Limit sets on Teichmüller space and on asymptotic Teichmüller space

Ege Fujikawa

Department of Mathematics, Sophia University

1 Introduction

This note is a survey of our recent researches on the action of the quasiconformal mapping class group on the Teichmüller space and on the asymptotic Teichmüller space.

We say that two quasiconformal homeomorphisms \(f_1\) and \(f_2\) on \(R\) are equivalent if there exists a conformal homeomorphism \(h : f_1(R) \to f_2(R)\) such that \(f_2^{-1} \circ h \circ f_1\) is homotopic to the identity. All homotopies are considered to be relative to the ideal boundary at infinity. The Teichmüller space \(T(R)\) of a Riemann surface \(R\) is the set of all equivalence classes \([f]\) of quasiconformal homeomorphisms \(f\) on \(R\). A distance between two points \([f_1]\) and \([f_2]\) in \(T(R)\) is defined by \(d([f_1],[f_2]) = (1/2) \log K(f)\), where \(f\) is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation \(K(f)\) is minimal in the homotopy class of \(f_2 \circ f_1^{-1}\). Then \(d\) is a complete distance on \(T(R)\) which is called the Teichmüller distance.

We say that a quasiconformal homeomorphism \(f\) on \(R\) is asymptotically conformal if for every \(\epsilon > 0\), there exists a compact subset \(V\) of \(R\) such that the maximal dilatation \(K(f|_{R-V})\) of the restriction of \(f\) to \(R-V\) is less than \(1+\epsilon\). We say that two quasiconformal homeomorphisms \(f_1\) and \(f_2\) on \(R\) are asymptotically equivalent if there exists an asymptotically conformal homeomorphism \(h : f_1(R) \to f_2(R)\) such that \(f_2^{-1} \circ h \circ f_1\) is homotopic to the identity. The asymptotic Teichmüller space \(AT(R)\) of a Riemann surface \(R\) is the set of all asymptotic equivalence classes \([f]\) of quasiconformal homeomorphisms \(f\) on \(R\). Since a conformal homeomorphism is asymptotically conformal, there is a natural projection \(\pi : T(R) \to AT(R)\) that maps each Teichmüller equivalence class \([f]\) in \(T(R)\) to the asymptotic Teichmüller equivalence class \([f]\) in \(AT(R)\). For a quasiconformal homeomorphism \(f\) of \(R\), the boundary dilatation of \(f\) is defined by \(H^*(f) = \inf K(f|_{R-V})\), where the infimum is taken over all compact subsets \(V\) of \(R\). Furthermore, for a Teichmüller equivalence class \([f]\) in \(T(R)\), the boundary dilatation of \([f]\) is defined by \(H([f]) = \inf H^*(g)\), where the infimum is taken over all elements \(g \in [f]\). A distance between two points \([f_1]\) and \([f_2]\) in \(AT(R)\) is defined by \(d_A([f_1],[f_2]) = (1/2) \log H([f_2 \circ f_1^{-1}])\), where \([f_2 \circ f_1^{-1}]\) is a Teichmüller equivalence class of \(f_2 \circ f_1^{-1}\) in \(T(f_1(R))\). Then \(d_A\)
is a complete distance on $AT(R)$, which is called the asymptotic Teichmüller distance. For every point $[[f]] \in AT(R)$, there exists an asymptotically extremal element $f_0 \in [[f]]$ in the sense that $H([f]) = H^*(f_0)$.

The \textit{quasiconformal mapping class} is the homotopy equivalence class $[g]$ of quasiconformal automorphisms $g$ of a Riemann surface, and the \textit{quasiconformal mapping class group} $\text{MCG}(R)$ of $R$ is the group of all quasiconformal mapping classes on $R$. Every element $[g] \in \text{MCG}(R)$ induces a biholomorphic automorphism $[g]_*$ of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$ and a biholomorphic automorphism $[g]_{**}$ of $AT(R)$ by $[[f]] \mapsto [[[f \circ g^{-1}]]]$. Let $\text{Aut}(T(R))$ and $\text{Aut}(AT(R))$ be the groups of all biholomorphic automorphisms of $T(R)$ and $AT(R)$ respectively. Then we have a homomorphism

$$\iota : \text{MCG}(R) \rightarrow \text{Aut}(T(R))$$

given by $[g] \mapsto [g]_*$ and a homomorphism

$$\iota_A : \text{MCG}(R) \rightarrow \text{Aut}(AT(R))$$

given by $[g] \mapsto [g]_{**}$. We define the Teichmüller modular group

$$\text{Mod}(R) := \iota(\text{MCG}(R))$$

and the asymptotic Teichmüller modular group (the geometric automorphism group)

$$\text{Mod}_A(R) := \iota_A(\text{MCG}(R)).$$

\section{Characterization of the asymptotically trivial mapping class group}

To observe the dynamics of the action of $\text{Mod}(R)$ on $T(R)$ and $\text{Mod}_A(R)$ on $AT(R)$, first we have to investigate properties of the homomorphisms $\iota$ and $\iota_A$. It is known that the homomorphism $\iota$ is injective (faithful) for all Riemann surfaces $R$ of non-exceptional type. Here we say that a Riemann surface $R$ is of \textit{exceptional type} if $R$ has finite hyperbolic area and satisfies $2g + n \leq 4$, where $g$ is the genus of $R$ and $n$ is the number of punctures of $R$. Furthermore, the homomorphism $\iota$ is also surjective for all Riemann surfaces $R$ of non-exceptional type. Thus we may identify $\text{Mod}(R)$ with $\text{MCG}(R)$. However, the homomorphism $\iota_A$ is not injective, namely $\text{Ker} \ iota_A \neq \{[id]\}$, unless $R$ is either the unit disc or a once-punctured disc. We give a geometric characterization of quasiconformal mapping classes belonging to $\text{Ker} \ iota_A$. To state our result, we define two subgroups of the quasiconformal mapping class group.

\textbf{Definition 2.1} The \textit{pure mapping class group} $P(R)$ is the group of all $[g] \in \text{MCG}(R)$ such that $g$ fixes all non-cuspidal ends of $R$. The \textit{eventually trivial mapping class group} $E(R)$ is the group of all eventually trivial mapping classes. Here $[g] \in \text{MCG}(R)$ is said to be \textit{eventually trivial} if there exists a compact subsurface $V_n$ of $R$ such that, for each connected component $W$ of $R - V_n$ that is not a cusp neighborhood, the restriction $g|_W : W \rightarrow R$ is homotopic to the inclusion map $id|_W : W \hookrightarrow R$. 

Then we have the following.

**Proposition 2.2 ([3])** The inclusion relation $E(R) \subset \text{Ker } \iota_A \subset P(R)$ holds.

Each inclusion in Proposition 2.2 is proper, in general. However, under a certain condition on hyperbolic geometry of Riemann surfaces, we have a characterization of $\text{Ker } \iota_A$.

**Definition 2.3** We say that a Riemann surface $R$ satisfies the **bounded geometry condition** if $R$ satisfies the following three conditions:

(i) the lower bound condition: the injectivity radius at any point of $R$ except cusp neighborhoods are uniformly bounded away from zero.

(ii) the upper bound condition: there exists a subdomain $\tilde{R}$ of $R$ such that the injectivity radius at any point of $\tilde{R}$ is uniformly bounded from above and that the simple closed curves in $\tilde{R}$ carry the fundamental group of $R$.

(iii) $R$ has no ideal boundary at infinity, namely the Fuchsian model of $R$ is of the first kind.

We state our result.

**Theorem 2.4 ([5])** Let $R$ be a Riemann surface satisfying the bounded geometry condition. Then $E(R) = \text{Ker } \iota_A$.

**Remark 2.5** If $R$ satisfies the bounded geometry condition, then $\text{Ker } \iota_A$ is a proper subset of $\text{MCG}(R)$. Namely, the action of $\text{MCG}(R)$ on $AT(R)$ is non-trivial. See [2, Corollary 3.5]. However, there exists a Riemann surface $R$ that does not satisfy the bounded geometry condition and that $E(R) = \text{Ker } \iota_A = \text{MCG}(R)$. See [7].

3 Discontinuity of mapping class group on Teichmüller space

We say that a subgroup $G \subset \text{MCG}(R)$ acts at a point $p \in T(R)$ discontinuously if there exists a neighborhood $U$ of $p$ such that the number of elements $g \in G$ satisfying $g_*(U) \cap U \neq \emptyset$ is finite, namely if the orbit $\{g_*(p) \mid g \in G\}$ is discrete and the stabilizer subgroup $\text{Stab}_G(p) = \{g \in G \mid g_*(p) = p\}$ is finite.

We consider the discontinuity of the pure and eventually trivial mapping class groups on the Teichmüller space. It was proved in [6] that, for a special planar Riemann surface, the pure mapping class group acts on the Teichmüller space discontinuously. We generalize this result in the following form.

**Theorem 3.1 ([3])** Let $R$ be a Riemann surface satisfying the bounded geometry condition and having more than two non-cuspidal ends. Then the pure mapping class group $P(R)$ acts on the Teichmüller space $T(R)$ discontinuously.

For Riemann surfaces the number of whose non-cuspidal ends are at most two, Theorem 3.1 is not true. However, concerning the action of $E(R)$, we always have the following.
Theorem 3.2 ([3]) Let $R$ be a Riemann surface satisfying the bounded geometry condition. Then the eventually trivial mapping class group $E(R)$ acts on $T(R)$ discontinuously.

By Theorems 2.4 and 3.2, we have the following corollary immediately.

Corollary 3.3 Let $R$ be a Riemann surface satisfying the bounded geometry condition. Then $\text{kerr}_A$ acts on $T(R)$ discontinuously.

We apply Corollary 3.3 in the next section.

4 Dynamics of mapping class group on asymptotic Teichmüller space

For the Teichmüller modular group $\text{Mod}(R)$, we define the limit set $\Lambda(\text{Mod}(R))$ on $T(R)$ as the set of points $p \in T(R)$ such that $\gamma_n(p) \to p$ ($n \to \infty$) for a sequence of distinct elements $\gamma_n \in \text{Mod}(R)$. Then for every point $p \in \Lambda(\text{Mod}(R))$, the action of $\text{MCG}(R)$ at $p$ is not discontinuous. Also, for the asymptotic Teichmüller modular group $\text{Mod}_A(R)$, we define the limit set $\Lambda(\text{Mod}_A(R))$ on $AT(R)$ as the set of points $\hat{p} \in AT(R)$ such that $\hat{\gamma}_n(\hat{p}) \to \hat{p}$ ($n \to \infty$) for a sequence of distinct elements $\hat{\gamma}_n \in \text{Mod}_A(R)$. The following theorem says that the actions of the quasiconformal mapping class group on $T(R)$ and on $AT(R)$ are quite different.

Proposition 4.1 ([2]) There exists a Riemann surface $R$ satisfying the bounded geometry condition such that $\Lambda(\text{Mod}(R)) = \emptyset$ and $\Lambda(\text{Mod}_A(R)) \neq \emptyset$.

On the other hand, if a Riemann surface has a certain property, then the dynamics of the actions of the quasiconformal mapping class group on $T(R)$ and on $AT(R)$ are same.

Proposition 4.2 ([2]) Let $R$ be a Riemann surface that does not satisfy the lower bound condition. Then $\Lambda(\text{Mod}(R)) = T(R)$ and $\Lambda(\text{Mod}_A(R)) = AT(R)$.

We consider the problem whether $\pi(\Lambda(\text{Mod}(R))) \subset \Lambda(\text{Mod}_A(R))$ for all Riemann surfaces $R$. Proposition 4.2 gives a partial answer to this problem. Furthermore, we have proved in [4] that this problem is also true for the limit sets of the subgroups of $\text{Mod}(R)$ and $\text{Mod}_A(R)$ induced by a cyclic group $\{[g]\} \subset \text{MCG}(R)$, where $g$ is a conformal automorphism of infinite order of a Riemann surface $R$. By Corollary 3.3, we have the following general statement.

Corollary 4.3 ([5]) Let $R$ be a Riemann surface satisfying the bounded geometry condition. Then $\pi(\Lambda(\text{Mod}(R))) \subset \Lambda(\text{Mod}_A(R))$.

Proof. We take a limit point $p \in \Lambda(\text{Mod}(R))$ arbitrarily. Then there exists a sequence $\{g_n\}$ of distinct elements of $\text{MCG}(R)$ such that $d([g_n],(p),p) \to 0$ as $n \to \infty$. Then $d_A([g_n]_*(\hat{p}),\hat{p}) \to 0$ for the projection $\hat{p} = \pi(p)$. We will show that $\{[g_n]_*\}_{n \in \mathbb{Z}} \subset \text{Mod}_A(R)$ contains infinitely many elements. Then
we conclude that \( \hat{p} \in \Lambda(\text{Mod}_A(R)) \). Suppose to the contrary that \( \{[g_n]_{**}\}_{n \in \mathbb{Z}} \) is a finite set \( \{[g_1]_{**}, \ldots, [g_k]_{**}\} \) for some \( k \geq 1 \). Then there exists an integer \( i \ (1 \leq i \leq k) \), say \( 1 \), such that \( [g_n]_{**} = [g_1]_{**} \) for infinitely many \( n \). Let \( \gamma_n := g_n \circ g_1^{-1}. \) Then \( [\gamma_n] \in \text{Ker}\iota_A \) and \( d([\gamma_n]_{*}(g_1(p)), p) = d([g_n]_{*}(p), p) \to 0. \) This means that the point \( p \in T(R) \) is a limit point for the subgroup \( \text{Ker}\iota_A \). This contradicts Corollary 3.3. Thus we conclude that \( \{[g_n]_{**}\}_{n \in \mathbb{Z}} \) contains infinitely many elements.

We have already proved in [1] that \( \Lambda(\text{Mod}(R)) \subsetneq T(R) \) for a Riemann surface \( R \) with the bounded geometry condition. A similar statement is true also for the limit set on the asymptotic Teichmüller space.

\textbf{Theorem 4.4 ([5])} Let \( R \) be a Riemann surface satisfying the bounded geometry condition. Then \( \Lambda(\text{Mod}_A(R)) \subsetneq AT(R) \).

\textbf{References}


