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<th>Title</th>
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</tr>
</thead>
<tbody>
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Kyoto University
Microfunctions and a transfer operator
for complex dynamical systems.

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§1. Functions with regular singularities

Let us begin with the following situation. Let \( R(z) = z^2 + c \) be a quadratic polynomial on the Riemann sphere. We assume that the complex dynamical system defined by this quadratic polynomial is postcritically finite, i.e., the forward orbit \( \{ f^n(0) \mid n \in \mathbb{N} \} \) of the critical point 0 of \( R(z) \) is a finite set. For the sake of simplicity, we denote by \( F \) the Fatou set of \( R(z) \) and by \( J \) the Julia set of \( R(z) \). In order to illustrate the situation, we consider especially the case \( c = i \). Then, the critical point is 0 and its forward orbit is \( \{ 0, i, -1, -i \} \) and \( R(0) \) and \( R(i) \) from a periodic cycle of period 2. \( R(z) \) has two fixed points, which we denote by \( \alpha \) and \( \beta \) as in the picture. These two fixed points are so-called the \( \alpha \)-fixed point and the \( \beta \)-fixed point.
As $R(\mathbb{C})$ is postcritically finite, there exists an external ray, say $\theta_0$, landing at the critical value $c$. We give an orientation to this curve as $\theta \to \theta_0$.

Note that this external ray is spiralling near $c$.

Let $\Omega_c = \mathbb{C} \setminus (\theta_0 \cup f(\theta))$ be a domain in the complex plane. This domain $\Omega_c$ has smooth boundaries along the external ray $\theta_0$.

We consider an abstract "neighborhood" $\Omega_c$ of $\Omega_c$ doubly sheeted near $\theta_0$.

The domain $\Omega_c$ has two smooth curves on the boundary. We add two curves $\Omega_+^c$ and $\Omega^-_c$ to this domain to each sides of the external ray $\theta_0$, and we denote this set by $\Omega_c$.

$\Omega_c$ is an open set containing $\Omega_c$. For domain $\Omega$, we denote by $O(\Omega)$ the set of holomorphic functions on $\Omega$.

Let $f: \Omega_c \to \mathbb{C}$ be a holomorphic function on $\Omega_c$ which can be extended holomorphically to some "neighborhood" $\Omega_c'$ of $\Omega_c$. Such a function $f$ is said to be equivalent to $g: \Omega_c \to \mathbb{C}$ which is a holomorphic function on $\Omega_c$ and extendable to some "neighborhood" $\Omega_c''$ of $\Omega_c$ holomorphically if there exists a "neighborhood" $\Omega_c'$ of $\Omega_c'$ such that $f$ and $g$ coincide on $\Omega_c''$. This equivalence relation defines a concept of germ. Note that $f: \Omega_c \to \mathbb{C}$ itself gives a representative of its germ since analytic continuation is unique if it exists. We call such function $f$ a (general)
premicrofunction along \( \Omega_c \). A (general) microfunction at \( \Omega_c \) is defined by an equivalence class of germs of (general) pre-microfunctions \( f : \Omega_c \to \mathbb{C} \) at \( \Omega_c \) modulo germs of holomorphic functions at \( \Omega_c \).

More precisely, (general) pre-microfunctions \( f : \Omega_c \to \mathbb{C} \)
and \( g : \Omega_c \to \mathbb{C} \) define the same microfunction at \( \Omega_c \) if there exists an open neighborhood \( U \) of \( \Omega_c \) in the complex plane \( \mathbb{C} \) and a holomorphic function \( h : U \to \mathbb{C} \) such that \( f(z) - g(z) = h(z) \) holds for \( z \in U \cap \Omega_c \).

The above definition of (general) microfunction is so general that the singularities of such functions at \( \Omega_c \) are too much complicated. So, we restrict our singularities to "regular singularities" defined as follows.

**Definition 1.1.** Premicrofunction \( f : \Omega_c \to \mathbb{C} \) is said to have a regular singularity at \( \Omega_c \) if there exist positive numbers \( \varepsilon \) and \( k \) such that inequality
\[
|f(z)| < k |z - c|^{-1+\varepsilon}
\]
holds near \( \Omega_c \).

**Definition 1.2.** Premicrofunction \( f : \Omega_c \to \mathbb{C} \) is said to have a regular singularity at \( \infty \) if there exist positive numbers \( \varepsilon \) and \( k \) such that inequality
\[
|f(z)| < k |z|^{-\varepsilon}
\]
holds near the infinity.

We denote by \( M_c \) the set of pre-microfunctions along \( \Omega_c \) with regular singularities both at \( \Omega_c \) and \( \infty \).

More precisely, we denote \( M_c \) instead of \( M_c \) when there are more than one external rays landing at \( \Omega_c \). The space of equivalence classes of germs of pre-microfunctions with regular singularities at \( \Omega_c \) modulo the space of germs of holomorphic functions \( O(\Omega_c) \) at \( \Omega_c \), will be denoted by \( M_c \).

Let \( P(R) \) denote the postcritical set. For each point
$p \in P(R)$, the space of pre-microfunctions along its external rays with regular singularities at both point $p$ is defined in a similar manner and will be denoted by $M_p$ for simplicity, and by $M_p$ when it is necessary to indicate the external ray.

For $p \in J$ with multiple external rays, say $\alpha_1, \ldots, \alpha_r$, landing at $p$, we define the space $M_p$ by the direct sum

$$M_p = \bigoplus_{k=1}^{r} M_{\alpha_k}$$

where the sum is taken as a formal sum, since each component belongs to different spaces. However, each element of $M_p$ defines a function holomorphic in the intersection of the domains of definitions and the decomposition of a holomorphic function

$$f : C \setminus \left( \bigcup_{k=1}^{r} \cup \{p_k\} \right) \to C$$

defined by an element of $M_p$ into components $f_k$ in $M_{\alpha_k}$, $f_k : C \setminus \{p_k\} \to C$

is unique since we are considering the pre-microfunctions with regular singularities at the infinity. We denote

$$M_+ = \bigoplus_{p \in P(R)} M_p$$

$$M_0 = M_{00} \oplus M_{0-}$$

$$M_- = \bigoplus_{k=1}^{\infty} \bigoplus_{p \in R^{\alpha_k}} M_p$$

and

$$M = M_+ \oplus M_0 \oplus M_-.$$
3.2 Difference operator and an exact sequence

Let \(O(\Omega_c)\) denote the space of holomorphic functions in a neighborhood of the external ray \(\Omega_c\). An element of \(O(\Omega_c)\) is represented by a continuous function \(f: \Omega_c \to \mathbb{C}\) which can be extended to some neighborhood of \(\Omega_c\) holomorphically. The space of holomorphic functions along the external ray \(\Omega_c\) with regular singularities at both \(c\) and the infinity is defined by the following.

\[ O_0(\Omega_c) = \left\{ f \in O(\Omega_c) \mid \exists \varepsilon > 0, \exists k > 0, \exists \text{nbhd of } \Omega_c \text{ s.t.} \right. \]
\[ \left. \begin{array}{c}
|f(z)| < k |z-c|^{-1+\varepsilon} \text{ near } c \\
|f(z)| < k |z|^{-\varepsilon} \text{ near } \infty
\end{array} \right\} \]

Now, we define a difference operator along an external ray.

**Definition 2.1** Difference operator \(\Delta_c: M_c \to O_0(\Omega_c)\) is defined by the difference of boundary values along \(\Omega_c\)

\[ \Delta_c (\Phi) = \Phi(z) - \Phi((z-c)e^{-2\pi i} + c). \]

Here \(z \in \Omega_c\) is considered as a point in the boundary of \(\Omega_c\) of the clockwise side, and \((z-c)e^{-2\pi i} + c\) represents the same point but considered as a point in the boundary of \(\Omega_c\) of the counterclockwise side.

For each \(p \in J\) and its external ray \(\delta_p\), difference operator \(\Delta_p: M_p \to O_0(\Omega_p)\) is defined in a similar way. We denote \(\Delta_p\) instead of \(\Delta_p\) if there are more than two external rays landing at \(p\) and we need to indicate it.

**Remark** Difference operator can be defined for functions holomorphic along \(\Omega_c\) in a doubly sheeted domains. The domain of definition of such a holomorphic function need not be connected.

Let us fix a double sheeted neighborhood \(\Omega'_c\) of our domain \(\Omega_c = \Omega \setminus (\delta_e \cup \delta_p)\), and let \(S_c\) denote the neighborhood of \(\Omega_c\) where \(\Omega'_c\) is double sheeted.
Theorem 2.2 The following sequence is exact:

\[ 0 \to \mathcal{O}(C \setminus \{c\}) \to \mathcal{O}(S_c) \xrightarrow{\Delta_c} \mathcal{O}(S_c^*) \to 0. \]

Proof. We gave an orientation to the external ray $S_c$ defining an order to the points in $S_c$ so that $\infty < p < c$. Take points $R_d, S_d \in S_c$ for $d \in \mathbb{Z}$ ordered along $S_c$ as

\[ \infty < \cdots < R_0 < S_0 < R_1 < S_1 < \cdots < c \]

and

\[ \lim_{k \to -\infty} S_k = \lim_{k \to -\infty} R_k = \infty, \]

\[ \lim_{k \to +\infty} S_k = \lim_{k \to +\infty} R_k = c. \]

Then open arcs $\overline{R_d S_d}$ ($d \in \mathbb{Z}$) form an open covering of the external ray $S_c$. The space $\mathcal{O}(C \setminus \{c\})$ of holomorphic functions on $C \setminus \{c\}$ can be injectively embedded in the space of holomorphic functions on $S_c$ and the values of such a function on the two sheets of $S_c$ coincide on the overlapped sector $S_c^*$, hence the difference of these values vanishes. So, we need only to prove the ontoeness of the difference operator $\Delta_c$. For $f \in \mathcal{O}(S_c)$, we want to construct a holomorphic function in $\mathcal{O}(S_c^*)$. Note that such a function is not unique since the kernel of $\Delta_c$ contains $\mathcal{O}(C \setminus \{c\})$.

Let

\[ F_d(z) = \frac{1}{2\pi i} \int_{R_d}^{S_d} \frac{\varphi(t)}{t-z} \, dt \]

for $d \in \mathbb{Z}$. Such integration is called a Cousin’s integral along the arc $R_d S_d$. Note that this arc includes the arc $R_d S_d$ in its interior. The function $F_d(z)$ is holomorphic in $C \setminus \overline{\{R_d S_d \cup R_{d+1} S_{d+1}\}}$.

By deforming the path of integration of the Cousin’s integral we see that $F_d(z)$ can be holomorphically extended beyond the arc from both sides into the other side, except
at $R_{d+1}$ and $S_{d+1}$. Next let us take a family of annuli in a $\mathcal{S}$ $y$ separating $c$ and $r$ with smooth boundaries as follows. We take annulus $B_j$ for each $j \in \mathbb{Z}$ so that the intersection of $B_j$ with the external ray $\gamma_c$ is the arc $\overline{R_{d+1}S_d}$, and $R_j, S_j$ belong to the outer and inner boundary of $B_j$ respectively. Furthermore, we for $j, k \in \mathbb{Z}$, $B_j \cap B_k$ is empty if $|k - j| > 1$ holds, and for each $j \in \mathbb{Z}$, $B_j \cap B_{d+1}$ is an annulus. We impose that

$$\bigcup_{j \in \mathbb{Z}} B_j = C \setminus \{c\}$$

For each $j$, we denote by $B_j$ a covering of $B_j$ such that $B_j$ covers twice on the sector $S_c \cap B_j$. An function $F_j(z)$ defined by Cauchy's integration can be extended holomorphically to $B_j$. It is further extendable to a wider domain $\overline{B_j} \cup B_{j-1} \cup B_{j+1}$. Hence $F_j(z)$ is bounded in $B_j$. As is easily verified by considering the integration, we have

$$F_j(z) - F_j((z - c) e^{-2\pi i} + c) = \Phi(z)$$

for $z \in S_c \cap B_j$.

For $j, k \in \mathbb{Z}$ with $B_j \cap B_k = \phi$, define a holomorphic function

$$H_{jk} : B_j \cap B_k \to C$$

by

$$H_{jk}(z) = F_j(z) - F_k(z).$$

$F_j(z)$ and $F_k(z)$ are holomorphic on $B_j \cap B_k$. But, as we have

$$F_j((z - c) e^{2\pi i} + c) - F_k((z - c) e^{-2\pi i} + c) = F_j(z) - F_k(z)$$
along $s_c$, $H_{2k}((z-c)e^{2\pi i}+c)=H_{2k}(z)$ holds on $S_c \cap B_1 \cap B_{1c}$, so that $H_{2k}(z)$ is well defined and holomorphic on the annulus $B_1 \cap B_{1c}$. This family of holomorphic functions $H_{2k}$ forms a "Cousin data" i.e. for $i,j,k \in \mathbb{Z}$,

$$H_j + H_{j+1} = 0 \quad \text{on} \quad B_j \cap B_{j+1}.$$ 

As we assumed $B_j \cap B_{j+1} = \emptyset$ if $|j-j'| > 1$, this above fact is easily verified.

For each $j \in \mathbb{Z}$, take a loop $l_j$ in $B_j \cap B_{j+1}$, making a clockwise turn once and define $r_j(z)$ and $k_j(z)$ by

$$r_j(z) = \frac{1}{2\pi i} \int_{l_j} \frac{H_{i,j+1}(t)}{t-z} \, dt$$

defined and holomorphic in $\bigcup_{k=-\infty}^{\infty} B_k \cup \cup c$ (outside of the annulus), and

$$k_{j+1}(z) = \frac{1}{2\pi i} \int_{l_j} \frac{H_{i,j+1}(t)}{t-z} \, dt$$

define an holomorphic in $\bigcup_{k=-\infty}^{\infty} B_k \cup \cup c$ (inside of the annulus).

By deforming the integration path we see that they are well defined and we have

$$H_{j+1}(z) = r_j(z) - k_{j+1}(z) \quad (z \in B_j \cap B_{j+1})$$

By Runge's theorem, $r_j : \bigcup_{k=-\infty}^{\infty} B_k \cup \cup c \to \mathbb{C}$ can be approximated by polynomials in the sense of uniform convergence on compact sets, and $k_j : \bigcup_{k=-\infty}^{\infty} B_k \cup \cup c$ can be approximated by rational functions with poles only at $c$.

For each $j > 0$, find a rational function $g_j : \cup \cup c \to \mathbb{C}$ such that

$$|g_j(z) - f_j(z)| < \frac{1}{2^j} \quad \text{for} \quad z \in \bigcup_{k=-\infty}^{\infty} B_k \cup \cup c.$$ 

And for each $j < 0$, find a polynomial $g_j : \mathbb{C} \to \mathbb{C}$ such that

$$|g_j(z) - k_{j+1}(z)| < \frac{1}{2^j} \quad \text{for} \quad z \in \bigcup_{k=-\infty}^{\infty} B_k \cup \cup c.$$ 

Note that these functions $g_j$ are all holomorphic in $\mathbb{C}\setminus \cup c$.
Let \( \tilde{\rho}_d^i = \rho_d^i - \rho_d^{i+1} : \bigcup_{i=0}^{\infty} B_{d^i} U + c y \to \mathbb{C} \)
for \( d \geq 0 \), and
\[ \tilde{\rho}^{i+1}_d = \rho^{i+1}_d - \rho^i_d : \bigcup_{i=0}^{\infty} B_{d^i} U + c y \to \mathbb{C}, \]
for \( d < 0 \). Then we have
\[ |\tilde{\rho}^i_d| < \frac{1}{2^i} \quad (d \geq 0) \]
and
\[ |\tilde{\rho}^{i+1}_d| < \frac{1}{2^i} \quad (d < 0). \]
We still have
\[ H_d^i(z) = \tilde{\rho}^i_d(z) - H_d^{i+1}(z) \quad \text{for } z \in B_d \cap B_{d+1}. \]
Now, we set
\[ H_d^j(z) = - \sum_{i=0}^{j} \tilde{\rho}^i_d(z) - \sum_{i=0}^{\infty} \tilde{\rho}^i_d(z). \]
For \( i \leq j \), \( \tilde{\rho}^i_d(z) \) is holomorphic in \( \bigcup_{i=0}^{\infty} B_{d^i} U + c y \),
hence they are all holomorphic in the smallest disk \( \bigcup_{i=0}^{\infty} B_{d^i} U + c y \) and that we have the estimate of
\[ \sup_{\|z\|} \|\tilde{\rho}^i_d(z)\| \]
the sum of \( \tilde{\rho}^i_d(z) \) is uniformly convergent on \( B_d \).
Similarly, the sum of \( \tilde{\rho}^i_d(z) \) converge uniformly convergent
on \( B_d \), too. Hence \( H_d^j(z) \) is holomorphic in \( B_d \).
In the overlapping annulus \( B_d \cap B_{d+1} \), we have
\[ H_d^{j+1} - H_d^j = - \sum_{i=0}^{j+1} \tilde{\rho}^i_d(z) - \sum_{i=0}^{\infty} \tilde{\rho}^i_d(z) + \sum_{i=0}^{j+1} \tilde{\rho}^i_d(z) + \sum_{i=0}^{\infty} \tilde{\rho}^i_d(z) = \tilde{\rho}^j_d - \tilde{\rho}^{j+1}_d = H_d^{j+1}. \]
Finally, in \( B_d \), let \( G_d^j(z) = H_d^j(z) + F_d^j(z) \). These
functions \( G_d^j \) on \( B_d \) defines a holomorphic function
\[ G_d^j : \mathbb{C} \to \mathbb{C} \]
define on the overlapped neighborhood \( \Omega_c \) of \( B_d \). We can verify that these functions coincide and \( G_d^j \) is well
defined by an immediate calculations as follows.
In \( B_d \cap B_{d+1} \),
\[ G_d^{j+1}(z) = H_d^{j+1}(z) + F_d^{j+1}(z) = H_d^j(z) + H_d^{j+1}(z) + F_d^{j+1}(z) \]
\[ = H_d^j(z) + F_d^j(z) - F_d^{j+1}(z) + F_d^{j+1}(z) = H_d^j(z) + F_d^j(z) = G_d^j(z). \]
Thus, we conclude that $G \in \mathcal{O}(\mathbb{C})$ and

$$\Delta_c G = \Phi$$

holds. This completes the proof of the our Theorem 2.2.

We remark that such a function $G$ satisfying $\Delta_c G = \Phi$ is not unique since $\text{ker } \Delta_c = \mathcal{O}(\mathbb{C} \setminus \{c\})$.

§3. Cousin's integral operator and decomposition of pre-microfunctions.

In the previous section, we discussed the surjectivity of the difference operator $\Delta_c$. In this section, we restrict the space of (general) pre-microfunctions to the space of pre-microfunctions with regular singularities, and consider an inverse operator of $\Delta_c$, which we call a Cousin's integral operator.

**Definition 3.1** $I_c : \mathcal{O}_0(\mathbb{C}_c) \to \mathcal{M}_c$ is defined by

$$I_c[\Phi](z) = \frac{1}{2\pi i} \int_{\gamma_c} \frac{\Phi(t)}{t-z} \, dt$$

for $\Phi \in \mathcal{O}_0(\mathbb{C}_c)$.

Here, we use notation $I_c[\Phi]$ as $I_c : \mathcal{O}_0(\mathbb{C}_c) \to \mathcal{M}_c$ is an operator and we want to emphasize it, i.e., the argument of the operator is a function and not its value.

**Definition 3.2** Let $f$ be a bi-valued function defined in a neighborhood of $\mathbb{C}_c$, both of the two branches are holomorphic and the difference $\Delta f$ of $f$ has regular singularities at $c$ and at $\infty$, i.e., $\Delta f \in \mathcal{O}_0(\mathbb{C}_c)$. The $\mathcal{M}_c$ component of $f$ is defined as

$$[f]_c = [f]_{\mathbb{C}_c} = I_c[\Delta f]$$

This mapping $[\cdot]_c$ is a projection map onto $\mathcal{M}_c$. 
We have the following identities.

**Theorem 3.3** \( \Delta_c \circ I_c = \text{id} \) on \( \mathcal{M}_c \),
\[
\Delta_c \circ I_c = \text{id} \quad \text{on} \quad \mathcal{O}_c(\delta_c).
\]

**Proof.** These identities are easily verified.

For each point \( p \in J \) (and on external ray \( \gamma_p \) landing at \( p \)), projection \( \gamma \) \( \gamma_p \) is similarly defined.

Let \( \mathcal{O}_c(\delta_c) \) denote the space of holomorphic functions \( f: \delta_c \to \mathbb{C} \) such that \( f \) can be extended holomorphically to some double-sheeted neighborhood \( \delta_c \) and satisfies \( \Delta f \in \mathcal{O}_c(\delta_c) \). Function \( f \in \mathcal{O}_c(\delta_c) \) is holomorphic in \( \delta_c = \mathbb{C} \setminus \{ \delta_c \cup \{ c \} \} \) and has singularities at \( c \) and at the infinity together with its difference along \( \delta_c \). Let \( \mathcal{H}_c \) denote the space of hyperfunctions supported at \( c \), i.e., \( \mathcal{H}_c \) if and only if \( f \) is holomorphic in \( (\mathbb{C} \setminus \{ 0 \}) \setminus \delta_c \). The space of entire functions is denoted by \( \mathcal{O}(\mathbb{C}) \). Let us define the operators that extract singularities of \( f \).

**Definition 3.4.** Operator \( \Gamma_c : \mathcal{O}_c(\delta_c) \to \mathcal{H}_c \) is defined by
\[
\Gamma_c[f](z) = \frac{1}{2\pi i} \left( \int_{|z-c|=\varepsilon} \frac{f(t)}{t-z} \, dt - \frac{1}{2\pi i} \int_{\delta_c} \frac{Af(t)}{t-z} \, dt \right)
\]

where \( z \in \mathbb{C} \setminus \{ c \} \), \( \varepsilon > 0 \) is chosen sufficiently small so that the \( \varepsilon \)-ball around \( c \) does not contain \( z \), and \( \delta_c \) is the intersection point of \( \delta_c \) and the circle \( |z-c|=\varepsilon \). The orientation of the path of integration along the circle is the counter-clockwise with respect to \( z \).
As $Δf$ has a regular singularity at $c$, this defines a holomorphic function on $(\Omega_c \setminus \{c\}) \setminus \{c\}. That is, $Γ_c[f] ∈ \mathcal{H}_c$.

**Definition 3.5.** Operator $Γ_∞ : \mathcal{O}_0(Ω_ε) \to \mathcal{O}(C)$ is defined by

$$Γ_∞[f](z) = \frac{1}{2πi} \oint_{|τ-c|=ω} \frac{f(τ)}{τ-z} \, dτ + \frac{1}{2πi} \int_{R_0} \frac{G_0(τ)}{τ-z} \, dτ$$

when $ω > 0$ is taken sufficiently large for each $z$, so that the circle of integration path surrounds $z$, and $R_0 \subset \Omega_ε$ is the intersection point of $R_0$ and the big circle. The orientation is taken as the counterclockwise with respect to $z$.

As $Δf$ has a regular singularity at $∞$, this defines an entire function. Hence $Γ_∞[f] ∈ \mathcal{O}(C)$.

Just for the sake of consistency of notation, we define

$$Γ_M : \mathcal{O}_0(Ω_ε) \to \mathcal{M}_ε$$

by $Γ_M[f] = [f]_ε$. We have the following decomposition.

**Theorem 3.6.** $\mathcal{O}_0(Ω_ε) = \mathcal{H}_c ⊕ \mathcal{M}_ε ⊕ \mathcal{O}(C)$

and $Γ_c : \mathcal{O}_0(Ω_ε) \to \mathcal{H}_c$, $Γ_M : \mathcal{O}_0(Ω_ε) \to \mathcal{M}_ε$,

$Γ_∞ : \mathcal{O}_0(Ω_ε) \to \mathcal{O}(C)$

gives the projections to components.

**Proof.** Clearly, the kernel of the difference operator $Δ_c$ is $\mathcal{O}(C \setminus \{c\})$ and $\mathcal{O}(C \setminus \{c\}) = \mathcal{H}_c ⊕ \mathcal{O}(C)$.

Note that these operators can be defined if $f$ is defined and holomorphic in a double covered neighborhood of $\overline{Ω}_c$. In this case, $Γ_c + Γ_M + Γ_∞$ defines a projection to $\mathcal{O}_0(\Omega_ε) = \mathcal{H}_c ⊕ \mathcal{M}_ε ⊕ \mathcal{O}(C)$ if $Δ_c f ∈ \mathcal{O}_0(\Omega_ε)$. 
§ 4 Space of pre-microfunctions and a transfer operator

Let us go back to our complex dynamical system \( R(z) \). Let \( J \) denote the Julia set of \( R(z) \) and let \( F \) denote the Fatou set of \( R(z) \). We suppose \( R(z) \) is postcritically finite, especially the case of \( c = i \). We denote by \( \mathcal{O}(J) \) the space of germs of continuous functions \( f : J \to \mathbb{C} \) which are holomorphic in some neighborhood of \( J \). The space of holomorphic functions \( f : F \to \mathbb{C} \) of the Fatou set satisfying \( f(0) = 0 \) will be denoted by \( \mathcal{O}_0(F) \). The postcritical set of \( R(z) \) is denoted by \( P(R) \). The space of pre-microfunctions at \( P(R) \) is defined by

\[
\mathcal{M}_{P(R)} = \bigoplus_{p \in P(R)} \mathcal{M}_p,
\]

and

\[
\mathcal{M} = \mathcal{M}_{P(R)} \oplus \bigoplus_{p \in P(R) \setminus P} \mathcal{M}_p
\]

denotes the space of formal sum of pre-microfunctions at the graph orbit of the critical point \( 0 \).

Let \( f \in \mathcal{O}(J) \oplus \mathcal{M}_{P(R)} \oplus \mathcal{O}_0(F) \) and \( p \in P(R) \)

with \( R_p \) its external ray. Then the \( M_p \)-component \( [f]_p \) of \( f \) is given by a projection

\[
[f]_p(z) = \frac{1}{2\pi i} \int \frac{dA(f)(z)}{z - z},
\]

Let us consider the most simple postcritically finite case (except \( c = -2 \) case) of \( R(z) = z^2 + i \). Fixed points of \( R \) are denoted by \( \alpha \) and \( \beta \). The preimage of the external ray \( \alpha \) consists of two external rays, say \( \alpha^+ \) and \( \alpha^- \), of the critical point \( 0 \), with external angles \( \frac{1}{2} \) and \( \frac{1}{12} \) respectively. These external rays are oriented as \( \infty \to 0 \). Let \( V_\alpha \) denote the upper connected component of \( \mathbb{C} \setminus (\alpha^+ \cup \beta^-) \) which contains the critical value \( c = i \). The \( \alpha \)-fixed point belongs to this domain. We denote the other connected point by \( T_\beta \). It contains the \( \beta \)-fixed point. The quadratic map \( R \) is of degree two. The critical value \( c \) is a branch point. We denote the two branches of \( R^{-1} \) by \( \beta_0 \) and \( \beta \) defined in \( \mathcal{R}_c = \mathbb{C} \setminus (\beta_0 \cup i\mathbb{C}) \).
\[ Q_\alpha : \mathbb{C} \setminus (\gamma_c \cup \{z\}) \rightarrow \bar{\Omega}_c \]
\[ Q_\beta : \mathbb{C} \setminus (\gamma_c \cup \{z\}) \rightarrow \bar{\Omega}_B \]

with \[ Q_\alpha(z) = -\sqrt{z - c}, \quad Q_\beta(z) = \sqrt{z - c}, \]
where the branch of the square root is chosen by assigning \[ Q_\alpha(z) \vert_{z=0} = 1. \]

If we regard \( Q_\alpha \) and \( Q_\beta \) as holomorphic functions on \( \Omega_c \), we can naturally consider holomorphic functions \( Q_\alpha(z)^s \) and \( Q_\beta(z)^s \) for \( 0 < s < 2 \). They can be extended to a double sheeted neighborhood \( \Omega_c \) holomorphically. We define a holomorphic function
\[ \gamma_s(z) = \frac{1}{(2Q_\beta(z))^s} \]
defined in \( \Omega_c \).

For \( z = c + r e^{i\theta} \in \Omega_c \), we have
\[ (\Delta_c \gamma_s)(z) = \gamma_s(z) - \gamma_s((z - c) e^{-2\pi i} + c) \]
\[ = \frac{1 - e^{5\pi i}}{(2\sqrt{r e^{2\pi i}})^s} \]

Hence \[ |\Delta_c \gamma_s| = \text{const.} \ r^{-\frac{s}{2}} \]
which implies \( \Delta_c \gamma_s \) has regular singularities at \( c \) and \( \infty \)
if \( 0 < s < 2 \). Therefore \( \gamma_s \in \mathcal{M}_c \).

Now, take a function \( f \in \mathcal{O}(\mathbb{C}) \oplus \mathcal{M}_B^{\mathbb{R}}(\mathbb{C}) \oplus \mathcal{O}(\mathbb{C}) \).

Here, we abuse the formal sum of function in different spaces and the sum as a function defined in the common domain of definition. So, \( f \) is defined and holomorphic in \( \mathbb{C} \setminus (\bigcup_{p \in P(\mathbb{R})} \gamma_p) \).

For an external ray \( \gamma_p \), we denote by...
The path of integration coming from $\infty$ to $p$ along $\gamma_p$ taking the clockwise side value of the integrand function and going back from $p$ to $\infty$ along $\gamma_p$ taking the counter-clockwise side value of the integrand function. That is,

$$\int_{\gamma_p} f(z) \, dz = \int_{\gamma_p} f(z) \, dz - \int_{\gamma_p} f((z-p)e^{2\pi i}+p) \, dz$$

$$= \int_{\gamma_p} \Delta_p f(z) \, dz.$$

By $\gamma_F$ we represent an integration path along the boundary of the Fatou set, passing near the Julia set and taking values of the function on the Fatou set. And finally by $\gamma_J$ we represent an integration path turning around the Julia set in the counter-clockwise direction.

Let $\mathcal{N}_+ = \mathcal{U}(C) \oplus \mathcal{M}(R) \oplus \mathcal{C}(F)$

Definition 4.1 Transfer operator $L_\Sigma : \mathcal{N}_+ \rightarrow \mathcal{N}_+$ is defined for $0 < \sigma < 2$ and for $\tau \in \mathcal{N}_+$ by

$$(L_\Sigma f)(\tau) = \sum_{R \in \Sigma(\tau)} \gamma_\Sigma(R(\tau)) \, f(\tau).$$

We can rewrite the transfer operator in an integral operator form as

$$(L_\Sigma f)(\tau) = \frac{1}{2\pi i} \oint_{\gamma_F + \gamma_R + \gamma_J} \frac{\gamma_\Sigma(\tau(\tau)) \, f(\tau)}{\tau(\tau) - \tau} \, d\tau$$

and as

$$(L_\Sigma f)(\tau) = \gamma_\Sigma(\tau) \left( f \circ Q_\Sigma(\tau) + f \circ Q_R(\tau) \right).$$

Definition 4.2 Push forward operator $R_\Sigma$ is defined by

$$R_\Sigma f = f \circ Q_\Sigma + f \circ Q_R.$$
For points \( \gamma \in \mathbb{C} \), we denote by \( \chi_\gamma(z) = \frac{1}{z - \gamma} \) the unit pole at \( \gamma \). If \( \gamma \in \mathbb{J} \) then \( \chi_\gamma \in \mathcal{O}_s(F) \). If \( \gamma \in \mathbb{R}_p \) then \( \chi_\gamma \in \mathcal{O}(\mathbb{R}_p) \). Note that \( \chi_2(z) = -\chi_\gamma(z) \).

**Proposition 4.3** For \( \gamma \in \mathbb{J} \cup \bigcup_{p \in \mathbb{R}_p} \mathbb{R}_p \),

\[
\mathcal{L}_s \chi_\gamma = R(\gamma) \chi_s(R(\gamma)) \chi_{R(\gamma)} + R(\gamma) \left[ \chi_s \cdot \chi_{R(\gamma)} \right]_c.
\]

**Proof.** By direct computation, we have

\[
\mathcal{L}_s \chi_{R(\gamma)}(z) = \chi_s(z) \left( \chi_{R(\gamma)} \circ \phi(z) + \chi_{R(\gamma)} \circ \phi_2(z) \right) = \chi_s(z) \sum_{\delta \in \mathbb{R}_p} \left( \chi_{R(\gamma)}(\gamma) \right)
\]

\[
= \chi_s(z) \sum_{\delta \in \mathbb{R}_p} \frac{1}{z - \gamma}.
\]

By the residue formula, we have

\[
o = \frac{1}{2\pi i} \int_{\gamma} \frac{R(\gamma)}{(z - \gamma)(y - \gamma)} dy = \sum_{\delta \in \mathbb{R}_p} \frac{1}{y - \gamma} + \frac{R(\gamma)}{z - R(\gamma)}.
\]

Hence

\[
\mathcal{L}_s \chi_{R(\gamma)}(z) = \chi_s(z) \left( \frac{R(\gamma)}{z - R(\gamma)} \right) = \chi_s(z) \left( \frac{R(\gamma)}{R(\gamma)} \chi_{R(\gamma)}(z) \right)
\]

\[
= \chi_s(R(\gamma)) \chi_{R(\gamma)}(z) + \left[ \chi_s \cdot \chi_{R(\gamma)} \right]_c.
\]

Here \( \left[ R(\gamma) \chi_{R(\gamma)} \right]_c(z) = (\chi_s(z) - \chi_s(\gamma)) \chi_{R(\gamma)}(z) \chi_{R(\gamma)}(2) \) is holomorphic near \( R(\gamma) \), so that it belongs to \( \mathcal{N}_c \). The first term \( \chi_s(R(\gamma)) \) \( \chi_{R(\gamma)}(z) \) is a multiple of unit pole at \( R(\gamma) \).

Unit poles \( \chi_\gamma, \gamma \in \mathbb{J} \) form a basis of function space \( \mathcal{O}(F) \) and the family of unit poles \( \chi_\gamma, \gamma \in \mathbb{R}_p \) form a basis of space of pre-maximizer \( \mathcal{M}_p \). This splitting of the transfer operator \( \mathcal{L}_s \) gives a decomposition of the operator into components of the direct sum decomposition of function spaces.

**Theorem 4.4** For \( p \in \mathbb{J} \) and \( g \in \mathcal{M}_p \) we have the following decomposition

\[
\mathcal{L}_s g = \left[ g \circ \phi \cdot \chi_s \right]_{R(\gamma)} + \left[ \chi_s \cdot [g \circ \phi] \right]_{R(\gamma)} \chi_c,
\]

where \( \phi = \phi_\omega \) or \( \phi_\beta \) according to \( p \in \mathbb{U}_\omega \) or \( \mathbb{U}_\beta \).
As \( g \in M_p \), \( g(z) \) can be expressed in an integration of Cauchy type for \( z \in \Omega_p = \mathbb{C} \setminus (\gamma_p \cup \gamma_F) \) as
\[
g(z) = \frac{1}{2\pi i} \int_{\gamma_p} (\gamma_R) (t) \frac{dt}{t-z} = \frac{1}{2\pi i} \int_{\gamma_p} (\gamma_R) (t) (-\gamma_t (z)) dt.
\]
Hence, we have
\[
\mathcal{L}_s g (z) = \frac{-1}{2\pi i} \int_{\gamma_p} (\gamma_R) (\gamma_s (x, y)) (\gamma_t (x, y)) (\gamma_R (x, y)) d\eta
\]
\[
= \frac{1}{2\pi i} \int_{\gamma_p} (\gamma_R) (\gamma_s (x, y)) (\gamma_t (x, y)) (\gamma_R (x, y)) d\eta - \frac{1}{2\pi i} \int_{\gamma_p} (\gamma_R) (\gamma_t (x, y)) (\gamma_R (x, y)) d\eta
\]
\[
= \frac{1}{2\pi i} \int_{\gamma_p} (\gamma_R R (\gamma_s (x, y)) (\gamma_t (x, y)) (\gamma_R (x, y)) d\eta + \frac{\gamma_s \cdot \gamma_t}{2\pi i} \int_{\gamma_p} (\gamma_R) (\gamma_t (x, y)) (\gamma_R (x, y)) d\eta
\]
\[
= \left[ \gamma^0 \gamma_p \right] \gamma_s (x, y) \gamma_p (x, y) + \left[ \gamma_s \cdot \gamma_t \right] \gamma_p (x, y)
\]
Note that in the above calculations, \( \gamma, \gamma_s \in \gamma_p \) is a variable along \( \gamma_p \) and we changed variables by \( x = \gamma (x) \) and \( y = \gamma (y) \).

This theorem shows that the transfer operator \( \mathcal{L}_s \) sends \( M_p \) into \( M_{R(p)} \oplus M_c \) for \( p \in J \). In the case of postcritically finite complex dynamical system case, the space of pre-microfunctions for postcritical set is invariant for \( \mathcal{L}_s \), i.e.

**Proposition.** \( \mathcal{L}_s (M_+) \subseteq M_+ \).

§5. Decomposition of the transfer operator

Our space of pre-microfunctions \( H_+ \) has a direct sum decomposition

\[
H_+ = C(\Omega) \oplus M_+ \oplus C(\Omega)
\]
and the transfer operator \( \mathcal{L}_s \) maps this space into itself.

We denote the components of \( \mathcal{L}_s \) in a matrix form as

\[
\mathcal{L}_s = \begin{pmatrix}
\mathcal{L}_{J} & \mathcal{L}_{M} & \mathcal{L}_{F} \\
\mathcal{L}_{MJ} & \mathcal{L}_{MN} & \mathcal{L}_{MF} \\
\mathcal{L}_{PJ} & \mathcal{L}_{PEU} & \mathcal{L}_{EF}
\end{pmatrix}
\]
For $\phi_j \in U(\mathbb{C})$ we have the following proposition.

**Proposition 5.1**

$$L_s f_j = \gamma_s \cdot R_s f_j = L_{\phi_j} f_j + L_{\gamma_s} f_j + L_{\gamma_j} f_j$$

$$L_{\phi_j} f_j = \gamma_s \cdot R_s f_j - \gamma_s \cdot R_s f_j$$

$$L_{\gamma_s} f_j = \gamma_s \cdot R_s f_j$$

$$L_{\gamma_j} f_j = 0.$$

**Proof.** As $0 < s < 1$, we have

$$\lim_{t \to 0} \frac{1}{2\pi i} \int_{C_0} \frac{\gamma_s(R(t)) \cdot R(t) f_j(t)}{R(t) - \lambda} dt = 0,$$

where $\lambda \in \mathbb{C}$ and $C_0$ the integration path $C_0$ is the circle of radius $r$ around the original point $0$, since the singularity at the origin is of order $1 - s$. In the next calculation, integration paths are as explained in the previous section. We have

$$\left( L_{\phi_j} f_j \right)(z) = \frac{1}{2\pi i} \int_{C_0} \frac{\gamma_s(R(t)) \cdot R(t) f_j(t)}{R(t) - \lambda} dt$$

$$= \frac{1}{2\pi i} \int_{C_0} \gamma_s(R(t)) \cdot \int_{\gamma_j} \frac{\gamma_s(R(t)) \cdot R(t) f_j(t)}{R(t) - \lambda} dt$$

$$= \gamma_s(\mathbb{R}) \sum_{\gamma \in K(\mathbb{R})} f_j(y) - \frac{1}{2\pi i} \int_{C_0} \gamma_s(R(t)) \cdot R(t) f_j(t) dt$$

$$= \gamma_s(\mathbb{R}) \sum_{\gamma \in K(\mathbb{R})} f_j(y) - \frac{1}{2\pi i} \int_{C_0} \gamma_s(R(t)) \cdot R(t) f_j(t) dt$$

$$= \gamma_s(\mathbb{R})(R_s f_j)(z) = -\frac{1}{2\pi i} \int_{C_0} \gamma_s(R(t)) \cdot R(t) f_j(t) dt$$

Here we made a change of variables $\gamma = R(t)$ and $dt = R(t) dt$. $Q^+ : \gamma_j \to \gamma_j^+$ and $Q^- : \gamma_j \to \gamma_j^-$ denotes the inverse branches of $R$ along $\gamma_j$. $R_s f_j = f_j \circ Q^+ + f_j \circ Q^-$ holds along $\gamma_j$ and $R_s f_j = f_j \circ Q^+ + f_j \circ Q^-$ elsewhere. We continue the calculation.

$$\left( L_{\gamma_s} f_j \right)(z) = \gamma_s'(\mathbb{R})(R_s f_j)(z) - \int_{C_0} \left( \gamma_s'R_s \cdot R_s f_j \right)$$

$$= \gamma_s'(\mathbb{R}) \cdot R_s f_j(z) - \left[ \gamma_s' \circ R_s f_j \right]_c.$$
\[
(L_{\text{MJ}} f_J)(z) = \frac{1}{2\pi i} \int_{\Gamma_M} \frac{\psi_s(R(t)) R(t) f_J(t)}{R(t) - z} \, dt
\]

and
\[
(L_{\text{FJ}} f_J)(z) = \frac{1}{2\pi i} \int_{\Sigma_F} \frac{\psi_s(R(t)) R(t) f_J(t)}{R(t) - z} \, dt
\]

\[
= \frac{1}{2\pi i} \int_{\Sigma_F} \frac{\psi_s(\sigma) R(t) f_J(\sigma)}{\sigma - z} \, d\sigma
\]

\[
= 0
\]

Since
\[
|\Sigma_{\text{FJ}} f_J| \leq \frac{2\pi P}{2\pi} \frac{|2P|^{-S}|P| + |P|^{-S}|P|}{|z - c| - |P|} \rightarrow 0 \quad \text{as } P \rightarrow 0
\]

For the second column of the operator matrix, we have the following.

**Proposition 5.2.** For \( f_M \in M_p \subset M^+ (p \neq 0) \),
\[
L_{\text{M}M} f_M = 0
\]
\[
L_{\text{FM}} f_M = [\psi_s \cdot R_y f_M]_{c} + [\psi_s \cdot f_T]_{R(p)}
\]
\[
L_{\text{FM}} f_M = 0
\]

**Remark** \( L_{\text{F}} f_M \in M_c \oplus M_{R(p)} \) if \( f_M \in M_p \)

**Proof.** As \( f_M \in M_p \), there exists a positive number \( \varepsilon \) and a positive constant \( K \) such that
\[
|f_M(z)| < K|z|^{-\varepsilon}
\]
holds near the infinity. Hence we have
\[
|L_{\text{M}M} f_M(2)| < \frac{1}{2\pi} \int_{\Sigma_F} \frac{|\psi_s(R(t)) R(t) f_M(2)|}{|R(t) - 2|} \, dt
\]
\[
\leq \frac{2\pi |2|^{-S}}{2\pi} \frac{1}{|1/2|^2 + \frac{c}{\varepsilon} \cdot \frac{2}{\varepsilon}^2}
\]
\[
\leq \text{const.} \cdot |2|^{-S-\varepsilon} \rightarrow 0 \quad (|2| \rightarrow \infty)
\]

and \( L_{\text{FM}} f_M = 0 \).

Next, we show that \( L_{\text{FM}} = 0 \) in the following.
\[(\mathcal{L}_{FM} f_M)(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\gamma_s(R(\tau), R(\tau), f_M(\tau))}{R(\tau) - z} d\tau \]
\[
= \frac{1}{2\pi i} \left( \int_{\mathbb{C}} \frac{\gamma_s(R(\tau), R(\tau), f_M(\tau))}{R(\tau) - z} d\tau \right)_{\text{int}} + \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\gamma_s(R(\tau), R(\tau), f_M(\tau))}{R(\tau) - z} d\tau.
\]

This goes to zero as \( p \to 0. \)

For, \( \gamma_s \) belongs to \( M_0 \), so that the second term vanishes and the first term vanishes since \( f_M \) belongs to \( M_p. \) Note that in this computation, \( z \) is taken from the Fatou set and the integration path \( R \) runs near the Julia set. Note that this argument cannot be applied if \( p = 0 \) since the integrand might have a singularity at \( p, \) which is not regular. We need regularity of the singular points to have this kind of integral vanish.

Finally,
\[(\mathcal{L}_{MN} f_M)(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\gamma_s(R(\tau), R(\tau), f_M(\tau))}{R(\tau) - z} d\tau \]
\[
= \frac{1}{2\pi i} \left( \int_{\mathbb{C}} \frac{\gamma_s(R(\tau), R(\tau), f_M(\tau))}{R(\tau) - z} d\tau \right)_{\text{int}} + \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\gamma_s(R(\tau), R(\tau), f_M(\tau))}{R(\tau) - z} d\tau.
\]

\[
= \mathcal{I} \left( \gamma_s R_y f_M \right) + \mathcal{I}_{\mathbb{C}} \left( \gamma_s \cdot f_M^0 q_p \right).
\]

This completes the proof of Proposition 5.2.

Remark. If \( f_M \in M_0, \) then the integrand may have a non-regular singularity since we have a product of two regular singularities at \( p = 0. \) Hence,

\[
\mathcal{L}_{FM} f_M = \gamma_s R_y f_M \mathcal{F},
\]

\[
\mathcal{L}_{MN} f_M = \gamma_s R_y f_M \mathcal{C},
\]

\[
\mathcal{L}_{JM} f_M = \gamma_s R_y f_M \mathcal{J}.
\]

For \( f_F \in \mathcal{C}_0(F), \) we have the following.

Proposition 5.3
\[
\mathcal{L}_{JF} f_F = 0,
\]

\[
\mathcal{L}_{MF} f_F = \gamma_s R_y f_F \mathcal{F},
\]

\[
\mathcal{L}_{FF} f_F = \gamma_s R_y f_F \mathcal{J}.
\]
Proof. As \( f_F \in U_0(F) \), we have an estimate \(|f_F| \leq C|t|^{-1}\) near \( \infty \). Hence

\[
|(R_{TF} f_F)(z)| \leq \frac{12\pi^1}{2\pi} \frac{|2|^{-1}}{|t^2| + \frac{8}{|t|^2}} \rightarrow 0 \quad (as \ t \to \infty)
\]

Therefore we have \( L_{TF} f_F = 0 \). For the second component,

\[
L_{MF} f_F = \int_0^1 \left[ \frac{d}{ds} \left( R_s f_F \right) \right] R_s \phi_F(t) \, ds = [\phi_F \cdot R_s f_F]_c.
\]

And the third component is computed similarly,

\[
(L_{FF} f_F)(z) = \frac{1}{2\pi i} \int_{\gamma_F} \frac{\psi(t) \phi_F(t)}{R(t) - z} \, dt
\]

\[
= \frac{1}{2\pi i} \int_{\gamma_F} \frac{\psi(t) \phi_F(t)}{t - z} \, dt
\]

\[
= \psi(z) \cdot (R_z f_F)(z) - [\psi \cdot R_z f_F]_c.
\]

Putting the above propositions together, we have the decomposition of our transfer operator into components.

\[
\begin{pmatrix}
L_{TT} & L_{TM} & L_{TF} \\
L_{MT} & L_{MM} & L_{MF} \\
L_{FT} & L_{FM} & L_{FF}
\end{pmatrix}
\begin{pmatrix}
f_T \\
f_M \\
f_F
\end{pmatrix}
= \begin{pmatrix}
[\psi \cdot R_z f_F]_c & [\psi \cdot R_z f_F]_c & [\psi \cdot R_z f_F]_c \\
[\psi \cdot R_z f_F]_c & [\psi \cdot R_z f_F]_c & [\psi \cdot R_z f_F]_c \\
[\psi \cdot R_z f_F]_c & [\psi \cdot R_z f_F]_c & [\psi \cdot R_z f_F]_c
\end{pmatrix}
\begin{pmatrix}
f_T \\
f_M \\
f_F
\end{pmatrix}
\]

Note that if \( f_T \) or \( f_F \) are not identically zero, then \([\psi \cdot R_z f_F]_c \neq 0 \) or \([\psi \cdot R_z f_F]_c \neq 0 \). Hence, there is no eigenfunction in subspace \( O(C) \oplus U_0(F) \).

§6. Invariant subspace of the transfer operator

Our transfer operator \( T_z : M_+ \rightarrow M_+ \) maps the space of pre-microfunctions supported on the forward orbit of the critical point. For \( h_0 \in M_0 \), \( h_0(t) \) can be written in a form of Cauchy integral.

\[
h_0(t) = \frac{1}{2\pi i} \int_{\gamma_0} (A_c h_0)(z) \phi_c(z) \, dz = \frac{1}{2\pi i} \int_{\gamma_c} (A_c h_0)(z) \phi_c(z) \, dz.
\]

In the following, we denote as \( \gamma_c \cdot (z) = \gamma_z (R_c(z)) \cdot \gamma_z (R_{c}(z)) \cdots \gamma_z (R(z)) \)
\[ R_k(\alpha) = R_k \cdot R_k(\alpha) \text{ denotes the } k\text{-th iteration of } R(\alpha). \]

Note that \( \Psi_k \circ R_k \in M_{R_k(\alpha)} \) and that \( \Psi_k \) are regular on \( Y_k(\alpha) \).

For each \( k = 1, 2, \ldots \), we consider a pre-microfunction \( \varphi_k \) in \( M_{R_k(\alpha)} \) expressed in terms of a pre-microfunction \( \varphi_k \) in \( M_{\alpha} \).

For \( \varphi_k \in M_{\alpha} \), let

\[ \varphi_k(\alpha) = \varphi_k(\alpha) \cdot \Psi_k(\alpha) \]

and define \( \varphi_k \in M_{R_k(\alpha)} \) by

\[ \varphi_k(\alpha) = \frac{-1}{2\pi i} \int_{\gamma_k} \left( \Delta \varphi_k(\alpha) \right)(t) R_k(\alpha) \right) Y_k(\alpha)(t) \, dt. \]

Let \( \varphi_k = (R_k \mid Y_k)^{-1} \colon Y_k(\alpha) \longrightarrow \gamma_k \) be the inverse branch of \( R_k \). By a change of variables \( s = R_k(t), \, ds = R_k(\alpha) \, dt \) and \( t = Q_k(s) \), we have

\[ \varphi_k(\alpha) = \frac{-1}{2\pi i} \int_{Q_k(\alpha)} \left( \Delta \varphi_k(\alpha) \left[ G_k \circ Q_k \right] \right)(s) Y_k(\alpha)(t) \, ds. \]

This implies \( \varphi_k = \left[ G_k \circ Q_k \right] R_k(\alpha) \) and \( \varphi_k \in M_{R_k(\alpha)} \).

We have

\[ \Delta \varphi_k = (\Delta R_k(\alpha) \varphi_k) \circ R_k \text{ along } \gamma_k. \]

So, this correspondence induces an isomorphism \( M_{R_k(\alpha)} \equiv M_{\alpha} \).

As \( \varphi_k \in M_{R_k(\alpha)} \), we have \( \varphi_k \in M_{R_k(\alpha)} \oplus M_{\alpha} \).

More precisely, we have the following explicit formula.

**Proposition 6.1** If \( \varphi_k = \left[ G_k \circ Q_k \right] R_k \in M_{R_k(\alpha)} \) with \( G \) as above, we have the following decomposition.

\[ \varphi_k = \left[ G_k \circ Q_k \right] R_k + \left[ \gamma_k \left[ G_k \circ Q_k \right] R_k \right] \]

**Proof** This is immediately verified by applying Theorem 4.2.

By an immediate calculation, we can obtain the proof as follows.

First consider of \( \varphi_k \) is given by

\[ \varphi_k = \left[ \frac{-1}{2\pi i} \int_{\gamma_k} \left( \Delta \varphi_k(\alpha) \left[ G_k \circ Q_k \right] \right)(s) Y_k(\alpha)(t) \, ds \right] \right) R_k(\alpha). \]

Thus, by an immediate calculation, we can obtain the proof as follows.

First consider of \( \varphi_k \) is given by

\[ \varphi_k = \left[ \frac{-1}{2\pi i} \int_{\gamma_k} \left( \Delta \varphi_k(\alpha) \left[ G_k \circ Q_k \right] \right)(s) Y_k(\alpha)(t) \, ds \right] \right) R_k(\alpha). \]
\[
\begin{align*}
&= \frac{-1}{2\pi i} \int_{\gamma_{R_{a+1}}(c)} \left( \Delta_{R_{a+1}}(c) \left[ G_{R_{a+1}} \circ Q_{a+1} \right] \right)(\sigma) \left[ L_S \chi_{R_{a+1}} \right]_{c} \ d\sigma \\
&= \frac{-1}{2\pi i} \int_{\gamma_{R_{a+1}}(c)} \left( \Delta_{R_{a+1}}(c) \left[ G_{R_{a+1}} \circ Q_{a+1} \right] \right)(\rho) \left[ \chi_{R_{a+1}} \right]_{c} \ d\rho \\
&= \left[ \chi_{S} \cdot \frac{1}{2\pi i} \int_{\gamma_{R_{a+1}}(c)} \left( \Delta_{R_{a+1}}(c) \left[ G_{R_{a+1}} \circ Q_{a+1} \right] \right)(\rho) \left[ \chi_{R_{a+1}} \right]_{c} \ d\rho \right]_{c}
\end{align*}
\]

where we made change of variables \( \rho = R_{a+1}(\sigma) \), \( d\rho = R_{a+1}(\sigma) d\sigma \). Similarly, the second component is computed as follows.

\[
\begin{align*}
&= \frac{-1}{2\pi i} \int_{\gamma_{R_{a+1}}(c)} \left( \Delta_{R_{a+1}}(c) \left[ G_{R_{a+1}} \circ Q_{a+1} \right] \right)(\sigma) \left[ L_S \chi_{R_{a+1}} \right]_{c} \ d\sigma \\
&= \frac{-1}{2\pi i} \int_{\gamma_{R_{a+1}}(c)} \left( \Delta_{R_{a+1}}(c) \left[ G_{R_{a+1}} \circ Q_{a+1} \right] \right)(\rho) \left[ \chi_{R_{a+1}} \right]_{c} \ d\rho \\
&= \left[ \chi_{S} \cdot \frac{1}{2\pi i} \int_{\gamma_{R_{a+1}}(c)} \left( \Delta_{R_{a+1}}(c) \left[ G_{R_{a+1}} \circ Q_{a+1} \right] \right)(\rho) \left[ \chi_{R_{a+1}} \right]_{c} \ d\rho \right]_{c}
\end{align*}
\]

\section{7 Eigenvalue problem}

In this section, we consider the eigenvalue problem for our transfer operator \( L_S \) restricted to an invariant subspace of pre-microfunctions \( M_{(R)}(c) \) defined in section 4 as

\[
M_{(R)}(c) = \bigoplus_{k=0}^{\infty} M_{R_{k}(c)}.
\]

Here, the sum is considered as formal sum. In the case of postcritically finite maps, the post critical set \( P(R) \) is a finite set and the sum is finite. In this case

\[
M_{P(R)}(c) = \bigoplus_{P \in P(R)} M_{P}.
\]
In order to analyze the eigenvalue problem of $L_s$, we consider a formal sum of pre-minifications.

$$
\hat{h} = \sum_{k=0}^{\infty} \hat{h}_k
$$

with $\hat{h}_k \in \mathcal{M}_R(c)$.

**Proposition 7.1.** If $h$ is an eigenfunction of $L_s$ satisfying

$$
I_{L_s} h = h
$$

and $P(R)$ is infinite,

then

$$
\lambda \left[ L_s \hat{h}_k \right]_{R_{k+1}(c)} = \hat{h}_{k+1}
$$

and

$$
\lambda \sum_{k=0}^{\infty} \left[ L_s \hat{h}_k \right]_c = \hat{h}_0.
$$

**Proof.** By a straightforward calculation, we have

$$
L_s \hat{h}_k = \left[ L_s \hat{h}_k \right]_{R_{k+1}(c)} + \left[ L_s \hat{h}_k \right]_c.
$$

**Theorem 7.2.** The eigenvalue problem $\lambda L_s h = h$ of the transfer operator $L_s : \mathcal{M}_R(c) \rightarrow \mathcal{M}_R(c)$ reduces to an "eigenvalue" problem of an integral operator

$$
\mathcal{T}_s : \mathcal{C}_0(Y_c) \rightarrow \mathcal{C}_0(Y_c)
$$

defined by

$$
(\mathcal{T}_s \varphi)(u) = (A_c \varphi, \psi) \cdot \frac{1}{2\pi i} \int_{Y_c} H_s(u, t; \lambda) \varphi(t) \, dt
$$

where

$$
H_s(u, t; \lambda) = -\sum_{k=0}^{\infty} \lambda^k \chi_{S_k}(t) R_{k+1}(t) \chi_{K_{k+1}}(u),
$$

with $\lambda \mathcal{T}_s \hat{h}_0 = \hat{h}_0$, $\hat{h}_0 = A_c \hat{h}_0$.

**Proof.** As $R_{k+1} = \lambda \left[ L_s \hat{h}_k \right]_{R_{k+1}(c)} = \lambda \left[ \chi_{S_k} \circ Q_{k+1} \right]_{R_{k+1}(c)}$

$$
= \lambda \left[ \chi_{S_k} \circ Q_{k+1} \cdot \chi_{S_{k+1}} \circ Q_{k+1} \cdot \chi_{S_{k+1}} \circ Q_{k+1} \right]_{R_{k+1}(c)}
$$

and $\hat{h}_{k+1} = \left[ \chi_{S_k} \circ Q_{k+1} \right]_{R_{k+1}(c)} = \left[ \chi_{S_k} \circ Q_{k+1} \cdot \chi_{S_{k+1}} \circ Q_{k+1} \right]_{R_{k+1}(c)}$,

we have

$$
g_{k+1} = \lambda^k \hat{h}_{k+1} = \lambda^k \hat{h}_0
$$

for $k \geq 0$.

Hence $g_k = \lambda^k \hat{h}_0$, which implies

$$
\hat{h}_k = \left[ \chi_{S_k} \circ Q_{k+1} \right]_{R_{k+1}(c)} = \left[ \lambda^k \hat{h}_0 \circ \chi_{S_{k+1}} \circ Q_{k+1} \right]_{R_{k+1}(c)}.
and \[ h_0 = \lambda \sum_{k=0}^{\infty} \left[ \psi_0 \cdot \left[ G \cdot Q_{\delta+1} \right] R_{\delta+1}(t) \right]_c = \lambda \sum_{k=0}^{\infty} \left[ \psi_0 \cdot \left[ G \cdot Q_{\delta+1} \right] R_{\delta+1}(t) \right] c \]
\[ = \lambda \sum_{k=0}^{\infty} \left[ \psi_0 \cdot \lambda^{k} \cdot \left[ \psi_0 \cdot Q_{\delta+1} \right] R_{\delta+1}(t) \right] c \]
\[ = \lambda \sum_{k=0}^{\infty} \left[ \psi_0 \cdot \lambda^{k} \cdot \frac{1}{2\pi i} \int_{C_\delta} \left( \Delta_{\psi_{\delta+k}} \left( \lambda \cdot Q_{\delta+1} \right) \right) \left( \psi_0 \cdot Q_{\delta+1} \right) (s) d\sigma \right] c \]
\[ = \lambda \sum_{k=0}^{\infty} \left[ \psi_0 \cdot \lambda^{k} \cdot \frac{1}{2\pi i} \int_{C_\delta} \left( \psi_{\delta+k} \cdot \lambda \cdot \left( \psi_{\delta+k} \cdot Q_{\delta+1} \right) \right) \left( \psi_0 \cdot Q_{\delta+1} \right) (s) d\sigma \right] c \]

(Here we changed variables \( t = Q_{\delta+1}(s) \) and \( d\sigma = R_{\delta+1}(t) dt \))

This yields an integral equation for \( x_0 \in M_c \):

\[ (\Delta_{\psi_0})_C = \lambda \sum_{k=0}^{\infty} \left[ \psi_0 \cdot \lambda^{k} \cdot \frac{1}{2\pi i} \int_{C_\delta} \left( \psi_{\delta+k} \cdot \lambda \cdot \left( \psi_{\delta+k} \cdot Q_{\delta+1} \right) \right) \left( \psi_0 \cdot Q_{\delta+1} \right) (s) d\sigma \right] c \]

more briefly

\[ x_0 = \lambda \sum_{k=0}^{\infty} \left[ \psi_0 \cdot \lambda^{k} \cdot \frac{1}{2\pi i} \int_{C_\delta} \left( \psi_{\delta+k} \cdot \lambda \cdot \left( \psi_{\delta+k} \cdot Q_{\delta+1} \right) \right) (s) d\sigma \right] c \]

by setting \( H_s(u, t; \lambda) = \frac{1}{2\pi i} \int_{C_\delta} \left( \psi_{\delta+k} \cdot \lambda \cdot \left( \psi_{\delta+k} \cdot Q_{\delta+1} \right) \right) (s) d\sigma \).


Let \( \mathcal{O}(J) \) denote the space of germs of holomorphic function along the Julia set \( J = J(\mathbb{C}) \). Each element of \( \mathcal{O}(J) \) has a representative \( f : J \to \mathbb{C} \) which can be extended to a holomorphic function in a neighborhood of \( J \). As \( J \) is a perfect set, the analytic continuation is uniquely determined by \( f \).

Topology in \( \mathcal{O}(J) \) is given by the uniform convergence on \( J \).

Linear functional \( G^T : \mathcal{O}(J) \to \mathbb{C} \) is said to be holomorphic if for any holomorphic family \( f : \lambda \to \mathcal{O}(J) \), \( G^T[f(\lambda)] : \lambda \to \mathbb{C} \) is holomorphic. We require the continuity of \( G^T \) with respect to the sup norms on \( \mathcal{O}(J) \). The space of continuous holomorphic linear functionals \( G^T : \mathcal{O}(J) \to \mathbb{C} \) will be denoted by \( \mathcal{O}^*(J) \).

As in the previous sections, \( \mathcal{O}_0(F) \) denotes the space of holomorphic functions in the Fatou set \( F \) vanishing at the infinity.

If \( p \in F \) then \( x_p = \frac{1}{2-p} \) belongs to \( \mathcal{O}(J) \). For holomorphic linear functional \( G^T \in \mathcal{O}_0(F) \), define a holomorphic function \( G^T \in \mathcal{O}_0(F) \) by \( G^T[z] = G^T[-\overline{z}] \). Then, family
of holomorphic functions \( F \to O(I) \) defined by \( z \mapsto z^2 \)

is a holomorphic family, \( g^F : F \to \mathbb{C} \) is a holomorphic function, since the functional \( g^F \) is holomorphic. By the continuity of \( g^F \), we see immediately \( g^F(z_0) = 0 \) and hence \( g^F \in O_0(F) \).

This correspondence between \( O^*(I) \) and \( O_0(F) \) is called the Cauchy transformation.

**Proposition 8.1.** For \( f_J \in O(I) \), \( g^F[f_J] \) can be expressed in a integration form

\[
g^F[f_J] = \oint_{\gamma_F} f_J(\tau) g^F(\tau) \, d\tau,
\]

where the integration path \( \gamma_F \) goes around the Julia set in the clockwise direction.

**Proof.** As \( f_J \) is holomorphic near \( J \), for \( z \) in a neighborhood of \( J \),

\[
f_J(z) = \frac{1}{2\pi i} \oint_{\gamma_J} f_J(\tau) \chi^J(\tau) \, d\tau = \frac{-1}{2\pi i} \oint_{\gamma_J} f_J(\tau) \bar{z}(\tau) \, d\tau,
\]

where \( \gamma_J \) runs around the Julia set in the counter-clockwise direction. The right hand side of this equality gives an expression of \( f_J(z) \) as a "linear combination" of unit poles.

We have

\[
g^F[f_J] = \frac{-1}{2\pi i} \oint_{\gamma_J} f_J(\tau) g^F[J^\tau] \, d\tau = \frac{-1}{2\pi i} \oint_{\gamma_J} f_J(\tau) g^F(\tau) \, d\tau
\]

\[
= \frac{1}{2\pi i} \oint_{\gamma_F} f_J(\tau) \bar{z}(\tau) \, d\tau.
\]

In the following, we shall denote such pairings of functions as

\[
\langle g^F, f_J \rangle_F = \frac{1}{2\pi i} \oint_{\gamma_F} f_J(\tau) g^F(\tau) \, d\tau.
\]

**Proposition 8.2.** \( O^*(I) \approx O_0(F) \)

**Proof.** The Cauchy transformation defines a complex linear map from \( O^*(I) \) to \( O_0(F) \), and the pairing along \( \gamma_F \) defines a complex linear map from \( O_0(F) \) to \( O^*(I) \). These two transformations are mutually inverse.
Let $\mathcal{O}_0^*(F)$ denote the space of holomorphic linear and continuous functional $G^F : \mathcal{O}_0(F) \to \mathbb{C}$.

**Proposition 8.3** $\mathcal{O}(J) \subset \mathcal{O}_0^*(F)$.

**Proof.** If $g^F \in \mathcal{O}(J)$, then $g^F \in \mathcal{O}_0(F)$. For $G^F \in \mathcal{O}_0^*(F)$, let $\chi_2^F(z) = G^F [-\chi_2]$. Then $g^F : J \to \mathbb{C}$ is a continuous function.

If $g^F \in \mathcal{O}(J)$, then $\exists f^F \in \mathcal{O}_0(F)$ with

$$f^F(z) = \frac{1}{2\pi i} \int_{r_f} f^F(t) \chi_2^F(t) \, dt = \frac{1}{2\pi i} \int_{r_f} f^F(t) \chi_2(t) \, dt,$$

we have

$$G^F \left[ f^F \right] = \frac{1}{2\pi i} \int_{r_f} f^F(t) G^F \left[ \chi_2^F \right] \, dt = \frac{1}{2\pi i} \int_{r_f} f^F(t) g^F(t) \, dt = \langle g^F, f^F \rangle_J,$$

where the integration path $r_f$ is given by $g^F$, which is holomorphic in a neighborhood of $J$. This pairing will be denoted as $\langle g^F, f^F \rangle_J$.

We define the third pairings $\langle \cdot, \cdot \rangle_J$ and $\langle \cdot, \cdot \rangle_F$ related to the external rays and pre-microfunctions. Let $M$ denote the space of pre-microfunctions and let $\mathcal{V}_M$ denote the "sum" of external rays supporting the pre-microfunctions. We use symbol $M$ to indicate the object is related to the pre-microfunction component. When we apply operations $\Delta_M$, $\Im_M$, etc. we take the "sum" of the objects over external rays. The dual space $M^\vee$ is the space of holomorphic linear and continuous functionals $G^M : M \to \mathbb{C}$.

For $\gamma_M$, we denote by $\tilde{r}_M$ the integration path passing along the external ray both sides of $\gamma_M$ coming from the infinity to $p$ on the negative side of $\gamma_M$ and coming back from $p$ to the infinity on the positive side of $\gamma_M$. If $g^M \in \mathcal{O}_0(\tilde{r}_M)$, then $g^M$ is a holomorphic function in a neighborhood of $\gamma_M$ with regular singularities at the infinity and each landing points.
For $f_M \in M$, we can rewrite it in the following form:

$$f_M(z) = \frac{1}{2\pi i} \int_{\gamma_M} (\Delta_M f_M)(\tau) \chi_M(\tau) d\tau = \frac{1}{2\pi i} \int_{\gamma_M} (\Delta_M f_M)(\tau) \chi_M(\tau) d\tau,$$

For $g^M_0 \in C(\gamma_M)$ define a holomorphic functional $G^M \in M^*$ by

$$G^M[f_M] = \langle g^M_0, \Delta_M f_M \rangle_M = \langle g^M, f_M \rangle_M^*.$$  

$G^M[f_M]$ is defined if $\Delta_M f_M, g^M_0 \in L^1(\gamma_M)$. Note that

$$G^M[-\chi_2] = \langle g^M, -\chi_2 \rangle_M^* = g^M(z).$$

by Cauchy's integration formula. Note that if $g^M_0 \in C(\gamma_M)$ and $g^M = \mathcal{T}_1 g^M_0$, then for $z \in C \setminus \gamma_M$

$$\mathcal{T}_1 g^M_0(z) = \langle g^M, \chi_3 \rangle_M^* = \langle g^M, \chi_3 \rangle_M^*$$

holds. If $f_M \in M$, then

$$G^M[f_M] = G^M[\frac{1}{2\pi i} \int_{\gamma_M} (\Delta_M f_M)(\tau) (-\chi_2^*) d\tau] = \frac{1}{2\pi i} \int_{\gamma_M} (\Delta_M f_M)(\tau) G^M[-\chi_2] d\tau = \langle g^M, \Delta_M f_M \rangle_M^*.$$

We have a splitting of pre-microfunctions,

$$f = f_J \oplus f_M \oplus f_F \in C_0(\gamma) \oplus M \oplus C_0(F)$$

and a splitting of its dual space

$$G = G_J \oplus G_M \oplus G_F \in C_0^*(\gamma) \oplus M^* \oplus C_0^*(F),$$

with Cauchy's transforms given by

$$G^J[-\chi_2] = g^J(z), \quad z \in F, \quad g^J \in C_0(F)$$

$$G^M[-\chi_2] = g^M(z), \quad z \in \gamma_M, \quad g^M \in C(\gamma_M)$$

$$G^M[\chi_5] = g^M(z), \quad z \in C \setminus \gamma_M, \quad g^M \in M$$

$$G^F[-\chi_2] = g^F(z), \quad z \in J, \quad g^F \in C(\gamma).$$

The pairing of $f$ on $G$ is defined by

$$G[f] = G^J[f_J] + G^M[f_M] + G^F[f_F] = \langle g^J, f_J \rangle_M + \langle g^M, \Delta_M f_M \rangle_M + \langle g^F, f_F \rangle_F.$$

The projections of $G = C_0(\gamma) \oplus M \oplus C_0(F)$ to components are denoted by $f \mapsto [f]_J, f \mapsto [f]_M, f \mapsto [f]_F$ respectively.
These projections are given by
\[ [f]_J(\tau) = \frac{1}{2\pi i} \int_{\gamma} f(\tau) \chi_2(\tau) \, d\tau \quad (\tau \in J), \]
\[ [f]_M(\tau) = \frac{1}{2\pi i} \int_{\gamma_M} (\lambda_M f)(\tau) \chi_2(\tau) \, d\tau \quad (\tau \in \gamma_M), \]
\[ [f]_F(\tau) = \frac{1}{2\pi i} \int_{\gamma} f(\tau) \chi_2(\tau) \, d\tau \quad (\tau \in F). \]

And the projections of \( \mathcal{N}^* = \mathcal{O}^*(J) \oplus \mathcal{M}^* \oplus \mathcal{O}^*(F) \) are denoted as
\[ [J]^J : \mathcal{N}^* \to \mathcal{O}^*(F) \subset \mathcal{O}^*(J), \]
\[ [J]^M : \mathcal{N}^* \to \mathcal{O}(\gamma_M) \subset \mathcal{M}^* \]
\[ [J]^F : \mathcal{N}^* \to \mathcal{O}(\gamma) \subset \mathcal{O}^*(F). \]

Let \( L^* \) denote the dual of our transfer operator \( L \). We abuse notations and confuse functionals and its Cauchy's transforms. \( L^* : \mathcal{C}_0(F) \oplus \mathcal{O}(\gamma_M) \oplus \mathcal{O}(\gamma) \to \mathcal{C}_0^* \) is decomposed as
\[ L^* = \begin{pmatrix} L^*_J & L^*_M & L^*_F \\ L^*_{JM} & L^*_M & L^*_{MF} \\ L^*_{JF} & L^*_{FM} & L^*_F \end{pmatrix}. \]

In the rest of this section, we compute these components more explicitly.

**Proposition 8.4.**
\[ L^*_J g^J = \chi_2 \circ \gamma \cdot R \cdot g^J \circ R - [\chi_2 \circ \gamma \cdot R \cdot g^J \circ R]_o, \]
\[ L^*_M g^J = \Delta_o [\chi_2 \circ \gamma \cdot R \cdot g^J \circ R], \]
\[ L^*_F g^J = 0. \]

**Proof.** For \( g^J \in \mathcal{O}^*(F) \), we compute \( (L^*_J g^J)(\gamma) \) for \( \gamma \in F.
\[ (L^*_J g^J)(\gamma) = \left[ (L^*_J g^J)(\chi_2) \right]^T = \left[ g^J \left[ \chi_2 \circ \gamma \cdot R \right] \chi_2 \right]^T. \]
\[ = \left[ g^J \left[ -\chi_2 (R(\gamma)) \cdot R(\gamma) \chi_2 (R(\gamma)) - [\chi_2 \circ \gamma \cdot R \cdot g^J \circ R]_o \right] \right]^T \]
\[ = \left[ g^J \left[ -\chi_2 (R(\gamma)) \cdot R(\gamma) \chi_2 (R(\gamma)) \right] \right]^T \quad \text{(since \( \left[\gamma_2 \circ \gamma \cdot R \right] \in M_\gamma \))}
\[ = \left[ \chi_2 (R(\gamma)) \cdot R(\gamma) \left[ -\chi_2 (R(\gamma)) \right] \right]^T = [\chi_2 (R(\gamma)) \cdot R(\gamma) \cdot g^J \left[ -\chi_2 (R(\gamma)) \right]_o. \]
\[ = \chi_2 (R(\gamma)) \cdot R(\gamma) \cdot g^J \left[ -\chi_2 (R(\gamma)) \right]_o. \]
Next, for $z \in \mathcal{M}$, $L^x_{\mathcal{L}^x} g^J \in \mathcal{U}(\mathcal{M})$ is defined as follows.

$$(L^x_{\mathcal{L}^x} g^J)(z) = [G^J [-\gamma_s(R(3)) \cdot R'(3) \cdot R(3) - \gamma_s \cdot R'(3) \cdot R(3)] \mathcal{J}_c]^M$$

$$= [G^J [-\gamma_s(R(3)) \cdot R'(3) \cdot G^J [-\gamma_s(R(3))] \mathcal{J}_c]^M = [-\gamma_s(R(3)) \cdot R'(3) \cdot G^J(R(3))] \mathcal{J}_c]$$

$$= \Delta_0 [\gamma_s \cdot R \cdot R' \cdot g^F \circ g \mathcal{J}_c](3) = \Delta_0 [\gamma_s \cdot R \cdot R' \cdot g^F \circ g \mathcal{J}_c](3)$$

For $z \in \mathcal{J}$, we have

$$(L^x_{\mathcal{L}^x} g^J)(z) = [G^J [-\gamma_s(R(3)) \cdot R'(3) \cdot R(3)] \mathcal{J}_c]^M$$

$$= [G^J [-\gamma_s(R(3)) \cdot R'(3) \cdot R(3) - \gamma_s \cdot R'(3) \cdot R(3)] \mathcal{J}_c]^M$$

$$= 0$$

The last equality holds since $\gamma_s(R(3)) \cdot R(3) \cdot R(3) \in \mathcal{U}(\mathcal{F})$ and $[\gamma_s \cdot R(3) \cdot R(3)] \mathcal{J}_c \in \mathcal{M}$.

**Proposition 8.5**

$L^x_{\mathcal{L}^x} g^F = 0$

$L^x_{\mathcal{L}^x} g^F = \Delta_0 [\gamma_s \cdot R \cdot R' \cdot g^F \circ g \mathcal{J}_c]$,

$L^x_{\mathcal{L}^x} g^F = \gamma_s \cdot R \cdot R' \cdot g^F \circ g - [\gamma_s \cdot R \cdot R' \cdot g^F \circ g \mathcal{J}_c]_0$.

**Proof.** For $z \in \mathcal{F}$, we have $-\gamma_s \notin \mathcal{O}(\mathcal{I})$. For $g^F \in \mathcal{O}(\mathcal{I})$,

$$(L^x_{\mathcal{L}^x} g^F)(3) = [G^x [-\gamma_s(R(3)) \cdot R'(3) \cdot R(3)] \mathcal{J}_c]^M = [G^F [-\gamma_s(R(3)) \cdot R'(3) \cdot R(3)] \mathcal{J}_c]^M$$

$$= [G^F [-\gamma_s(R(3)) \cdot R'(3) \cdot R(3) - \gamma_s \cdot R'(3) \cdot R(3)] \mathcal{J}_c]^M$$

$$= [-\gamma_s(R(3)) \cdot R'(3) \cdot G^F [-\gamma_s(R(3))] \mathcal{J}_c]^M = 0$$

Here, the last equality holds since $\gamma_s(R(3)) \cdot R(3) \cdot R(3) \in \mathcal{O}(\mathcal{I})$,

$[\gamma_s \cdot R(3) \cdot R(3) \cdot R(3)] \mathcal{J}_c \in \mathcal{M}$ and $G^F [-\gamma_s(R(3))] = 0$.

For $z \in \mathcal{J}$, $-\gamma_s \notin \mathcal{O}(\mathcal{F})$. Hence

$$(L^x_{\mathcal{L}^x} g^F)(3) = [G^x [-\gamma_s(R(3)) \cdot R'(3) \cdot R(3)] \mathcal{J}_c]^F = [G^F [-\gamma_s(R(3))] \mathcal{J}_c]^F$$

$$= [\gamma_s(R(3)) \cdot R(3) \cdot G^F [-\gamma_s(R(3))] \mathcal{J}_c]^F = [\gamma_s(R(3)) \cdot R(3) \cdot G^F(R(3))] \mathcal{J}_c$$

$$= \gamma_s \cdot R(3) \cdot R'(3) \cdot g^F \circ g \mathcal{J}_c(3).$$

We used the fact $-\gamma_s(R(3)) \cdot R(3) \cdot R(3) \in \mathcal{O}(\mathcal{F})$ and $[\gamma_s \cdot R(3) \cdot R(3)] \mathcal{J}_c \in \mathcal{M}$. 
To compute $L^*_{M_F} g^F$, we take $\mathfrak{z} \in \mathcal{Y}_M \subset F$. Then,

$$(L^*_{M_F} g^F)(\mathfrak{z}) = \left[ (L^* G^F) [-\chi_5] \right] M$$

$$= \left[ G^F \left[ -\chi_5 (R(3)) \cdot R(3) \chi_{R(3)} \right] - \left[ \chi_5 \cdot R(3) \chi_{R(3)} \right] \right] c \quad \left[ \mathfrak{z} \in \mathcal{Y}_M \right]$$

$$= \left[ \chi_5 (R(3)) \cdot R(3) \cdot G^F \left[ -\chi_{R(3)} \right] \right] M = \left[ \chi_5 (R(3)) \cdot R(3) \cdot g^F (R(3)) \right] M$$

$$= \Delta_\phi \left[ \chi_5 \cdot R \cdot R' \cdot g^F (R) \right] (\mathfrak{z})$$

In the above calculations, we used the fact $[\chi_5 \cdot R(3) \cdot \chi_{R(3)}] c = M_0$. During the computations, $\mathfrak{z}$ is regarded as constant and the final result gives the formula as a function of $\mathfrak{z}$.

Proposition 8.6. $L^*_{M_F} g^M = [\chi_5 \cdot R \cdot R' \cdot (I_M g^M) \cdot R] F$

$L^*_{MM} g^M = \left[ \chi_5 \cdot R \cdot R' \cdot g^M \cdot R \right]^M_P$

$L^*_{FM} g^M = [\chi_5 \cdot R \cdot R' \cdot (I_M g^M) \cdot R] F$

Proof. For $G^M \in M^*$, let $g^M (z) = G^M [-\chi_z]$, $z \in M$, $g^M \in C(M)$. For $\mathfrak{z} \in F$, (and $\mathfrak{z} \in C \setminus M$),

$$(L^*_{FM} g^M)(\mathfrak{z}) = \left[ (L^* G^M) [\chi_z] \right] F = \left[ G^M [-\chi_z \chi_z] \right] F$$

$$= \left[ G^M \left[ -\chi_5 (R(3)) \cdot R(3) \chi_{R(3)} \right] - \chi_5 \cdot R(3) \chi_{R(3)} \right] c \quad \left[ \mathfrak{z} \in \mathcal{Y}_\mathfrak{z} \right]$$

$$= \left[ \chi_5 (R(3)) \cdot R(3) \cdot G^M \left[ -\chi_{R(3)} \right] \right] F + \left[ \chi_5 \cdot R(3) \chi_{R(3)} \right] c$$

$$= \left[ \chi_5 (R(3)) \cdot R(3) \cdot g^M (R(3)) \chi_{R(3)} \right] F + \left[ \chi_5 \cdot R(3) \chi_{R(3)} \right] c$$

Here, the last equality holds since $\mathfrak{z} \in F (C \setminus M)$ and $\mathfrak{z}$ moves near $\mathfrak{z}$ along $F$. Next, we compute $L^*_{FM} g^M$. For $\mathfrak{z} \in F$, note that $R(3) \in J$ and $[\chi_5 \cdot R(3) \cdot \chi_{R(3)}] c = M$. Hence,
\[(\mathcal{L}^*_{FM} G^M)(3) = \left[ (\mathcal{L}^* G^M) R_{3} \right] F \]

\[= \left[ G^M \left[ \chi_s(3), R(3) \cdot \chi_{R(3)} \right] + \left[ \chi_s \cdot R(3) \cdot \chi_{R(3)} \right] \right] F \]

\[= \left[ \chi_s(3), R(3) \cdot \left( \mathcal{G} M R(3) \right) \right] F + \left[ \frac{1}{2\pi i} \int_{\gamma} g^M(t) R(3) \chi_{R(3)}(t) \right] F \]

The second term of the above line is computed as follows:

\[= \frac{1}{2\pi i} \int_{\gamma} g^M(t) R(3) \chi_{R(3)}(t) \int_{\gamma} d\tau \]

\[= \frac{1}{2\pi i} \int_{\gamma} g^M(t) R(3) \chi_{R(3)}(t) d\tau \]

\[= 0. \]

The last equality holds since \( R(3) \) is of degree one and the denominator \((z - 2)(t - R(3))\) is of degree three with respect to the variable of integration and the integration path \( \gamma \) turns around the Julia set along a circle of infinitely large radius. Hence we have:

\[(\mathcal{L}^*_{FM} G^M)(3) = \left[ \chi_s(3), R(3) \right] (x^M G^M) R(3) \]

Finally, let us compute \( \mathcal{L}^*_{HM} G^M \in \mathcal{O}^M(R) \).

For \( z \in \gamma_p \) with \( p \in P(R) \), the component \( \left( \mathcal{L}^*_{HM} G^M \right)_p \in \mathcal{O}^M(R) \) is computed as follows.

\[(\mathcal{L}^*_{HM} G^M)(3) = \left[ (\mathcal{L}^* G^M) [-\chi_s] \right]_{\gamma_p} = \left[ G^M [-\chi_s \chi_{R(3)}] \right]_{\gamma_p} \]

\[= \left[ \chi_s(3), R(3) \cdot \left( G^M [-\chi_{R(3)}] \right) + \left< \chi_{R(3)} \cdot R(3) \cdot \chi_{R(3)} \right] \right]_{\gamma_p} \]

\[= \left[ \chi_s(3), R(3) \right] \mathcal{G} M \left[ \frac{1}{2\pi i} \int_{\gamma} g^M(t) \left( \Delta t \gamma_s(t) \cdot R(3) \right) \chi_{R(3)}(t) \right] \]

\[= \left[ \chi_s \cdot R(3) \cdot \left( \mathcal{G} M \right) \right] \left( R \cdot \left( \mathcal{G} M \right) \right) \left( \chi_s \right) + \left[ R \cdot \left( \mathcal{G} M \right) \right] \chi_s \cdot \chi_{R(3)} \]

\[= \left[ \chi_{R(3)} \cdot \chi_s \cdot \chi_{R(3)} \right] \]

In the case \( p = 0, \ldots, p \), for \( z \in \gamma_p \), we have:

\[(\mathcal{L}^*_{HM} G^M)(3) = \left[ (\mathcal{L}^* G^M) [-\chi_s] \right]_{\gamma_p} = \left[ G^M [-\chi_s \chi_{R(3)}] \right]_{\gamma_p} \]

\[= \left[ G^M \left[ \chi_s \cdot R(3) \right] \chi_{R(3)} \right] \]
\[
= \left[ \frac{1}{2\pi i} \int \chi_3^M(\tau) \, d\tau \right]_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
= \left[ \frac{1}{2\pi i} \int \chi_3^M(R(\tau)) \, d\tau \right]_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
= R(\tau(3)) \left( \Delta_0 \left[ \chi_3^M \cdot \chi_{R(3)} \right]_0 R \right)(\tau(3)).
\]

8.9. Example

In this section, we compute the operator $L^*_{MM}$ more precisely for $R(\tau) = \mathbb{Z}^2 + i$ case. In this case, the critical value $c = i$ is periodic and the postcritical set $P(R) = \{ i, i-1, -i \}$ consists of three points.

Let us compute $L^*_{MM} g^M$ for $g^M \in \mathcal{O}(c)$, where $G_c^M \subseteq M_c^+$ and $g_3^M(\tau) = \tilde{G}_c^M[\chi_3^M]$ for $\tau \in \mathcal{X}_c$.

For $\tau \in \mathcal{X}_c$ with $p \neq 0$, $\tau \in P(R)$
\[
(L^*_{MM} g^M)(\tau) = \left[ \left( L^V \tilde{G}_c^M \right)[\chi_3^M] \right]_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
= \left[ G_c^M \left[ -\gamma \cdot R(\tau) \chi_{R(3)} \right] \right]_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
\]
Here, as $p \neq 0$, $R(\tau) \in \mathcal{X}_c$, $\chi_{R(3)} \in M_c$. And so, $[\chi_3^M \cdot R(\tau) \chi_{R(3)}]_c$ belongs to $M_c$, we have
\[
(L^*_{MM} g^M)(\tau) = \left[ G_c^M \left[ -\gamma \cdot R(\tau) \chi_{R(3)} \right] \right]_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
= \left[ \frac{1}{2\pi i} \int \chi_3^M(\tau) \, d\tau \right]_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
= \left[ R(\tau) \left[ \frac{1}{2\pi i} \int \chi_3^M(\tau) \, d\tau \right] \right]_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
= \left[ R(\tau) \left[ \frac{1}{2\pi i} \int \chi_3^M(\tau) \, d\tau \right] \right]_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
= \left( R^* \cdot I_c \left[ \frac{1}{2\pi i} \int \chi_3^M(\tau) \, d\tau \right] \right)_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
= \left( R^* \cdot I_c \left[ \frac{1}{2\pi i} \int \chi_3^M(\tau) \, d\tau \right] \right)_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
\]
For $\tau \in \mathcal{X}_c$
\[
(L^*_{MM} g^M)(\tau) = \left[ \left( L^V \tilde{G}_c^M \right)[\chi_3^M] \right]_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
= \left[ G_c^M \left[ -\gamma \cdot R(\tau) \chi_{R(3)} \right] \right]_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
= \left[ R(\tau) \left[ \frac{1}{2\pi i} \int \chi_3^M(\tau) \, d\tau \right] \right]_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
= \left[ R(\tau) \left[ \frac{1}{2\pi i} \int \chi_3^M(\tau) \, d\tau \right] \right]_{\tau = \tau(3)}^{\tau = \tau(3)} M^0
\]
\[
\left[ R(\beta) \left\{ \frac{1}{2\pi i} \int_{\Gamma_0} \frac{g_c^M(\tau) \psi_c(\tau)}{t-R(\beta)} d\tau \right\} \right]^{M_0}
= \left[ R(\beta) \left\{ \psi_s \cdot g_c^M \circ R(\beta) \right\} \right]^{M_0},
\]

where \( \{ \psi_s \cdot g_c^M \} \) denotes the regular part of \( \psi_s \cdot g_c^M \) along \( \Gamma_c \).

\[ \{ \psi_s \cdot g_c^M \} = \psi_s \cdot g_c^M - \int_c A_c \psi_s \cdot g_c^M \]

and is defined as
\[ \int_c A_c \psi_c = \frac{1}{2\pi i} \int_{\Gamma_0} \psi_s(\tau) \cdot \frac{g_c^M(\tau) d\tau}{t-\tau} \quad \text{for } \tau \in \Gamma_c \]

We have a decomposition
\[ \psi_s \cdot g_c^M = \{ \psi_s \cdot g_c^M \} + \int_c A_c \psi_c \]

with \( \{ \psi_s \cdot g_c^M \} \in M_c \) and \( \int_c A_c \psi_c \in \mathcal{C}(\Gamma_c) \).

Section 10. Complex conformal measures.

Let \( G \in \mathcal{B}(\mathcal{T}) \oplus \mathcal{M} \oplus \mathcal{O}_b^{\mathcal{F}} \). Let \( A \subset C \) be an open set with smooth boundary \( \partial A \) (oriented by the counter-clockwise direction). The characteristic function \( \chi_A(z) \) of \( A \) is expressed as
\[ \chi_A(z) = \frac{1}{2\pi i} \int_{\partial A} \chi_\eta(z) d\eta = \frac{1}{2\pi i} \int_{\partial A} \frac{d\eta}{z-\eta} \]

So, we can rewrite
\[ \chi_A = \frac{1}{2\pi i} \int_{\partial A} \chi_\eta d\eta. \]

Hence,
\[ G[\chi_A] = \frac{1}{2\pi i} \int_{\partial A} G[\chi_\eta] d\eta \]
defines a set function. If \( G = G^J + G^M + G^F \), then
\[ G[\chi_\eta] = g^J + g^M + g^F \]
with \( g^J \in \mathcal{O}_b^{\mathcal{T}} \), \( g^M \in \mathcal{M} \), \( g^F \in \mathcal{O}_b^{\mathcal{F}} \), and
\[ G[\chi_A] = \frac{1}{2\pi i} \int_{\partial A} (g^J(\eta) + g^M(\eta) + g^F(\eta)) d\eta \]
defines an additive set function. Suppose \( \lambda \) be a characteristic value of our transfer operator \( L_s \) and let \( f \in \mathcal{H} = \mathcal{B}(\mathcal{T}) \oplus \mathcal{M} \oplus \mathcal{O}_b^{\mathcal{F}} \) be an eigenfunction
Define a set function $M_g$ by
\[
M_g(A) = \frac{1}{2\pi i} \int_{\partial A} f(\tau) \, g(\tau) \, d\tau.
\]
Then, we have
\[
M_g(R^{-1}(A)) = \frac{1}{2\pi i} \int_{\partial R^{-1}(A)} f(\tau) \, g(\tau) \, d\tau
\]
\[
= \frac{1}{2\pi i} \int_{\partial R^{-1}(A)} f(\tau) \, \lambda(\mathcal{L}_\tau g)(\tau) \, d\tau
\]
\[
= \frac{1}{2\pi i} \int_{\partial R^{-1}(\partial A)} \lambda f(\tau) \, R(\tau) \, \mathcal{F} \left( R(\tau) \right) \, g(\tau) \, d\tau.
\]
Then by a change of variables $\zeta = R(\tau)$, we have
\[
M_g(R^{-1}(A)) = \frac{1}{2\pi i} \int_{\partial A} \lambda \mathcal{Y}_s(\zeta) \cdot g(\zeta) \left( \sum_{\tau \in R^{-1}(\partial A)} f(\tau) \right) \, d\zeta
\]
\[
= \frac{1}{2\pi i} \int_{\partial A} \lambda \mathcal{L}_s f(\zeta) \cdot g(\zeta) \, d\zeta
\]
\[
= M_g(A),
\]
where we used $\mathcal{L}_s f = \mathcal{Y}_s \cdot R \cdot f$ and $\mathcal{L}_s g = R^{-1}(\mathcal{Y}_s \circ g) \circ R$.

Our set function $M_g$ is backward invariant under $R$.

Finally, we consider the pull-back of the set function defined by the co-eigenfunction $g$. Suppose $\mathcal{L}_s^* g = g$; then, for $A$ with $R|_A : A \rightarrow R(A)$ injective, we have
\[
M_g(R(A)) = \frac{1}{2\pi i} \int_{\partial R(A)} g(\zeta) \, d\zeta
\]
\[
= \frac{1}{2\pi i} \int_{\partial A} \mathcal{Y}_s(R(\zeta))^{-1} \cdot R(\zeta) \cdot \mathcal{Y}_s(R(\zeta)) \cdot g(\zeta) \, d\zeta
\]
\[
= \frac{1}{2\pi i} \int_{\partial A} \mathcal{Y}_s(R(\zeta))^{-1} \cdot g(\zeta) \, d\zeta.
\]
This shows a kind of complex conformal property of the set function $M_g$ for co-invariant function $g$. 