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Limit sets on Teichmüller space and on asymptotic Teichmüller space

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1 Introduction

This note is a survey of our recent researches on the action of the quasiconformal mapping class group on the Teichmüller space and on the asymptotic Teichmüller space.

We say that two quasiconformal homeomorphisms f_1 and f_2 on R are *equivalent* if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. All homotopies are consider to be relative to the ideal boundary at infinity. The *Teichmüller space* $T(R)$ of a Riemann surface R is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f on R. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d([f_1],[f_2]) = (1/2)\log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d is a complete distance on $T(R)$ which is called the Teichmüller distance.

We say that a quasiconformal homeomorphism f on R is *asymptotically conformal* if for every $\epsilon > 0$, there exists a compact subset V of R such that the maximal dilatation $K(f|_{R-V})$ of the restriction of f to $R - V$ is less than $1 + \epsilon$. We say that two quasiconformal homeomorphisms f_1 and f_2 on R are *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. The *asymptotic Teichmüller space* $AT(R)$ of a Riemann surface R is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal homeomorphisms f on R. Since a conformal homeomorphism is asymptotically conformal, there is a natural projection $\pi : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic Teichmüller equivalence class $[[f]] \in AT(R)$. For a quasiconformal homeomorphism f of R, the *boundary dilatation* of f is defined by $H^*(f) = \inf K(f|_{R-V})$, where the infimum is taken over all compact subsets V of R. Furthermore, for a Teichmüller equivalence class $[f] \in T(R)$, the *boundary dilatation* of $[f]$ is defined by $H([f]) = \inf H^*(g)$, where the infimum is taken over all elements $g \in [f]$. A distance between two points $[[f_1]]$ and $[[f_2]]$ in $AT(R)$ is defined by $d_A([[f_1]],[[f_2]]) = (1/2)\log H([f_2 \circ f_1^{-1}])$, where $[f_2 \circ f_1^{-1}]$ is a Teichmüller equivalence class of $f_2 \circ f_1^{-1}$ in $T(f_1(R))$. Then d_A

is a complete distance on $AT(R)$, which is called the asymptotic Teichmüller distance. For every point $[[f]] \in AT(R)$, there exists an asymptotically extremal element $f_0 \in [[f]]$ in the sense that $H([f]) = H^*(f_0)$.

The *quasiconformal mapping class* is the homotopy equivalence class $[g]$ of quasiconformal automorphisms g of a Riemann surface, and the *quasiconformal mapping class group* $\mathrm{MCG}(R)$ of R is the group of all quasiconformal mapping classes on R. Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_*$ of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$ and a biholomorphic automorphism $[g]_{**}$ of $AT(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$. Let $\mathrm{Aut}(T(R))$ and $\mathrm{Aut}(AT(R))$ be the groups of all biholomorphic automorphisms of $T(R)$ and $AT(R)$ respectively. Then we have a homomorphism

$$\iota : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$$

given by $[g] \mapsto [g]_*$ and a homomorphism

$$\iota_A : \mathrm{MCG}(R) \to \mathrm{Aut}(AT(R))$$

given by $[g] \mapsto [g]_{**}$. We define the Teichmüller modular group

$$\mathrm{Mod}(R) := \iota(\mathrm{MCG}(R))$$

and the asymptotic Teichmüller modular group (the geometric automorphism group)

$$\mathrm{Mod}_A(R) := \iota_A(\mathrm{MCG}(R)).$$

2 Characterization of the asymptotically trivial mapping class group

To observe the dynamics of the action of $\mathrm{Mod}(R)$ on $T(R)$ and $\mathrm{Mod}_A(R)$ on $AT(R)$, first we have to investigate properties of the homomorphisms ι and ι_A. It is known that the homomorphism ι is injective (faithful) for all Riemann surfaces R of non-exceptional type. Here we say that a Riemann surface R is of *exceptional type* if R has finite hyperbolic area and satisfies $2g + n \leq 4$, where g is the genus of R and n is the number of punctures of R. Furthermore, the homomorphism ι is also surjective for all Riemann surfaces R of non-exceptional type. Thus we may identify $\mathrm{Mod}(R)$ with $\mathrm{MCG}(R)$. However, the homomorphism ι_A is not injective, namely $\mathrm{Ker}\, \iota_A \neq \{[id]\}$, unless R is either the unit disc or a once-punctured disc. We give a geometric characterization of quasiconformal mapping classes belonging to $\mathrm{Ker}\, \iota_A$. To state our result, we define two subgroups of the quasiconformal mapping class group.

Definition 2.1 The *pure mapping class group* $P(R)$ is the group of all $[g] \in \mathrm{MCG}(R)$ such that g fixes all non-cuspidal ends of R. The *eventually trivial mapping class group* $E(R)$ is the group of all eventually trivial mapping classes. Here $[g] \in \mathrm{MCG}(R)$ is said to be *eventually trivial* if there exists a compact subsurface V_g of R such that, for each connected component W of $R - V_g$ that is not a cusp neighborhood, the restriction $g|_W : W \to R$ is homotopic to the inclusion map $id|_W : W \hookrightarrow R$.

Then we have the following.

Proposition 2.2 ([3]) *The inclusion relation* $E(R) \subset \mathrm{Ker}\,\iota_A \subset P(R)$ *holds.*

Each inclusion in Proposition 2.2 is proper, in general. However, under a certain condition on hyperbolic geometry of Riemann surfaces, we have a characterization of $\mathrm{Ker}\,\iota_A$.

Definition 2.3 We say that a Riemann surface R satisfies the *bounded geometry condition* if R satisfies the following three conditions:

(i) *the lower bound condition*: the injectivity radius at any point of R except cusp neighborhoods are uniformly bounded away from zero.

(ii) *the upper bound condition*: there exists a subdomain $\check{R}$ of R such that the injectivity radius at any point of $\check{R}$ is uniformly bounded from above and that the simple closed curves in $\check{R}$ carry the fundamental group of R.

(iii) R has no ideal boundary at infinity, namely the Fuchsian model of R is of the first kind.

We state our result.

Theorem 2.4 ([5]) *Let* R *be a Riemann surface satisfying the bounded geometry condition. Then* $E(R) = \mathrm{Ker}\,\iota_A$.

Remark 2.5 If R satisfies the bounded geometry condition, then $\mathrm{Ker}\,\iota_A$ is a proper subset of $\mathrm{MCG}(R)$. Namely, the action of $\mathrm{MCG}(R)$ on $AT(R)$ is non-trivial. See [2, Corollary 3.5]. However, there exists a Riemann surface R that does not satisfy the bounded geometry condition and that $E(R) = \mathrm{Ker}\,\iota_A = \mathrm{MCG}(R)$. See [7].

3 Discontinuity of mapping class group on Teichmüller space

We say that a subgroup $G \subset \mathrm{MCG}(R)$ acts at a point $p \in T(R)$ *discontinuously* if there exists a neighborhood U of p such that the number of elements $g \in G$ satisfying $g_*(U) \cap U \neq \emptyset$ is finite, namely if the orbit $\{g_*(p) \mid g \in G\}$ is discrete and the stabilizer subgroup $\mathrm{Stab}_G(p) = \{g \in G \mid g_*(p) = p\}$ is finite. We consider the discontinuity of the pure and eventually trivial mapping class groups on the Teichmüller space. It was proved in [6] that, for a special planar Riemann surface, the pure mapping class group acts on the Teichmüller space discontinuously. We generalize this result in the following form.

Theorem 3.1 ([3]) *Let* R *be a Riemann surface satisfying the bounded geometry condition and having more than two non-cuspidal ends. Then the pure mapping class group* $P(R)$ *acts on the Teichmüller space* $T(R)$ *discontinuously.*

For Riemann surfaces the number of whose non-cuspidal ends are at most two, Theorem 3.1 is not true. However, concerning the action of $E(R)$, we always have the following.

Theorem 3.2 (**[3]**) *Let R be a Riemann surface satisfying the bounded geometry condition. Then the eventually trivial mapping class group $E(R)$ acts on $T(R)$ discontinuously.*

By Theorems 2.4 and 3.2, we have the following corollary immediately.

Corollary 3.3 *Let R be a Riemann surface satisfying the bounded geometry condition. Then* $\operatorname{Ker} \iota_A$ *acts on $T(R)$ discontinuously.*

We apply Corollary 3.3 in the next section.

4 Dynamics of mapping class group on asymptotic Teichmüller space

For the Teichmüller modular group $\mathrm{Mod}(R)$, we define the limit set $\Lambda(\mathrm{Mod}(R))$ on $T(R)$ as the set of points $p \in T(R)$ such that $\gamma_n(p) \to p$ $(n \to \infty)$ for a sequence of distinct elements $\gamma_n \in \mathrm{Mod}(R)$. Then for every point $p \in \Lambda(\mathrm{Mod}(R))$, the action of $\mathrm{MCG}(R)$ at p is not discontinuous. Also, for the asymptotic Teichmüller modular group $\mathrm{Mod}_A(R)$, we define the limit set $\Lambda(\mathrm{Mod}_A(R))$ on $AT(R)$ as the set of points $\hat{p} \in AT(R)$ such that $\hat{\gamma}_n(\hat{p}) \to \hat{p}$ $(n \to \infty)$ for a sequence of distinct elements $\hat{\gamma}_n \in \mathrm{Mod}_A(R)$. The following theorem says that the actions of the quasiconformal mapping class group on $T(R)$ and on $AT(R)$ are quite different.

Proposition 4.1 (**[2]**) *There exists a Riemann surface R satisfying the bounded geometry condition such that $\Lambda(\mathrm{Mod}(R)) = \emptyset$ and $\Lambda(\mathrm{Mod}_A(R)) \neq \emptyset$.*

On the other hand, if a Riemann surface has a certain property, then the dynamics of the actions of the quasiconformal mapping class group on $T(R)$ and on $AT(R)$ are same.

Proposition 4.2 (**[2]**) *Let R be a Riemann surface that does not satisfy the lower bound condition. Then $\Lambda(\mathrm{Mod}(R)) = T(R)$ and $\Lambda(\mathrm{Mod}_A(R)) = AT(R)$.*

We consider the problem whether $\pi(\Lambda(\mathrm{Mod}(R))) \subset \Lambda(\mathrm{Mod}_A(R))$ for all Riemann surfaces R. Proposition 4.2 gives a partial answer to this problem. Furthermore, we have proved in [4] that this problem is also true for the limit sets of the subgroups of $\mathrm{Mod}(R)$ and $\mathrm{Mod}_A(R)$ induced by a cyclic group $\langle [g] \rangle \subset \mathrm{MCG}(R)$, where g is a conformal automorphism of infinite order of a Riemann surface R. By Corollary 3.3, we have the following general statement.

Corollary 4.3 (**[5]**) *Let R be a Riemann surface satisfying the bounded geometry condition. Then $\pi(\Lambda(\mathrm{Mod}(R))) \subset \Lambda(\mathrm{Mod}_A(R))$.*

Proof. We take a limit point $p \in \Lambda(\mathrm{Mod}(R))$ arbitrarily. Then there exists a sequence $[g_n]$ of distinct elements of $\mathrm{MCG}(R)$ such that $d([g_n]_*(p), p) \to 0$ as $n \to \infty$. Then $d_A([g_n]_{**}(\hat{p}), \hat{p}) \to 0$ for the projection $\hat{p} = \pi(p)$. We will show that $\{[g_n]_{**}\}_{n \in \mathbf{Z}} \subset \mathrm{Mod}_A(R)$ contains infinitely many elements. Then we conclude that $\hat{p} \in \Lambda(\mathrm{Mod}_A(R))$. Suppose to the contrary that $\{[g_n]_{**}\}_{n\in\mathbf{Z}}$ is a finite set $\{[g_1]_{**}, \ldots, [g_k]_{**}\}$ for some $k \geq 1$. Then there exists an integer i $(1 \leq i \leq k)$, say 1, such that $[g_n]_{**} = [g_1]_{**}$ for infinitely many n. Let $\gamma_n := g_n \circ g_1^{-1}$. Then $[\gamma_n] \in \mathrm{Ker}\,\iota_A$ and $d([\gamma_n]_*(g_1(p)), p) = d([g_n]_*(p), p) \to 0$. This means that the point $p \in T(R)$ is a limit point for the subgroup $\mathrm{Ker}\,\iota_A$. This contradicts Corollary 3.3. Thus we conclude that $\{[g_n]_{**}\}_{n\in\mathbf{Z}}$ contains infinitely many elements. ∎

We have already proved in [1] that $\Lambda(\mathrm{Mod}(R)) \subsetneqq T(R)$ for a Riemann surface R with the bounded geometry condition. A similar statement is true also for the limit set on the asymptotic Teichmüller space.

Theorem 4.4 ([5]) *Let R be a Riemann surface satisfying the bounded geometry condition. Then $\Lambda(\mathrm{Mod}_A(R)) \subsetneqq AT(R)$.*

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