

Some properties of Julia sets of semi-hyperbolic entire functions

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Abstract

We investigate some dynamical properties of semi-hyperbolic transcendental entire functions. Under some additional conditions, we show that

(1) $f^n(z) \rightarrow \infty$ and $d_E(f^n(z), P(f)) \rightarrow \infty$ for almost every $z \in J(f)$,

moreover

(2) if $J(f)$ is thin at ∞ , then the Lebesgue measure of $J(f)$ is zero.

1 Introduction

In this paper, we consider the dynamics of entire functions f . Our main concern is the case of transcendental functions. Denote the Fatou set of f by $F(f)$ and the Julia set by $J(f)$.

Definition 1.1. (1) f is *semi-hyperbolic* at $z \in J(f)$ if there exists a neighborhood U of z and an $N \in \mathbb{N}$ such that for any connected component V of $f^{-n}(U)$ ($\forall n \in \mathbb{N}$), $f^n|_V : V \rightarrow U$ satisfies

$$\deg(f^n|_V : V \rightarrow U) \leq N.$$

In the case that f is transcendental, we add the following property:

$$f^n|_V : V \rightarrow U \text{ is proper for every } V.$$

Recall that a map $f : X \rightarrow Y$ is called *proper* if $f^{-1}(K) \subset X$ is compact for every compact subset $K \subset Y$. Note that this property is automatically satisfied when f is a polynomial.

(2) f is called *semi-hyperbolic* if f is semi-hyperbolic at every point $z \in J(f)$.

(3) f is called N -semi-hyperbolic ($N \in \mathbb{N}$) if f is semi-hyperbolic and we can take an N in the definition of semi-hyperbolicity independently for $z \in J(f)$. f is called *uniformly semi-hyperbolic* if f is N -semi-hyperbolic for some $N \in \mathbb{N}$.

When f is a polynomial or, more in general, a rational map, f is semi-hyperbolic at $z \in J(f)$ if and only if z is not a parabolic periodic point and satisfies

$$z \notin \bigcup_{c \in \text{Rec} \cap J_f} \omega(c), \text{ where } \text{Rec} = \{\text{recurrent critical points of } f\}.$$

This is the famous result by Mañé ([Ma]). In the case of transcendental entire functions, it is known that there are some other obstructions which break semi-hyperbolicity. For the details, see [K, p.37, Theorem A].

2 Known Results

2.1 Lebesgue measure of $J(f)$

When f is rational, the following is known by Yin ([Y, p.560, 定理 1.3]).

Theorem 2.1 (Yin (1999)). *If f is a semi-hyperbolic rational map and $J(f) \neq \widehat{\mathbb{C}}$, then the Lebesgue measure of $J(f)$ is 0.*

But in the case of transcendental entire functions, the Lebesgue measure of $J(f)$ is not always equal to 0 in general, even if f is semi-hyperbolic (or moreover, hyperbolic), as the following example shows ([Mc, p.329, Theorem 1.1]):

Example 2.2 (McMullen (1987)). Consider $f(z) = \lambda \sin z$, ($\lambda \neq 0$). McMullen showed that $\text{Leb}(J(f)) > 0$ for any λ , where “Leb” denotes the Lebesgue measure on \mathbb{C} . In particular, if $0 < \lambda < 1$, then there exists

an attractive fixed point and the post-singular set $P(f)$ is compact and contained in $F(f)$, that is, f is “hyperbolic”. But $\text{Leb}(J(f)) \neq 0$ in this case. This shows that semi-hyperbolicity (or even, hyperbolicity) does not imply $\text{Leb}(J(f)) = 0$ in the transcendental case.

On the other hand, $\text{Leb}(J(f))$ can be 0 for some (semi)-hyperbolic transcendental entire function under some additional conditions ([Mc, p.329, Theorem 1.3]):

Example 2.3 (McMullen (1987)). Consider $f(z) = \lambda e^z$, ($0 < \lambda < \frac{1}{e}$). Then f is hyperbolic in the sense in the above example and $J(f)$ is thin at ∞ . So it follows that $\text{Leb}(J(f)) = 0$. Here we say $J(f)$ is *thin at ∞* if there exist an $R > 0$ and an $\varepsilon > 0$ such that for every $z \in \mathbb{C}$ and Euclidean ball $D_E(z, r)$ for $\forall r > R$, we have

$$\frac{\text{Leb}(J(f) \cap D_E(z, r))}{\text{Leb}(D_E(z, r))} < 1 - \varepsilon.$$

More generally, McMullen showed the following result in the case that f is “hyperbolic” transcendental entire function ([Mc, p.333, Proposition 7.3]):

Theorem 2.4 (McMullen (1987)). *Let $P(f)$ be the post-singular set of f . If*

(a) $P(f) \subset F(f)$, (b) $P(f)$ is compact, (c) $J(f)$ is thin at ∞ ,
then $\text{Leb}(J(f)) = 0$.

In 1990, Stallard weakened the assumptions in the above theorem and obtained the similar result ([S, p.552, Theorem B]):

Theorem 2.5 (Stallard (1990)). *Assumptions (a) and (b) above can be replaced by $d_E(J(f), P(f)) > 0$, where d_E denotes the Euclidean distance on \mathbb{C} . That is, $d_E(J(f), P(f)) > 0$ together with (c) in Theorem 2.4 implies $\text{Leb}(J(f)) = 0$.*

2.2 The dynamics of f on $J(f)$

If f is a semi-hyperbolic entire function with $J(f) \neq \mathbb{C}$, the dynamics of f on $J(f)$ is very simple at least from measure theoretical point of view ([K, p.42, Corollary D]).

Theorem 2.6 (Kisaka (2006)). *If f is a semi-hyperbolic entire function with $J(f) \neq \mathbb{C}$, then $f^n(z) \rightarrow \infty$ for almost every $z \in J(f)$.*

It seems from this theorem that there is nothing more to investigate for measure theoretical properties of the dynamics of semi-hyperbolic entire functions on $J(f)$. But Stallard showed some refinement of the above result for hyperbolic (in the sense in Theorem 2.5 above) case. She essentially proved the following in the same paper, although she did not state it as a theorem ([S, p.555]).

Theorem 2.7 (Stallard (1990)). *If f satisfies $d_E(J(f), P(f)) > 0$, then $f^n(z) \rightarrow \infty$ and $d_E(f^n(z), P(f)) \rightarrow \infty$ for almost every $z \in J(f)$.*

In this paper, we try to generalize these results in the case of semi-hyperbolic entire functions.

3 Results

First we consider a function f in class S , which means that f has only finitely many singular values.

Theorem A. *Let f be a transcendental entire function in class S and assume that*

- (a) f is semi-hyperbolic, (b) $P(f) \cap J(f)$ is bounded, (c) $J(f) \neq \mathbb{C}$.

Then

- (1) $f^n(z) \rightarrow \infty$ and $d_E(f^n(z), P(f)) \rightarrow \infty$ for almost every $z \in J(f)$.
 (2) Moreover, if $J(f)$ is thin at ∞ , then $\text{Leb}(J(f)) = 0$.

For more general transcendental entire functions, we prove the following:

Theorem B. *Let f be a transcendental entire function and assume that*

- (a) f is uniformly semi-hyperbolic, (b) $P(f) \cap J(f)$ is bounded,
 (c) $d_E(J(f), P(f) \cap F(f)) > 0$, (d) $J(f) \neq \mathbb{C}$.

Then the same conclusion as in Theorem A holds.

Remark 3.1. We can show that if f is in class S , then f is uniformly semi-hyperbolic.

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