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# A subfamily of complex error functions

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## 1 Introduction

A complex error function is a transcendental entire function given by the form

$$C_{a,b}(z) = a \int_0^z e^{-w^2} dw + b$$

with  $a \in \mathbb{C} \setminus \{0\}$  and  $b \in \mathbb{C}$ . It has two asymptotic values  $\pm a\sqrt{\pi}/2 + b$  and has no other singular value. In [3], a subfamily of complex error functions given by the form

$$C_{a,\sqrt{B}}(z) = a \int_0^z e^{-w^2} dw + \sqrt{B}$$

with  $a \in \mathbb{R} \setminus \{0\}$  and  $B \in \mathbb{R}$  is considered. Hence the family is described by two real parameters. Fatou components of some functions of this family have common boundary curves. In this note, we consider a subfamily of complex error functions given by the form

$$f_a(z) = a \int_0^z e^{-w^2} dw$$

with  $a \in \mathbb{C} \setminus \{0\}$ . Hence the family is described by one holomorphic parameter. A well-known family of transcendental entire functions with one complex parameter is an exponential family. It is studied by Baker and Rippon [1], Devaney [2] and others.

## 2 Results

We say that  $f_a$  is hyperbolic if the orbit of each asymptotic value accumulates to attracting cyclic points. A connected component of the set of parameters

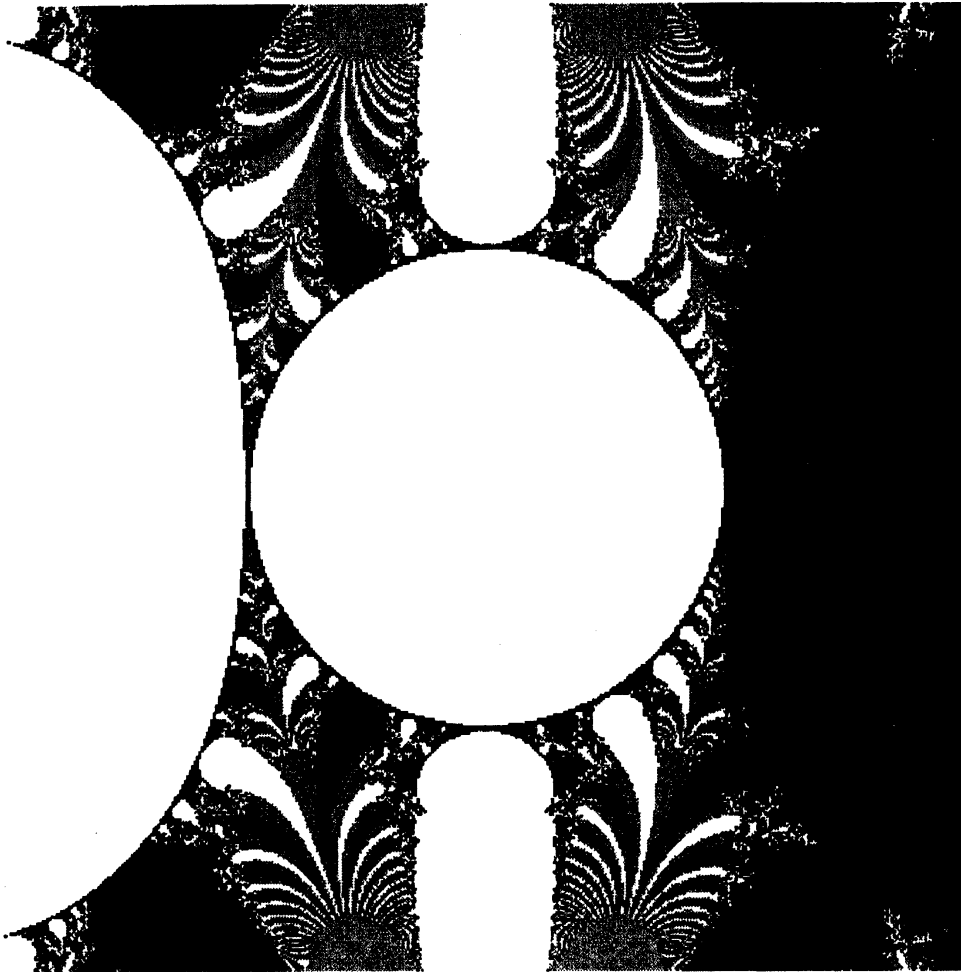


Figure 1: The parameter space of  $f_a(z)$ . The range shown is  $|\Re a| \leq 2$ ,  $|\Im a| \leq 2$ . The disk in the center is  $A$ . Hyperbolic components of  $B_n$  are colored white and those of  $D_n$  are colored black.

for which  $f_a$  is hyperbolic is called a hyperbolic component. It is known that hyperbolic components are open.

We define subsets in the parameter space of  $f_a$  as follows:

$$\begin{aligned} A &= \{a \mid f_a \text{ has a completely invariant component.}\}, \\ B_n &= \{a \mid f_a \text{ has only one attracting cycle with the period } 2n.\}, \\ D_n &= \{a \mid f_a \text{ has two attracting cycles with the period } n.\}, \end{aligned}$$

for  $n \in \mathbb{N}$ .

If there exists a cycle  $\{z_1, z_2, \dots, z_n\}$ , then  $\{-z_1, -z_2, \dots, -z_n\}$  is also a cycle from the equation

$$f_a(-z) = -f_a(z).$$

Furthermore, we see that if the cycle is attracting, repelling or indifferent, then so is the corresponding one, respectively. The Maclaurin expansion of

$Er(z) = f_1(z)$  is of the form

$$Er(z) = \int_0^z e^{-w^2} dw = z - \frac{z^3}{3} + \dots$$

Adding further investigation on properties of  $Er(z)$ , we have the following theorem.

**Theorem 1.** *Every hyperbolic component is contained in one of  $A$ ,  $B_n$  and  $D_n$ . Furthermore,  $A$  is also described by  $\{a \mid 0 < |a| < 1\}$ . Each of  $B_1$  and  $D_1$  consists of only one component.*

By the arguments similar to those in [1], we have the following theorems.

**Theorem 2.** *Every hyperbolic component except  $A$  is simply-connected and unbounded.*

**Theorem 3.** *Each of  $B_n$  and  $D_n$  contains a component which is tangent to  $A$ .*

Cyclic Fatou components of the function belonging to a hyperbolic component tangent to  $A$  attach to each other at the origin. By the arguments similar to those in [3], we have the following theorem.

**Theorem 4.** *Fatou components of  $f_a$  belonging to a hyperbolic component tangent to  $A$  have common boundary curves.*

## References

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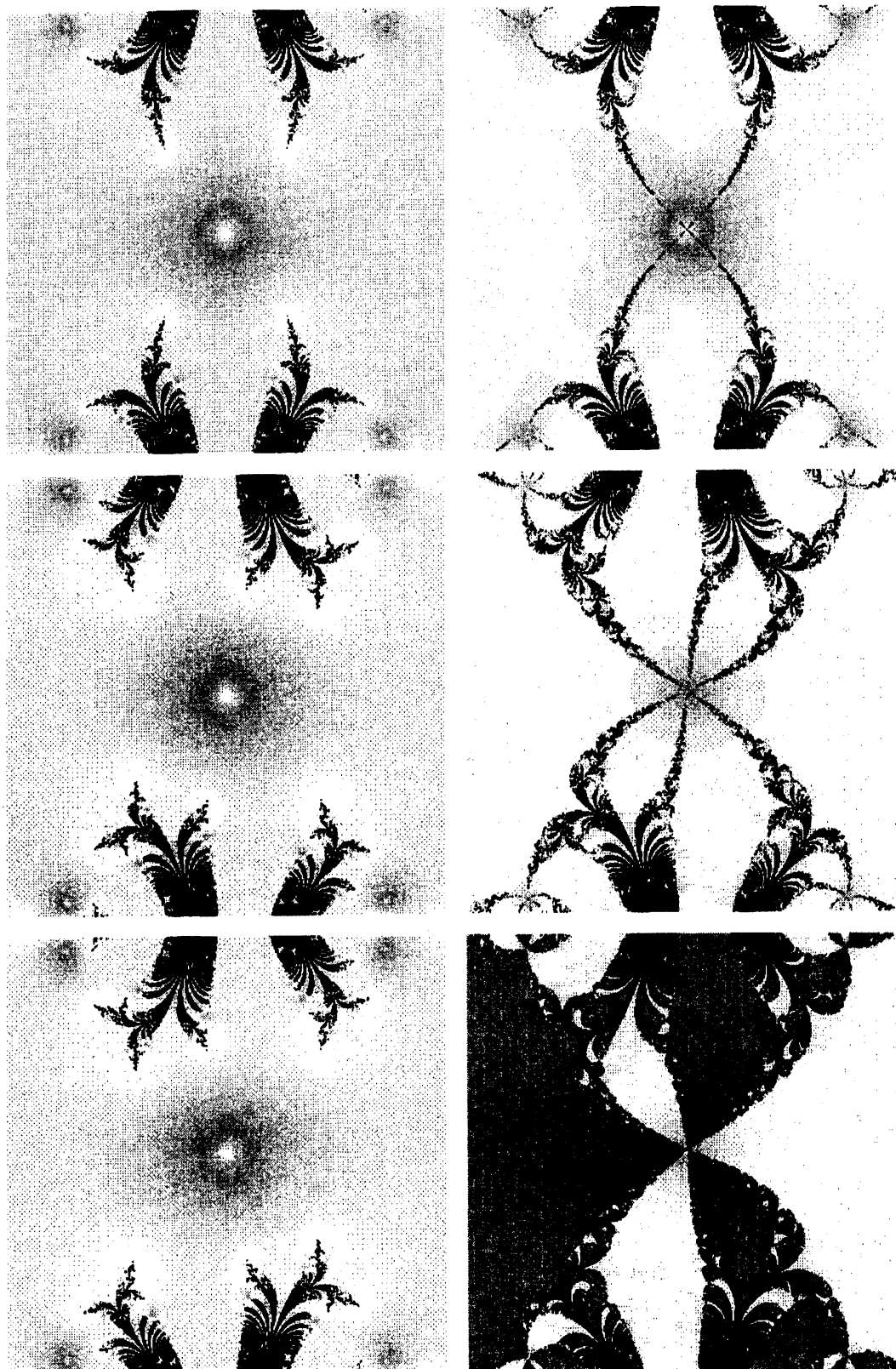


Figure 2: The Julia sets of  $f_a(z)$ . The range shown is  $|\Re z| \leq 2$ ,  $|\Im z| \leq 2$ . Upper left:  $a = 0.95i$ . Upper right:  $a = 1.05i$ . Middle left:  $a = 0.475 + 0.8227241i$ . Middle right:  $a = 0.55 + 0.952628i$ . Lower left:  $a = -0.475 + 0.8227241i$ . Lower right:  $a = -0.55 + 0.952628i$ .