty} \left[ \mathcal{P}_k \rho_k \left[ \mathcal{P}_k \circ Q_{2k+1} \right] \right] \left[ \Delta_{R_{2k+1}}(c) \right]_c \left[ \mathcal{P}_k \circ Q_{2k+1} \right]_c$

$= \lambda \sum_{k=0}^{\infty} \left[ \mathcal{P}_k \rho_k \left[ \mathcal{P}_k \circ Q_{2k+1} \right] \left[ \Delta_{R_{2k+1}}(c) \right]_c \left[ \mathcal{P}_k \circ Q_{2k+1} \right] \right]_c$

$\left( \text{Here we changed variables } \tau = R_{2k+1}(c) \text{ and } d\tau = R'_{2k+1}(c) d\tau \right)$

$= \frac{1}{2\pi i} \left( \mathcal{P}_k \rho_k \right) (\tau) \left[ \Delta_{R_{2k+1}}(c) \right] \left[ \mathcal{P}_k \circ Q_{2k+1} \right]_c$

This yields an integral equation for $h_0 \in \mathcal{N}$:

$\left( \Delta_{R_{2k+1}}(c) \right) \left( \mathcal{P}_k \rho_k \left[ \mathcal{P}_k \circ Q_{2k+1} \right] \right) = \lambda \left( \mathcal{P}_k \rho_k \right) (\tau) \left[ \Delta_{R_{2k+1}}(c) \right]_c \left[ \mathcal{P}_k \circ Q_{2k+1} \right]_c$

more briefly

$\left( \Delta_{R_{2k+1}}(c) \right) = \lambda \left( \mathcal{P}_k \rho_k \right) (\tau) \left[ \Delta_{R_{2k+1}}(c) \right]_c \left[ \mathcal{P}_k \circ Q_{2k+1} \right]_c$

by setting

$h_s(u, t; \lambda) = -\frac{1}{2\pi i} \lambda \sum_{k=0}^{\infty} \mathcal{P}_k \rho_k \left[ \mathcal{P}_k \circ Q_{2k+1} \right] \left[ \Delta_{R_{2k+1}}(c) \right]_c \left[ \mathcal{P}_k \circ Q_{2k+1} \right]_c$

88. Dual spaces and Cauchy's transformations.

Let $\mathcal{O}(f)$ denote the space of germs of holomorphic function along the Julia set $J = J(f)$. Each element of $\mathcal{O}(f)$ has a representative $f : J \rightarrow \mathbb{C}$ which can be extended to a holomorphic function in a neighborhood of $J$. As $J$ is a perfect set, the analytic continuation is uniquely determined by $f$. Topology in $\mathcal{O}(f)$ is given by the uniform convergence on $J$.

Linear functional $G^* : \mathcal{O}(f) \rightarrow \mathbb{C}$ is said to be holomorphic if for any holomorphic family $f_2 : \Lambda \rightarrow \mathcal{O}(f)$, $G^*[f_2] : \Lambda \rightarrow \mathbb{C}$ is holomorphic. We require the continuity of $G^*$ with respect to the sup norms on $\mathcal{O}(f)$. The space of continuous holomorphic linear functionals $G^* : \mathcal{O}(f) \rightarrow \mathbb{C}$ will be denoted by $\mathcal{O}^*(f)$. As in the previous sections, $\mathcal{C}_0(f)$ denotes the space of holomorphic functions in the Fatou set $F$ vanishing at the infinity.

If $p \in F$ then $\chi_p = \frac{1}{2-p}$ belongs to $\mathcal{O}(f)$. For holomorphic linear functional $G^* \in \mathcal{C}_0(f)$, define a holomorphic function $g^* \in \mathcal{C}_0(f)$ by $g^*(z) = G^*[\chi_z]$. Then, family
of holomorphic functions $F \rightarrow \mathcal{O}(F)$ defined by $z \mapsto z^2$

in a holomorphic family, $g^T : F \rightarrow \mathbb{C}$ is a holomorphic function, since the functional $\mathcal{G}^T$ is holomorphic. By the continuity of $g^T$, we see immediately $g^T(\infty) = 0$, and hence $g^T \in L_0(F)$.

This correspondence between $\mathcal{O}^*(F)$ and $L_0(F)$ is called the Cauchy transformation.

**Proposition 8.1.** For $f \in \mathcal{O}(F)$, $\mathcal{G}^T[f]$ can be expressed in a integration form
\[
\mathcal{G}^T[f] = \frac{-1}{2\pi i} \oint_{\gamma_F} f(z) g^T(\tau) \, dz,
\]
where the integration path $\gamma_F$ goes around the Julia set in the clockwise direction.

**Proof.** As $f$ is holomorphic near $I$, for $z$ in a neighborhood of $I$,
\[
f(z) = \frac{1}{2\pi i} \oint_{\gamma_I} f(\tau) \chi_2(\tau) \, d\tau = \frac{-1}{2\pi i} \oint_{\gamma_I} f(\tau) \chi_2(\tau) \, d\tau,
\]
where $\gamma_I$ runs around the Julia set in the counterclockwise direction. The right hand side of this equality gives an expression of $f(z)$ as a "linear combination" of unit poles. We have
\[
\mathcal{G}^T[f] = \frac{-1}{2\pi i} \oint_{\gamma_F} f(\tau) \mathcal{G}^T[\chi_2] \, d\tau = \frac{-1}{2\pi i} \oint_{\gamma_I} f(\tau) g^T(\tau) \, d\tau
\]
\[
= \frac{-1}{2\pi i} \oint_{\gamma_F} f(\tau) g^T(\tau) \, d\tau.
\]

In the following, we shall denote such pairings of functions as
\[
\langle g^T, f \rangle_F = \frac{1}{2\pi i} \oint_{\gamma_F} f(\tau) g^T(\tau) \, d\tau.
\]

**Proposition 8.2** $\mathcal{O}^*(F) \cong \mathcal{O}_0(F)$

**Proof.** The Cauchy transformation defines a complex linear map from $\mathcal{O}^*(F)$ to $\mathcal{O}_0(F)$, and the pairing along $\gamma_F$ defines a complex linear map from $\mathcal{O}_0(F)$ to $\mathcal{O}^*(F)$. These two transformation are mutually inverse.
Let $O_0^*(F)$ denote the space of holomorphic linear and continuous functional $G^F: O_0(F) \to \mathbb{C}$.

**Proposition 8.3** \( O(I) \subset O_0^*(F) \).

**Proof.** If \( z \in I \) then \( \varphi(z) \in O_0(F) \). For \( G^F \in O_0^*(F) \), set \( g^F(z) = G^F[\varphi(z)] \). Then \( g^F: I \to \mathbb{C} \) is a continuous function.

If \( g^F \in O(I) \), then there \( f_F \in O_0(F) \) with

\[
f_F(z) = \frac{1}{2\pi i} \int_{I_F} f_F(t) \varphi(t) \, dt = \frac{1}{2\pi i} \int_{I_F} f_F(t) \varphi(t) \, dt \]

we have

\[
G^F[f_F] = \frac{1}{2\pi i} \int_{I_F} f_F(t) G^F[\varphi(t)] \, dt = \frac{1}{2\pi i} \int_{I_F} f_F(t) g^F(t) \, dt
\]

where the integration path \( I_F \) is given by \( g^F \) which is holomorphic in a neighborhood of \( I \). This pairing will be denoted as \( \langle g^F, f_F \rangle_I \).

We define the third pairings \( \langle \cdot, \cdot \rangle_M \) and \( \langle \cdot, \cdot \rangle_M^* \) related to the external rays and pre-microfunctions. Let \( M \) denote the space of pre-microfunctions and let \( \gamma_M \) denote the "sum" of external rays supporting the pre-microfunctions. We use symbol \( M \) to indicate the object is related to the pre-microfunction component. When we apply operators \( \Delta M \), \( \Delta_M^* \), etc. we take the "sum" of the objects over external rays. The dual space \( M^* \) is the space of holomorphic linear and continuous functionals \( G^M: M \to \mathbb{C} \).

For \( \gamma_M \), we denote \( \tilde{I}_M \) the integration path passing along the external ray both sides of \( \gamma_M \) coming from the infinity to \( p \) in the negative side of \( \gamma_M \) and coming back from \( p \) to the infinity on the positive side of \( \gamma_M \). If \( g^M \in O_0(\tilde{I}_M) \), that is, \( g^M \) is a holomorphic function in a neighborhood of \( \gamma_M \) with regular singularities at the infinity and each landing points.
For \( f_m \in M \), we can rewrite it in the following form.

\[
\begin{align*}
  f_m(z) &= \frac{1}{2\pi i} \int_{\gamma_m} (\Delta_m f_m)(\tau) \chi_{\gamma}(\tau) d\tau
  = \frac{1}{2\pi i} \int_{\gamma_m} (\Delta_m f_m)(\tau) \chi_{\gamma}(\tau) d\tau
\end{align*}
\]

For \( g_m \in \mathcal{O}(\gamma_m) \), define a holomorphic functional \( G^M : M^* \to \mathbb{C} \) by

\[
G^M[g_m] = < g_m^M, \Delta_m f_m > M
\]

\( G^M[g_m] \) is defined if \( \Delta_m f_m, g_m \in \mathcal{L}_1(\gamma_m) \). Note that

\[
G^M[-\chi_z] = < g_m^M, -\chi_z > M = g_m^M(z)
\]

by Cauchy's integration formula. Note that if \( g_m \in \mathcal{O}(\gamma_m) \) and \( g_m^M = \mathcal{F}_m g_m \), then for \( z \in \mathcal{C} \setminus \gamma_m \)

\[
G^M[g_m] = < g_m^M, \chi_z > M = < g_m^M, \chi_z > M
\]

holds. If \( f_m \in M \), then

\[
G^M[f_m] = G^M\left[ \frac{1}{2\pi i} \int_{\gamma_m} (\Delta_m f_m)(\tau) (-\chi_{\gamma}(\tau)) d\tau \right] = \frac{1}{2\pi i} \int_{\gamma_m} (\Delta_m f_m)(\tau) G^M[-\chi_{\gamma}(\tau)] d\tau
\]

\[
= \frac{1}{2\pi i} \int_{\gamma_m} (\Delta_m f_m)(\tau) g_m^M(\tau) d\tau = < g_m^M, \Delta_m f_m > M.
\]

We have a splitting of pre-microfunctions,

\[
f = f_J \oplus f_M \oplus f_F \in \mathcal{O}_0(\mathbb{J}) \oplus M \oplus \mathcal{O}_0(\mathbb{F})
\]

and a splitting of its dual space

\[
G = G_J^* \oplus G^M \oplus G^F \in \mathcal{G}_0^*(\mathbb{J}) \oplus M^* \oplus \mathcal{G}_0^*(\mathbb{F}),
\]

with Cauchy's transforms given by

\[
G^J[-\chi_z] = g_J^J(z), \quad z \in \mathbb{F}, \quad g_J \in \mathcal{O}_0(\mathbb{F})
\]

\[
G^M[-\chi_z] = g_m^M(z), \quad z \in \gamma_m, \quad g_m \in \mathcal{O}(\gamma_m)
\]

\[
G^F[-\chi_z] = g_F^F(z), \quad z \in \mathbb{J}, \quad g_F \in \mathcal{O}(\mathbb{J})
\]

The pairing of \( f \) on \( G \) is defined by

\[
\]

\[
= < g_J^J, f_J > J + < g_m^M, \Delta_m f_m > M + < g_F^F, f_F > F.
\]

Projections of \( G = \mathcal{O}_0(\mathbb{J}) \oplus M \oplus \mathcal{O}_0(\mathbb{F}) \) to components are denoted by

\[
f \mapsto [f]_J, \quad f \mapsto [f]_M, \quad f \mapsto [f]_F
\]

respectively.
These projections are given by

\[
\begin{align*}
[f]_T(z) &= \frac{1}{2\pi i} \int_{\partial T} f(\tau) \chi_z(\tau) \, d\tau \quad (z \in T), \\
[f]_M(z) &= \frac{1}{2\pi i} \int_{\partial M} (\Delta_M f)(\tau) \chi_z(\tau) \, d\tau \quad (z \in \partial M), \\
[f]_E(z) &= \frac{1}{2\pi i} \int_{\partial E} f(\tau) \chi_z(\tau) \, d\tau \quad (z \in E).
\end{align*}
\]

And the projections of \( \mathcal{H}^* = C^0(T) \oplus M^* \oplus C_0^*(E) \) are denoted as

\[
\begin{align*}
J^T : \mathcal{H}^* &\rightarrow C_0^*(E) \subset C^0(T), \\
J^M : \mathcal{H}^* &\rightarrow C_0(R_M) \subset M^*, \\
J^E : \mathcal{H}^* &\rightarrow C_0(T) \subset C^0(E).
\end{align*}
\]

Let \( L_s^* \) denote the dual of our transfer operator \( L_s \). We abuse notations and confuse functionals and its Cauchy's transforms. Let \( C_0(E) \oplus C(\partial M) \oplus C(T) \geq \mathcal{L}^T_s \) is decomposed as

\[
\mathcal{L}_s^* = \begin{pmatrix}
\mathcal{L}_{sT}^* & \mathcal{L}_{sM}^* & \mathcal{L}_{sE}^* \\
\mathcal{L}^{*T}_{sM} & \mathcal{L}^{*M}_{sM} & \mathcal{L}^{*E}_{sM} \\
\mathcal{L}^{*T}_{sE} & \mathcal{L}^{*M}_{sE} & \mathcal{L}^{*E}_{sE}
\end{pmatrix}.
\]

In the rest of this section, we compute these components more explicitly.

**Proposition 8.4.**

\[
\begin{align*}
\mathcal{L}_{sT}^* g^T &= \gamma_s \circ R \cdot R^* \circ g^T \circ R - [\gamma_s \circ R \cdot R^* \circ g^T \circ R], \\
\mathcal{L}_{sM}^* g^T &= \Delta_c [\gamma_s \circ R \cdot R^* \circ g^T \circ R], \\
\mathcal{L}_{sE}^* g^T &= 0.
\end{align*}
\]

**Proof.** For \( g^T \in C_0(E) \), we compute \( (\mathcal{L}_{sT}^* g^T)(z) \) for \( z \in E \).

\[
\begin{align*}
(\mathcal{L}_{sT}^* g^T)(z) &= \left[ (\mathcal{L}_s^* G^T)[\chi_z] \right]^T \\
&= \left[ G^T[\chi_s - \chi_s(\gamma_{R(3)}) \cdot R(3) \chi_{R(3)} - \chi_s \cdot R' \chi_{R(3)}] \right]^T \\
&= \left[ G^T[\chi_s(\gamma_{R(3)}) \cdot R(3) \chi_{R(3)}] \right]^T \\
&= \left[ \gamma_s(\gamma_{R(3)}) \cdot R(3) G^T[\chi_{R(3)}] \right]^T \\
&= \gamma_s(\gamma_{R(3)}) \cdot R(3) G^T[\chi_{R(3)}] \\
&= \gamma_s(\gamma_{R(3)}) \cdot R(3) G^T[\chi_{R(3)}] \\
&= \gamma_s(\gamma_{R(3)}) \cdot R(3) G^T[\chi_{R(3)}].
\end{align*}
\]
Next, for $3 \in \mathcal{J}_1$, $L^x_{ij}g^j \in \mathcal{O}(\mathcal{V}_1)$ is limited as follows:

$$(L^x_{ij}g^j)(z) = \left[ \mathcal{L}^x_{ij}g^j \right][-\chi^T_z]$$

$$= \left[ g^j \left[ -\chi_z(R^{(3)}) \cdot R^{(3)} \cdot \chi_{R^{(3)}} - \chi_z \cdot R^{(3)} \cdot \chi_{R^{(3)}} \right] \right]$$

$$= \left[ \chi_z(R^{(3)}) \cdot R^{(3)} \cdot g^j \left[ -\chi_{R^{(3)}} \right] \right] = \left[ \chi_z(R^{(3)}) \cdot R^{(3)} \cdot g^j \right]$$

$$= \left[ \chi_z \cdot R^{(3)} \cdot g^j \right] \left[ -\chi_{R^{(3)}} \right] = \Delta_0 \left[ \chi_z \cdot R^{(3)} \cdot g^j \right].$$

For $3 \in \mathcal{J}$, we have

$$(L^x_{ij}g^j)(3) = \left[ \mathcal{L}^x_{ij}g^j \right][-\chi^T_3]$$

$$= \left[ g^j \left[ -\chi_3(R^{(3)}) \cdot R^{(3)} \cdot \chi_{R^{(3)}} - \chi_3 \cdot R^{(3)} \cdot \chi_{R^{(3)}} \right] \right]$$

$$= 0.$$

The last equality holds since $\chi_3(R^{(3)}) \cdot R^{(3)} \cdot \chi_{R^{(3)}} \in \mathcal{O}(F)$ and $\left[ \chi_3 \cdot R^{(3)} \cdot \chi_{R^{(3)}} \right] \in \mathcal{M}_c$.

**Proposition 8.5**

$$L^x_{ij}g^j = 0.$$

**Proof.** For $3 \in \mathcal{F}$, we have $-\chi_3 \in \mathcal{O}(T)$. For $g^j \in \mathcal{O}(T)$,

$$(L^x_{ij}g^j)(z) = \left[ \mathcal{L}^x_{ij}g^j \right][-\chi^T_3] = \left[ g^j \left[ -\chi_3 \right] \right]$$

$$= \left[ g^j \left[ -\chi_3(R^{(3)}) \cdot R^{(3)} \cdot \chi_{R^{(3)}} - \chi_3 \cdot R^{(3)} \cdot \chi_{R^{(3)}} \right] \right]$$

$$= \left[ \chi_3 \cdot R^{(3)} \cdot g^j \right].$$

Here, the last equality holds since $\chi_3(R^{(3)}) \cdot R^{(3)} \cdot \chi_{R^{(3)}} \in \mathcal{O}(T)$, $\left[ \chi_3 \cdot R^{(3)} \cdot \chi_{R^{(3)}} \right] \in \mathcal{M}_c$ and $g^j [-\chi_{R^{(3)}}] = 0$.

For $3 \in \mathcal{J}$, $-\chi_3$ belongs to $\mathcal{O}(F)$. Hence

$$(L^x_{ij}g^j)(3) = \left[ \mathcal{L}^x_{ij}g^j \right][-\chi^T_3] = \left[ g^j \left[ -\chi_3 \right] \right]$$

$$= \left[ g^j \left[ -\chi_3(R^{(3)}) \cdot R^{(3)} \cdot \chi_{R^{(3)}} - \chi_3 \cdot R^{(3)} \cdot \chi_{R^{(3)}} \right] \right]$$

$$= \left[ \chi_3 \cdot R^{(3)} \cdot g^j \right].$$

We used the fact $-\chi_3(R^{(3)}) \cdot R^{(3)} \cdot \chi_{R^{(3)}} \in \mathcal{O}(F)$ and $\left[ \chi_3 \cdot R^{(3)} \cdot \chi_{R^{(3)}} \right] \in \mathcal{M}_c$. 
To compute \( \hat{\phi}_{\text{FM}} g^M \), we take \( z \in \Gamma_M \subset F \). Then,
\[
\hat{\phi}_{\text{FM}} g^M(\chi) = \left[ \left( \hat{\phi}^* G^F \right) \left[ -X_z \right] \right]^M
\]
\[
= \left[ G^F \left[ -X_z (R(3)) \cdot R(3) \cdot X_{R(3)} \right] \right]^M
\]
\[
= \left[ X_z (R(3)) \cdot R(3) \cdot G^F \left[ -X_{R(3)} \right] \right]^M = \left[ \hat{\phi}_z (R(3)) \cdot R(3) \cdot g^F(\ell(3)) \right]^M
\]
\[
= \Delta_x \left[ \left[ -X_z \right] \cdot R \cdot g^F(R) \right](\chi).
\]

In the above calculations, we used the fact \( \left[ \hat{\phi}_z \cdot R(3) \cdot X_{R(3)} \right] \in M \).

During the computations, \( z \) is regarded as constant and the final result gives the formula as a function of \( z \).

**Proposition 8.6.**
\[
\hat{\phi}_{\text{FM}} g^M = \left[ \hat{\phi}_z \cdot R \cdot R' \cdot (I_M g^M) \cdot R \right]_F
\]
\[
\hat{\phi}_{\text{FM}} g^M = \left\{ \left[ \hat{\phi}_z \cdot R \cdot R' \cdot g^M \cdot R \right]^M + \left[ I_0 \left[ g^M \cdot R \cdot X z \right] \cdot R \right]^M \right\} \left( z \in V \right)
\]
\[
\hat{\phi}_{\text{FM}} g^M = \left[ \hat{\phi}_z \cdot R \cdot R' \cdot (I_M g^M) \cdot R \right]_F
\]

**Proof.** For \( g^M \in M^+ \), let \( g^M(\ell(2)) = G^M \left[ -X_\ell \right] \), \( \ell \in M \), \( g^M \in \mathcal{O}(M) \).

For \( z \in F \), (and \( z \in C \setminus M \)),
\[
\left( \hat{\phi}_{\text{FM}} g^M \right)(\ell(3)) = \left[ \left( \hat{\phi}^* G^F \right) \left[ -X_\ell \right] \right]^M = \left[ G^M \left[ -X_\ell \right] \right]^M
\]
\[
= \left[ G^M \left[ X_z (R(3)) \cdot R(3) \cdot X_{R(3)} \right] \right] + \left[ \hat{\phi}_z \cdot R(3) \cdot X_{R(3)} \right]_C
\]
\[
= \left[ X_z (R(3)) \cdot R(3) \cdot G^M \left[ X_{R(3)} \right] \right] + \left[ \hat{\phi}_z \cdot R(3) \cdot X_{R(3)} \right]_C
\]
\[
= \left[ X_z (R(3)) \cdot R(3) \cdot G^M \left[ X_{R(3)} \right] \right] + \left[ \hat{\phi}_z \cdot R(3) \cdot X_{R(3)} \right]_C
\]
\[
= \left[ X_z (R(3)) \cdot R(3) \cdot G^M \left[ X_{R(3)} \right] \right] + \frac{1}{2 \pi i} \int_{X_\ell} \frac{d \tau}{\tau - 3} \left[ g^M(\tau) \left( \Delta z \right) \right] R(3) \cdot X_{R(3)} \left( \tau - 3 \right) d \tau
\]
\[
= \left[ X_z (R(3)) \cdot R(3) \cdot G^M \left[ X_{R(3)} \right] \right] + \frac{1}{2 \pi i} \int_{X_\ell} \frac{d \tau}{\tau - 3} \left[ g^M(\tau) \left( \Delta z \right) \right] \cdot R(3) \cdot X_{R(3)} \left( \tau - 3 \right) d \tau
\]
\[
= \left[ X_z (R(3)) \cdot R(3) \cdot G^M \left[ X_{R(3)} \right] \right] + \frac{1}{2 \pi i} \int_{X_\ell} \frac{d \tau}{\tau - 3} \left[ g^M(\tau) \left( \Delta z \right) \right] \cdot R(3) \cdot X_{R(3)} \left( \tau - 3 \right) d \tau
\]
\[
= \left[ X_z (R(3)) \cdot R(3) \cdot G^M \left[ X_{R(3)} \right] \right] + \frac{1}{2 \pi i} \int_{X_\ell} \frac{d \tau}{\tau - 3} \left[ g^M(\tau) \left( \Delta z \right) \right] \cdot R(3) \cdot X_{R(3)} \left( \tau - 3 \right) d \tau
\]

Here, the last equality holds since \( z \in F \cap \left( C \setminus M \right) \) and \( z \) moves near \( \Delta F \) along \( Y_\ell \).

Next, we compute \( \hat{\phi}_{\text{FM}} \). For \( z \in J \), note that \( R(3) \in J \) and \( \left[ \hat{\phi}_z \cdot R(3) \cdot X_{R(3)} \right]_C \in M \). Hence,
\( (\mathcal{L}_{FM}^* G^M)(\beta) = [G^M[\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [G^M[\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]

The second term of the above line is computed as follows.
\[ \frac{1}{2\pi i} \int_{\gamma^R(\beta)} \frac{d\beta}{(2\beta - 1)^2} \]
\( = \frac{1}{2\pi i} \int_{\gamma^R(\beta)} \frac{d\beta}{(2\beta - 1)^2} \]
\( = 0. \]

The last equality holds since \( R(\beta) \) is of degree one and the denominator \( (2\beta - 1) \) is of degree three with respect to the variable of integration and the integration path \( \gamma^R(\beta) \) turns around the Julia set along a circle of infinitely large radius.

Therefore, we have
\( (\mathcal{L}_{FM}^* G^M)(\beta) = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]

Finally, let us consider \( \mathcal{L}_{MM}^* G^M \in \mathcal{L}^*(\mathbb{R}) \).

For \( \beta \in \gamma^R(\beta) \) with \( \beta \in \gamma^R(\beta) \), the component \( (\mathcal{L}_{MM}^* G^M)_m \) is computed as follows.
\( (\mathcal{L}_{MM}^* G^M)(\beta) = [G^M[-\chi^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]

In the case \( \beta = 0 \), i.e., for \( \beta \in \gamma^R(\beta) \), we have
\( (\mathcal{L}_{MM}^* G^M)(\beta) = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]

In the case \( \beta = 0 \), i.e., for \( \beta \in \gamma^R(\beta) \), we have
\( (\mathcal{L}_{MM}^* G^M)(\beta) = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]

In the case \( \beta = 0 \), i.e., for \( \beta \in \gamma^R(\beta) \), we have
\( (\mathcal{L}_{MM}^* G^M)(\beta) = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]

In the case \( \beta = 0 \), i.e., for \( \beta \in \gamma^R(\beta) \), we have
\( (\mathcal{L}_{MM}^* G^M)(\beta) = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]

In the case \( \beta = 0 \), i.e., for \( \beta \in \gamma^R(\beta) \), we have
\( (\mathcal{L}_{MM}^* G^M)(\beta) = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]

In the case \( \beta = 0 \), i.e., for \( \beta \in \gamma^R(\beta) \), we have
\( (\mathcal{L}_{MM}^* G^M)(\beta) = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]

In the case \( \beta = 0 \), i.e., for \( \beta \in \gamma^R(\beta) \), we have
\( (\mathcal{L}_{MM}^* G^M)(\beta) = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\( = [\chi^R(\beta); \gamma^R(\beta), \gamma^R(\beta), \gamma^R(\beta)]^F \]
\[
\left[ \frac{1}{2\pi i} \int_{\Gamma_0} g^M(\tau) \, \phi_\tau(\tau) \cdot R(\tau) \frac{d\tau}{\tau - R(\tau)} \right]^{M_0} \\
= \left[ \frac{1}{2\pi i} \int_{\Gamma_0} g^M(R(\tau)) \, \phi_\tau(R(\tau)) \cdot R(\tau) \chi_\tau(\tau) \, d\tau \right]^{M_0} \\
= R(\tau) \cdot \left( \Delta_0 \left[ \left( g^M \cdot \phi \right)_0 \right] R \right)(\tau).
\]

\section{Example}

In this section, we compute the operator \( L^*_{\text{MM}} \) more precisely for \( R(\tau) = \mathbb{Z}^2 + i \). In this case, the critical value \( \infty \) is preperiodic and the postcritical set \( P(R) = \{ i, -1, -i \} \) consists of three points.

Let us compute \( L^*_{\text{MM}} g^M \) for \( g^M \in \mathcal{C}(X_c) \), where \( G^M_c \in M_c^* \) and \( \phi_\tau(\tau) = G^M_c \cdot \chi_\tau \) for \( \tau \in X_c \).

For \( \tau \in \mathcal{Y}_p \) with \( p \neq 0 \), let \( \mathcal{P}(R) \) be the \( \mathbb{C} \)-linear mapping
\[
(L^*_{\text{MM}} G^M_c)(\tau) = \left[ \left( L^0_c G^M_c \right) [- \chi_\tau] \right]^{M_0} = \left[ G^M_c [- \mathcal{L}_c \chi_\tau] \right]^{M_0}
\]
\[
= \left[ G^M_c [- \chi_\tau(R(\tau)) \cdot R(\tau) \chi_{R(\tau)}] \right]^{M_0}
\]
Here, as \( p \neq 0 \), \( R(\tau) \) is \( X_c \), \( \chi_{R(\tau)} \) and \( M_c \). Also \( \left[ \phi \cdot R(\tau) \chi_{R(\tau)} \right] \) belongs to \( M_c \), we have
\[
(L^*_{\text{MM}} G^M_c)(\tau) = \left[ G^M_c [- \chi_\tau(R(\tau)) \chi_{R(\tau)}] \right]^{M_0}
\]
\[
= \left[ \frac{1}{2\pi i} \int_{\Gamma_0} g^M(\tau) \cdot \Delta_c \chi_\tau(\tau) \cdot R(\tau) \chi_{R(\tau)}(\tau) \, d\tau \right]^{M_0}
\]
\[
= \left[ R(\tau) \frac{1}{2\pi i} \int_{\Gamma_0} g^M(\tau) \cdot \Delta_c \chi_\tau(\tau) \chi_{R(\tau)}(\tau) \, d\tau \right]^{M_0}
\]
\[
= \left[ R(\tau) \left[ \phi \cdot g^M \cdot \Delta_c \chi_\tau \right] \cdot R(\tau) \right]^{M_0}
\]
For \( \tau \in \mathcal{Y}_c \)
\[
(L^*_{\text{MM}} G^M_c)(\tau) = \left[ \left( L^0_c G^M_c \right) [- \chi_\tau] \right]^{M_0} = \left[ G^M_c [- \mathcal{L}_c \chi_\tau] \right]^{M_0}
\]
\[
= \left[ G^M_c [- \chi_\tau(R(\tau)) \chi_{R(\tau)}] \right]^{M_0}
\]
\[
= \left[ R(\tau) \frac{1}{2\pi i} \int_{\Gamma_0} g^M(\tau) \cdot \Delta_c \chi_\tau(\tau) \chi_{R(\tau)}(\tau) \, d\tau \right]^{M_0}
\]
\[
\begin{align*}
&= \left[ R(3) \frac{-1}{2\pi i} \int_{\gamma_c} \gamma_c^M(\tau) \gamma_s(\tau) \frac{d\tau}{T-R(3)} \right]^{M_0} \\
&= \left[ R(3) \{ \gamma_s \cdot \gamma_c^M \circ R(3) \} \right]^{M_0}
\end{align*}
\]

where \( \{ \gamma_s \cdot \gamma_c^M \} \) denotes the regular part of \( \gamma_s \cdot \gamma_c^M \) along \( \gamma_c \).

i.e.
\[
\{ \gamma_s \cdot \gamma_c^M \} = \gamma_s \cdot \gamma_c^M - \int_c [\lambda_c \gamma_s \cdot \gamma_c^M]
\]

and is defined as
\[
\{ \gamma_s \cdot \gamma_c^M \} = \frac{1}{2\pi i} \int_{\gamma_c} \gamma_s(\tau) \cdot \gamma_c^M(\tau) \frac{d\tau}{T-\tau} \quad \text{for } \gamma_c \in C
\]

We have a decomposition
\[
\gamma_s \cdot \gamma_c^M = [\gamma_s \cdot \gamma_c^M] + \{ \gamma_s \cdot \gamma_c^M \}
\]

with \( [\gamma_s \cdot \gamma_c^M] \in M_c \) and \( \{ \gamma_s \cdot \gamma_c^M \} \in \mathcal{O}(\gamma_c) \).

810. Complex conformal measures.

Let \( G \in \mathcal{O}^*(T) \oplus M^* \oplus \mathcal{O}^*(F) \). Let \( A \subset C \) be an open set with smooth boundary \( \partial A \) (oriented by the counter clockwise direction). The characteristic function \( \chi_A(\gamma) \) of \( A \) is expressed as
\[
\chi_A(\gamma) = \frac{1}{2\pi i} \int_{\partial A} \chi_A(\eta) \frac{d\eta}{\gamma - \eta}
\]

So, we can rewrite
\[
\chi_A = \frac{1}{2\pi i} \int_{\partial A} \chi_A \frac{d\eta}{\gamma - \eta}
\]

Hence,
\[
G[\chi_A] = \frac{1}{2\pi i} \int_{\partial A} G[\chi_A] \frac{d\eta}{\gamma - \eta}
\]

defines a set function. If \( G = G^J + G^M + G^F \), then
\[
G[\chi_A] = G^J + G^M + G^F
\]

with \( g^J \in \mathcal{O}_b(T) \), \( g^M \in M \), \( g^F \in \mathcal{O}_b(F) \), and
\[
G[\chi_A] = \frac{1}{2\pi i} \int_{\partial A} (g^J(\eta) + g^M(\eta) + g^F(\eta)) \frac{d\eta}{\gamma - \eta}
\]
defines an additive set function. Suppose \( \lambda \) be a characteristic value of our transfer operator \( \mathcal{L}_5 \) and let \( f \in \mathcal{F} = \mathcal{O}^*(T) \oplus M \oplus \mathcal{O}^*(F) \) be an eigenfunction
of \( \Sigma \) for singular value \( \lambda \), i.e., \( \lambda \Sigma f = f \). And let \( G \in \mathcal{M} = (\mathcal{R}^c) \oplus \mathcal{M}^* \oplus (\mathcal{Q}^c)^* \) to the co-eigenfunctional \( \lambda \Sigma^* G = G \), with \( g(z) = G \Sigma^* (z) \) for \( z \in \mathcal{M} \).

Define a set function \( M \) by
\[
M(A) = \frac{1}{2\pi i} \int_{\partial A} f(t) g(t) dt.
\]
Then, we have
\[
M(A) = \frac{1}{2\pi i} \int_{\partial R^{-1}(A)} f(z) g(z) dz
\]
\[
= \frac{1}{2\pi i} \int_{\partial R^{-1}(A)} f(z) \lambda (\Sigma_0 G)(z) dz
\]
\[
= \frac{1}{2\pi i} \int_{\partial R^{-1}(A)} \lambda f(z) R(z) \Sigma_0 (R(z)) g(R(z)) dz.
\]
Then, by a change of variables \( \zeta = R(z) \) with \( d\zeta = R(z) dz \),
we have
\[
M(A) = \frac{1}{2\pi i} \int_{\partial R^{-1}(A)} \lambda \Sigma_0 G(\zeta) \cdot g(\zeta) d\zeta
\]
\[
= M(A),
\]
where we used \( L \Sigma f = \Sigma_0 f \) and \( L \Sigma g = R^{-1}(\Sigma_0 g) \circ R \).

Our set function \( M \) is backward invariant under \( R \).

Finally, we consider the pull-back of the set function defined by the co-eigenfunctional \( g \). Suppose \( \Sigma_0^* g = g \); then, for \( A \) with \( R^{-1} A : A \to R(A) \) injective, we have
\[
M(A) = \int_{\partial R^{-1}(A)} g(\zeta) d\zeta
\]
\[
= \int_{\partial R^{-1}(A)} R^{-1}(\zeta) \cdot g(R(z)) d\zeta
\]
\[
= \int_{\partial R^{-1}(A)} \Sigma_0^* g(R(z)) d\zeta.
\]
This shows a kind of complex conformal property of the set function \( M \) for co-invariant function \( g \).