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PARABOLIC RENORMALIZATION AND ITS CONSEQUENCES

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This is a report on a joint work with Hiroyuki Inou (Kyoto University). Detailed statements and proofs will be published elsewhere. (See [IS]*.)*

Let $f(z)$ be a holomorphic function defined near $z_0 \in \mathbb{C}$ and suppose z_0 is a fixed point. Its *multiplier* is $\lambda = f'(z_0)$ and the fixed point z_0 is called *parabolic* if λ is a root of unity. We will mainly consider the case $\lambda = 1$. In this case, for simplicity we say z_0 is *1-parabolic* and we call it *non-degenerate* if $f''(z_0) \neq 0$. Introduce a coordinate change $w = -\frac{1}{a_2 z}$, which sends the fixed point to ∞. The dynamics in this coordinate is

$$F(w) = -\frac{1}{a_2 f(-\frac{1}{a_2 w})} = w + 1 + \frac{b_1}{w} + O(\frac{1}{w^2})$$

near ∞. For a sufficiently large L, there exist injective holomorphic functions $\Phi_{attr} = \Phi_{attr,F} : \{w : \operatorname{Re} w > L\} \to \mathbb{C}$ and $\Phi_{rep} = \Phi_{rep,F} : \{w : \operatorname{Re} w < -L\} \to \mathbb{C}$ such that they satisfy the functional equation

$$\Phi_s(F(w)) = \Phi_s(w) + 1 \quad (s = attr, rep) \tag{1}$$

in the region where both sides are defined. Φ_{attr} and Φ_{rep} are unique up to addition of constant. The functions Φ_{attr} and Φ_{rep} are called *attracting* and *repelling Fatou coordinates* respectively. They are considered to be coordinates for half-neignborhoods (*"petals"*) of the fixed point such that the dynamics is conjugated to the translation $T : z \mapsto z + 1$. In the regions $V_\pm = \{w : \pm \operatorname{Im} w > |w| + L'\}$, both Fatou coordinates are defined. Now define the *horn map* E_F on $\Phi_{rep,F}(V_\pm)$ to be

$$E_F = \Phi_{attr} \circ \Phi_{rep}^{-1} \tag{2}$$

Denote $\operatorname{Exp}^\sharp(z) = e^{2\pi i z}$ and $\operatorname{Exp}^\flat(z) = e^{-2\pi i z}$. Both functions induce isomorphisms from $\mathbb{C}/\mathbb{Z}$ onto $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$; $\operatorname{Exp}^\sharp$ sends upper end $+i\infty$ to 0 and lower end $-i\infty$ to ∞, and for $\operatorname{Exp}^\flat$, the role of the ends is interchanged.

Suppose f has a non-degenerate parabolic fixed point at 0. Its *parabolic renormalization* is defined to be

$$\mathcal{R}_0 f = \mathcal{R}_0^\sharp f = \operatorname{Exp}^\sharp \circ E_f \circ \left(\operatorname{Exp}^\sharp\right)^{-1},$$

where E_f is the horn map of f, defined above and normalized as $E_f(z) = z + o(1)$ as $\operatorname{Im} z \to +\infty$. Then $\mathcal{R}_0 f$ extends holomorphically to 0 and $\mathcal{R}_0 f(0) = 0$, $(\mathcal{R}_0 f)'(0) = 1$.

So 0 has again a 1-parabolic fixed point at 0. Similarly the parabolic renormalization for lower end is defined as

$$\mathcal{R}_0^\flat f = c \operatorname{Exp}^\flat \circ E_f \circ \left(\operatorname{Exp}^\flat\right)^{-1},$$

where $c \in \mathbb{C}^*$ is chosen so that $(\mathcal{R}_0^\flat f)'(0) = 1$. It has been known [Sh1] that the following class $\mathcal{F}_0$ is invariant under $\mathcal{R}_0$.

$$\mathcal{F}_0 = \left\{ f : Dom(f) \to \mathbb{C} \;\middle|\; \begin{array}{l} 0 \in Dom(f) \text{ open } \subset \mathbb{C}, \quad f \text{ is holomorphic in } Dom(f), \\ \quad f(0) = 0,\ f'(0) = 1,\ f : Dom(f) \smallsetminus \{0\} \to \mathbb{C}^* \text{ is} \\ \text{a branched covering map with a unique critical value } cv_f, \\ \text{all critical points are of local degree 2} \end{array} \right\}.$$

Suppose that $f(z) = e^{2\pi i \alpha} z + O(z^2)$ with $\alpha \neq 0$ and has suitable fundamental domains and return map $h = \chi_f \circ E_f$. Its *near-parabolic renormalization* (or also called cylinder renormalization) is defined by

$$\mathcal{R}f = \mathcal{R}^\sharp f = \operatorname{Exp}^\sharp \circ \chi_f \circ E_f \circ \left(\operatorname{Exp}^\sharp\right)^{-1}.$$

Then $\mathcal{R}f$ extends to 0 and $\mathcal{R}f(0) = 0$, $(\mathcal{R}f)'(0) = e^{-2\pi i \frac{1}{\alpha}}$. For lower end, set $\mathcal{R}^\flat f = \operatorname{Exp}^\flat \circ \chi_f \circ E_f \circ \left(\operatorname{Exp}^\flat\right)^{-1}$.

Continued fraction: Any irrational number $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ can be written as an accelerated continued fraction of the form:

$$(3) \qquad \alpha = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{\ddots}}}, \qquad \text{where} \quad \begin{array}{l} a_n \in \mathbb{Z},\ \ \varepsilon_n = \pm 1 \ (n = 0, 1, 2, \dots), \\ a_n \geq 2 \ (n \geq 1). \end{array}$$

Denote $||x|| = \min\{|x - n| : n \in \mathbb{Z}\}$ and define $\alpha_0 = ||\alpha||$, $\alpha_{n+1} = \left\|\frac{1}{\alpha_n}\right\|$. Then $\alpha_n \in (0, \frac{1}{2})$ and a_n and ε_n are determined by $\frac{1}{\alpha_{n-1}} = a_n + \varepsilon_n \alpha_n$.

Successive renormalizations: Let $f(z) = e^{2\pi i \alpha} z + O(z^2)$ with $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ as above. We are interested in the construction of successive renormalizations:

$$(4) \qquad f_0(z) = \begin{cases} f(z) & (\varepsilon_0 = +1) \\ \overline{f(\overline{z})} & (\varepsilon_0 = -1) \end{cases} \qquad f_n(z) = \begin{cases} \mathcal{R}f_{n-1}(z) & (\varepsilon_n = -1) \\ \overline{\mathcal{R}f_{n-1}(\overline{z})} & (\varepsilon_n = +1) \end{cases} \quad (n \geq 1).$$

Here the complex conjugation is taken so that $f_n'(0) = e^{2\pi i \alpha_n}$ with $\alpha_n \in (0, \frac{1}{2})$. If such a construction is possible, we hope that the dynamics of f, whose irrationally indifferent fixed point causes recurrent behavior for nearby orbits, can be studied through the sequence $\{f_n\}$. In fact, problems involving high iterates of f_{n-1} often reduce to simpler problems on fewer iterates of f_n. The geometric structure near recurrent orbits may be "magnified" by the renormalization process. Hence it is natural to ask:

Question: When is it possible to define the sequence $\{f_n\}_{n=0}^{\infty}$?

Definition (P and Class $\mathcal{F}_1$). Let $P(z) = z(1+z)^2$. The polynomial P has a parabolic fixed point at 0 and critical points $-\frac{1}{3}$ and -1 with $P(-\frac{1}{3}) = -\frac{4}{27}$ and $P(-1) = 0$. Let V be a domain of $\mathbb{C}$ containing 0 and define

$$\mathcal{F}_1 = \left\{ f = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi : V \to \mathbb{C} \text{ is univalent, } \varphi(0) = 0,\ \varphi'(0) = 1 \\ \text{and } \varphi \text{ has a quasiconformal extension to } \mathbb{C} \end{array} \right\},$$

where univalent means holomorphic and injective. Note that if $f \in \mathcal{F}_1$, 0 is a 1-parabolic fixed point of f. If $-\frac{1}{3} \in V$, then $cp_f := \varphi(-\frac{1}{3})$ is a critical point and $-\frac{4}{27}$ is a critical value of f.

We have the following result:

Main Thorem 1 (Invariance of $\mathcal{F}_1$). *There exist a Jordan domain V containing* 0 *and* $-\frac{1}{3}$ *with a smooth boundary and an open set V' containing $\overline{V}$ such that the above $\mathcal{F}_1$ satisfies the following:*

(1) $f''(0) \neq 0$ *(in fact, $|f''(0) - 4.91| \leq 1.14$). $cp_f \in Basin(0)$.*
(2) $(\mathcal{F}_0 \setminus \{quadratic\ polynomial\}) / \sim_{linear}$ *can be naturally included into $\mathcal{F}_1$.*
(3) $\mathcal{R}_0(\mathcal{F}_1) \subset \mathcal{F}_1$. *That is, for $f \in \mathcal{F}_1$, the parabolic renormalization $\mathcal{R}_0 f$ is well-defined so that $\mathcal{R}_0 f = P \circ \psi^{-1} \in \mathcal{F}_1$. Moreover ψ extends to a univalent function from V' to $\mathbb{C}$.*
(4) $\mathcal{R}_0$ *is holomorphic in the following sense: Suppose a family $f_\lambda = P \circ \varphi_\lambda^{-1}$ is given by a holomorphic function $\varphi_\lambda(z)$ in two variables $(\lambda, z) \in \Lambda \times V$, where Λ is a complex manifold. Then the renormalization can be written as $\mathcal{R}_0 f_\lambda = P \circ \psi_\lambda^{-1}$ with $\psi_\lambda(z)$ holomorphic in $(\lambda, z) \in \Lambda \times V'$.*

Main Thorem 2 (Contraction). *There exists a one to one correspondence between $\mathcal{F}_1$ and the Teichmüller space of $\mathbb{C} \setminus \overline{V}$. Let $d(\cdot,\cdot)$ be the distance on $\mathcal{F}_1$ induced from the Teichmüller distance, which is complete. Then $\mathcal{R}_0$ is a uniform contraction;*

$$d(\mathcal{R}_0(f), \mathcal{R}_0(g)) \leq \lambda\, d(f,g) \quad \text{for } f, g \in \mathcal{F}_1$$

where $\lambda = e^{-2\pi \operatorname{mod}(V' \setminus \overline{V})} < 1$. The convergence with respect to d implies the uniform convergence on compact sets (but not vice versa).

The proof will be given in § and basic facts about the Teichmüller space is also summarized there. An immediate consequence, together with Theorem , is the following:

Corollary 1. *The parabolic renormalization $\mathcal{R}_0$ on $\mathcal{F}_1$ has a unique fixed point, which belongs to $\mathcal{F}_0$. For any $f \in \mathcal{F}_1$, $\{\mathcal{R}_0^n f\}_{n=0}^{\infty}$ converges to the fixed point exponentially fast with respect to the metric defined in Main Theorem 2. Moreoevr, if $f \in \mathcal{F}_0$, then the renormalizations $\mathcal{R}_0^n f$ considered as elements of $\mathcal{F}_0$ converge to the fixed point uniformly on compact sets.*

We can derive similar results for the near-parabolic renormalization $\mathcal{R}$ and the fiber renormalization $\mathcal{R}_\alpha$ defined in the previous section, provided that α is small.

Definition. For $\alpha_* > 0$, denote

$$(0, \alpha_*] * \mathcal{F}_1 = \{e^{2\pi i \alpha} h(z) \mid 0 < \alpha \leq \alpha_*, h \in \mathcal{F}_1\}.$$

The distance on $(0, \alpha_*] * \mathcal{F}_1$ is defined by $d(f, g) = d(\frac{1}{f'(0)} f, \frac{1}{g'(0)} g) + |f'(0) - g'(0)|$, where d on the right hand side is the one for $\mathcal{F}_1$ defined in Main Theorem 2.

For an integer N, let $Irrat_{\geq N}$ be the set of irrational numbers α such that the continued fraction expansion has coefficients $a_n \geq N$.

Main Thorem 3 (Invariance of $\mathcal{F}_1$ under $\mathcal{R}_\alpha$ and hyperbolicity). *There exists $\alpha_* > 0$ such that if $\alpha \in \mathbb{C}$, $|\arg \alpha| < \pi/4$ and $0 < |\alpha| \leq \alpha_*$, then $\mathcal{R}_\alpha$ can be defined in $\mathcal{F}_1$ so that (c) and (d) of Main Theorem 1 hold for $\mathcal{R}_\alpha$. Moreover $\mathcal{R}_\alpha$ is a contraction as in Main Theorem 2 with the same λ. Hence $\mathcal{R}$ is hyperbolic in $(0, \alpha_*] * \mathcal{F}_1$.*

In particular, there exists an integer $N \geq 2$ for which the following holds: If $f(z) = e^{2\pi i \alpha} h(z)$ with $h \in \mathcal{F}_1$ and $\alpha \in Irrat_{\geq N}$, then the sequence of renormalizations can be defined and f_n's belong to $(0, \alpha_*] * \mathcal{F}_1$. If $g(z)$ is another map of the same type with the same α, then $d(\mathcal{R}^n f, \mathcal{R}^n g) \to 0$ as $n \to \infty$ exponentially fast.

We obtain these α_* and N by a continuity argument, so we do not have explicit bounds. It will be important to know how big α_* can be.

Corollary 2. *There exists an N (may be larger than the one in Main Theorem 3) such that if $f(z) = e^{2\pi i \alpha} h(z)$ with $h \in \mathcal{F}_1$ and $\alpha \in Irrat_{\geq N}$, then the critical orbit of f stays in the domain of definition of f and can be iterated infinitely many times. Moreover there exists an infinite sequence of periodic orbits to which the critical orbit does not accumulate.*

The same conclusion holds for $f(z) = e^{2\pi i \alpha} z + z^2$ provided that $\alpha \in Irrat_{\geq N}$ and α itself is sufficiently small. Hence the critical orbit is not dense in J_f.

This provides another approach for the renormalization around the boundary of quadratic Siegel disks (bounded type rotation numbers) obtained by McMullen. Our result covers the case of irrational numbers with large coefficients as above , in particular includes unbounded types.

There is a remarkable application of our result:

Theorem (Buff–Chéritat). *There exists an irrational number α such that $f(z) = e^{2\pi i \alpha} z + z^2$ has Julia set of positive Lebesgue measure.*

(See [BC] and their forthcoming paper for details.)

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