A subfamily of complex error functions

Shunsuke MOROSAWA
Department of Mathematics and Information Science,
Faculty of Science, Kochi University
morosawa@math.kochi-u.ac.jp

1 Introduction

A complex error function is a transcendental entire function given by the form

\[ C_{a,b}(z) = a \int_{0}^{z} e^{-w^2} dw + b \]

with \( a \in \mathbb{C} \setminus \{0\} \) and \( b \in \mathbb{C} \). It has two asymptotic values \( \pm a\sqrt{\pi}/2 + b \) and has no other singular value. In [3], a subfamily of complex error functions given by the form

\[ C_{a,\sqrt{B}}(z) = a \int_{0}^{z} e^{-w^2} dw + \sqrt{B} \]

with \( a \in \mathbb{R} \setminus \{0\} \) and \( B \in \mathbb{R} \) is considered. Hence the family is described by two real parameters. Fatou components of some functions of this family have common boundary curves. In this note, we consider a subfamily of complex error functions given by the form

\[ f_{a}(z) = a \int_{0}^{z} e^{-w^2} dw \]

with \( a \in \mathbb{C} \setminus \{0\} \). Hence the family is described by one holomorphic parameter. A well-known family of transcendental entire functions with one complex parameter is an exponential family. It is studied by Baker and Rippon [1], Devaney [2] and others.

2 Results

We say that \( f_{a} \) is hyperbolic if the orbit of each asymptotic value accumulates to attracting cyclic points. A connected component of the set of parameters
Figure 1: The parameter space of $f_a(z)$. The range shown is $|\Re a| \leq 2$, $|\Im a| \leq 2$. The disk in the center is $A$. Hyperbolic components of $B_n$ are colored white and those of $D_n$ are colored black.

for which $f_a$ is hyperbolic is called a hyperbolic component. It is known that hyperbolic components are open.

We define subsets in the parameter space of $f_a$ as follows:

$A = \{a \mid f_a$ has a completely invariant component.$\}$,

$B_n = \{a \mid f_a$ has only one attracting cycle with the period $2n$.\}$,

$D_n = \{a \mid f_a$ has two attracting cycles with the period $n$.\}$,

for $n \in \mathbb{N}$.

If there exists a cycle $\{z_1, z_2, \cdots, z_n\}$, then $\{-z_1, -z_2, \cdots, -z_n\}$ is also a cycle from the equation

$f_a(-z) = -f_a(z)$.

Furthermore, we see that if the cycle is attracting, repelling or indifferent, then so is the corresponding one, respectively. The Maclaurin expansion of
$Er(z) = f_1(z)$ is of the form

$$Er(z) = \int_0^z e^{-w^2} dw = z - \frac{z^3}{3} + \cdots.$$  

Adding further investigation on properties of $Er(z)$, we have the following theorem.

**Theorem 1.** Every hyperbolic component is contained in one of $A$, $B_n$ and $D_n$. Furthermore, $A$ is also described by $\{a \mid 0 < |a| < 1\}$. Each of $B_1$ and $D_1$ consists of only one component.

By the arguments similar to those in [1], we have the following theorems.

**Theorem 2.** Every hyperbolic component except $A$ is simply-connected and unbounded.

**Theorem 3.** Each of $B_n$ and $D_n$ contains a component which is tangent to $A$.

Cyclic Fatou components of the function belonging to a hyperbolic component tangent to $A$ attach to each other at the origin. By the arguments similar to those in [3], we have the following theorem.

**Theorem 4.** Fatou components of $f_a$ belonging to a hyperbolic component tangent to $A$ have common boundary curves.

**References**


Figure 2: The Julia sets of $f_a(z)$. The range shown is $|\Re z| \leq 2$, $|\Im z| \leq 2$. Upper left: $a = 0.95i$. Upper right: $a = 1.05i$. Middle left: $a = 0.475 + 0.8227241i$. Middle right: $a = 0.55 + 0.952628i$. Lower left: $a = -0.475 + 0.8227241i$. Lower right: $a = -0.55 + 0.952628i$. 