Limit sets on Teichmüller space and on asymptotic Teichmüller space

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1 Introduction

This note is a survey of our recent researches on the action of the quasiconformal mapping class group on the Teichmüller space and on the asymptotic Teichmüller space.

We say that two quasiconformal homeomorphisms $f_1$ and $f_2$ on $R$ are equivalent if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. All homotopies are considered to be relative to the ideal boundary at infinity. The Teichmüller space $T(R)$ of a Riemann surface $R$ is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms $f$ on $R$. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d([f_1],[f_2]) = (1/2) \log K(f)$, where $f$ is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then $d$ is a complete distance on $T(R)$ which is called the Teichmüller distance.

We say that a quasiconformal homeomorphism $f$ on $R$ is asymptotically conformal if for every $\epsilon > 0$, there exists a compact subset $V$ of $R$ such that the maximal dilatation $K(f|_{R-V})$ of the restriction of $f$ to $R-V$ is less than $1+\epsilon$. We say that two quasiconformal homeomorphisms $f_1$ and $f_2$ on $R$ are asymptotically equivalent if there exists an asymptotically conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. The asymptotic Teichmüller space $AT(R)$ of a Riemann surface $R$ is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal homeomorphisms $f$ on $R$. Since a conformal homeomorphism is asymptotically conformal, there is a natural projection $\pi : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic Teichmüller equivalence class $[[f]] \in AT(R)$.

For a quasiconformal homeomorphism $f$ of $R$, the boundary dilatation of $f$ is defined by $H^*(f) = \inf K(f|_{R-V})$, where the infimum is taken over all compact subsets $V$ of $R$. Furthermore, for a Teichmüller equivalence class $[f] \in T(R)$, the boundary dilatation of $[f]$ is defined by $H([f]) = \inf H^*(g)$, where the infimum is taken over all elements $g \in [f]$. A distance between two points $[[f_1]]$ and $[[f_2]]$ in $AT(R)$ is defined by $d_A([[f_1]], [[f_2]]) = (1/2) \log H([f_2 \circ f_1^{-1}])$, where $[f_2 \circ f_1^{-1}]$ is a Teichmüller equivalence class of $f_2 \circ f_1^{-1}$ in $T(f_1(R))$. Then $d_A$
is a complete distance on $AT(R)$, which is called the asymptotic Teichmüller distance. For every point $[[f]] \in AT(R)$, there exists an asymptotically extremal element $f_0 \in [[f]]$ in the sense that $H([f]) = H^*(f_0)$.

The quasiconformal mapping class is the homotopy equivalence class $[g]$ of quasiconformal automorphisms $g$ of a Riemann surface, and the quasiconformal mapping class group $MCG(R)$ of $R$ is the group of all quasiconformal mapping classes on $R$. Every element $[g] \in MCG(R)$ induces a biholomorphic automorphism $[g]^*$ of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$ and a biholomorphic automorphism $[g]_{**}$ of $AT(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$. Let $Aut(T(R))$ and $Aut(AT(R))$ be the groups of all biholomorphic automorphisms of $T(R)$ and $AT(R)$ respectively. Then we have a homomorphism

$$\iota: MCG(R) \rightarrow Aut(T(R))$$

given by $[g] \mapsto [g]^*$ and a homomorphism

$$\iota_A : MCG(R) \rightarrow Aut(AT(R))$$

given by $[g] \mapsto [g]_{**}$. We define the Teichmüller modular group

$$Mod(R) := \iota(MCG(R))$$

and the asymptotic Teichmüller modular group (the geometric automorphism group)

$$Mod_A(R) := \iota_A(MCG(R)).$$

## 2 Characterization of the asymptotically trivial mapping class group

To observe the dynamics of the action of $Mod(R)$ on $T(R)$ and $Mod_A(R)$ on $AT(R)$, first we have to investigate properties of the homomorphisms $\iota$ and $\iota_A$. It is known that the homomorphism $\iota$ is injective (faithful) for all Riemann surfaces $R$ of non-exceptional type. Here we say that a Riemann surface $R$ is of exceptional type if $R$ has finite hyperbolic area and satisfies $2g + n \leq 4$, where $g$ is the genus of $R$ and $n$ is the number of punctures of $R$. Furthermore, the homomorphism $\iota$ is also surjective for all Riemann surfaces $R$ of non-exceptional type. Thus we may identify $Mod(R)$ with $MCG(R)$. However, the homomorphism $\iota_A$ is not injective, namely $\text{Ker} \ i_A \neq \{[id]\}$, unless $R$ is either the unit disc or a once-punctured disc. We give a geometric characterization of quasiconformal mapping classes belonging to $\text{Ker} \ i_A$. To state our result, we define two subgroups of the quasiconformal mapping class group.

**Definition 2.1** The pure mapping class group $P(R)$ is the group of all $[g] \in MCG(R)$ such that $g$ fixes all non-cuspidal ends of $R$. The eventually trivial mapping class group $E(R)$ is the group of all eventually trivial mapping classes. Here $[g] \in MCG(R)$ is said to be eventually trivial if there exists a compact subsurface $V_0$ of $R$ such that, for each connected component $W$ of $R - V_0$ that is not a cusp neighborhood, the restriction $g|_W : W \rightarrow R$ is homotopic to the inclusion map $id|_W : W \hookrightarrow R$. 


Then we have the following.

**Proposition 2.2 ([3])** The inclusion relation \( E(R) \subseteq \text{Ker} \iota_A \subseteq P(R) \) holds.

Each inclusion in Proposition 2.2 is proper, in general. However, under a certain condition on hyperbolic geometry of Riemann surfaces, we have a characterization of \( \text{Ker} \iota_A \).

**Definition 2.3** We say that a Riemann surface \( R \) satisfies the **bounded geometry condition** if \( R \) satisfies the following three conditions:

(i) **the lower bound condition**: the injectivity radius at any point of \( R \) except cusp neighborhoods are uniformly bounded away from zero.

(ii) **the upper bound condition**: there exists a subdomain \( \hat{R} \) of \( R \) such that the injectivity radius at any point of \( \hat{R} \) is uniformly bounded from above and that the simple closed curves in \( \hat{R} \) carry the fundamental group of \( R \).

(iii) \( R \) has no ideal boundary at infinity, namely the Fuchsian model of \( R \) is of the first kind.

We state our result.

**Theorem 2.4 ([5])** Let \( R \) be a Riemann surface satisfying the bounded geometry condition. Then \( E(R) = \text{Ker} \iota_A \).

**Remark 2.5** If \( R \) satisfies the bounded geometry condition, then \( \text{Ker} \iota_A \) is a proper subset of \( \text{MCG}(R) \). Namely, the action of \( \text{MCG}(R) \) on \( AT(R) \) is non-trivial. See [2, Corollary 3.5]. However, there exists a Riemann surface \( R \) that does not satisfy the bounded geometry condition and that \( E(R) = \text{Ker} \iota_A = \text{MCG}(R) \). See [7].

3 Discontinuity of mapping class group on Teichmüller space

We say that a subgroup \( G \subseteq \text{MCG}(R) \) acts at a point \( p \in T(R) \) **discontinuously** if there exists a neighborhood \( U \) of \( p \) such that the number of elements \( g \in G \) satisfying \( g_* (U) \cap U \neq \emptyset \) is finite, namely if the orbit \( \{g_* (p) \mid g \in G \} \) is discrete and the stabilizer subgroup \( \text{Stab}_G (p) = \{g \in G \mid g_* (p) = p\} \) is finite.

We consider the discontinuity of the pure and eventually trivial mapping class groups on the Teichmüller space. It was proved in [6] that, for a special planar Riemann surface, the pure mapping class group acts on the Teichmüller space discontinuously. We generalize this result in the following form.

**Theorem 3.1 ([3])** Let \( R \) be a Riemann surface satisfying the bounded geometry condition and having more than two non-cuspidal ends. Then the pure mapping class group \( P(R) \) acts on the Teichmüller space \( T(R) \) discontinuously.

For Riemann surfaces the number of whose non-cuspidal ends are at most two, Theorem 3.1 is not true. However, concerning the action of \( E(R) \), we always have the following.
Theorem 3.2 ([3]) Let $R$ be a Riemann surface satisfying the bounded geometry condition. Then the eventually trivial mapping class group $E(R)$ acts on $T(R)$ discontinuously.

By Theorems 2.4 and 3.2, we have the following corollary immediately.

Corollary 3.3 Let $R$ be a Riemann surface satisfying the bounded geometry condition. Then $\text{Ker} \, \iota_A$ acts on $T(R)$ discontinuously.

We apply Corollary 3.3 in the next section.

4 Dynamics of mapping class group on asymptotic Teichmüller space

For the Teichmüller modular group $\text{Mod}(R)$, we define the limit set $\Lambda(\text{Mod}(R))$ on $T(R)$ as the set of points $p \in T(R)$ such that $\gamma_n(p) \to p$ ($n \to \infty$) for a sequence of distinct elements $\gamma_n \in \text{Mod}(R)$. Then for every point $p \in \Lambda(\text{Mod}(R))$, the action of $\text{MCG}(R)$ at $p$ is not discontinuous. Also, for the asymptotic Teichmüller modular group $\text{Mod}_A(R)$, we define the limit set $\Lambda(\text{Mod}_A(R))$ on $AT(R)$ as the set of points $\hat{p} \in AT(R)$ such that $\hat{\gamma}_n(\hat{p}) \to \hat{p}$ ($n \to \infty$) for a sequence of distinct elements $\hat{\gamma}_n \in \text{Mod}_A(R)$. The following theorem says that the actions of the quasiconformal mapping class group on $T(R)$ and on $AT(R)$ are quite different.

Proposition 4.1 ([2]) There exists a Riemann surface $R$ satisfying the bounded geometry condition such that $\Lambda(\text{Mod}(R)) = \emptyset$ and $\Lambda(\text{Mod}_A(R)) \neq \emptyset$.

On the other hand, if a Riemann surface has a certain property, then the dynamics of the actions of the quasiconformal mapping class group on $T(R)$ and on $AT(R)$ are same.

Proposition 4.2 ([2]) Let $R$ be a Riemann surface that does not satisfy the lower bound condition. Then $\Lambda(\text{Mod}(R)) = T(R)$ and $\Lambda(\text{Mod}_A(R)) = AT(R)$.

We consider the problem whether $\pi(\Lambda(\text{Mod}(R))) \subset \Lambda(\text{Mod}_A(R))$ for all Riemann surfaces $R$. Proposition 4.2 gives a partial answer to this problem. Furthermore, we have proved in [4] that this problem is also true for the limit sets of the subgroups of $\text{Mod}(R)$ and $\text{Mod}_A(R)$ induced by a cyclic group $\langle g \rangle \subset \text{MCG}(R)$, where $g$ is a conformal automorphism of infinite order of a Riemann surface $R$. By Corollary 3.3, we have the following general statement.

Corollary 4.3 ([5]) Let $R$ be a Riemann surface satisfying the bounded geometry condition. Then $\pi(\Lambda(\text{Mod}(R))) \subset \Lambda(\text{Mod}_A(R))$.

Proof. We take a limit point $p \in \Lambda(\text{Mod}(R))$ arbitrarily. Then there exists a sequence $[g_n]$ of distinct elements of $\text{MCG}(R)$ such that $d([g_n]_*(p), p) \to 0$ as $n \to \infty$. Then $d_A([g_n]_*(\hat{p}), \hat{p}) \to 0$ for the projection $\hat{p} = \pi(p)$. We will show that $\{[g_n]_*\}_{n \in \mathbb{Z}} \subset \text{Mod}_A(R)$ contains infinitely many elements. Then
we conclude that \( \hat{p} \in \Lambda(\text{Mod}_{A}(R)) \). Suppose to the contrary that \( \{[g_{n}]_{**}\}_{n \in \mathbb{Z}} \) is a finite set \( \{[g_{1}]_{**}, \ldots, [g_{k}]_{**}\} \) for some \( k \geq 1 \). Then there exists an integer \( i (1 \leq i \leq k) \), say 1, such that \( [g_{n}]_{**} = [g_{1}]_{**} \) for infinitely many \( n \). Let \( \gamma_{n} := g_{n} \circ g_{1}^{-1} \). Then \( \gamma_{n} \in \text{Ker} \iota_{A} \) and \( d([\gamma_{n}]_{*}(g_{1}(p)), p) = d([g_{n}]_{*}(p), p) \rightarrow 0 \).

This means that the point \( p \in T(R) \) is a limit point for the subgroup \( \text{Ker} \iota_{A} \). This contradicts Corollary 3.3. Thus we conclude that \( \{[g_{n}]_{**}\}_{n \in \mathbb{Z}} \) contains infinitely many elements.

We have already proved in [1] that \( \Lambda(\text{Mod}(R))_{\neq} \subset T(R) \) for a Riemann surface \( R \) with the bounded geometry condition. A similar statement is true also for the limit set on the asymptotic Teichmüller space.

**Theorem 4.4 ([5])** Let \( R \) be a Riemann surface satisfying the bounded geometry condition. Then \( \Lambda(\text{Mod}_{A}(R))_{\neq} \subset AT(R) \).

**References**


