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Biaccessibility in unicritical Julia sets

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Abstract

We consider the dynamics of a degree $d \geq 2$ polynomial with only one bounded critical point. Such a polynomial is often said to be a unicritical polynomial. We can conjugate the polynomial to $f(z) = z^d + c$ by an appropriate transformation. If f has an irrationally indifferent fixed point then the filled Julia set K is connected. So we can use external rays for K. We are interested in how many rays land at a common point for studying the topology of the Julia set. In fact, points which are landing points of two or more rays are cut points of the Julia set. Such points are said to be biaccessible. D. Schleicher and S. Zakeri studied which points are biaccessible when $d = 2$. We extend the result and consider when $d \geq 2$.

1 Preliminaries

In this paper, we set $f(z) = z^d + c$ for some d greater than or equal to 2. Thus the bounded critical point is 0. Recall that the filled Julia set of f is

$$K \overset{\text{def}}{=} \{z \in \mathbb{C} : \{f^n(z)\}_{n \geq 0} \text{ is bounded}\}$$

and the Julia set of f is $J \overset{\text{def}}{=} \partial K$. Let $\tau_j(z) \overset{\text{def}}{=} e^{2\pi i \frac{j}{d}} z$ $(0 \leq j \leq d-1)$ be a $\dfrac{j}{d}$-rotation. $f(\tau_j(z)) = f(z)$ implies $\tau_j(K) = K$ and thus $\tau_j(J) = J$. Let us now assume that K is connected. Then there exists a unique conformal isomorphism $\psi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K$ such that $\displaystyle\lim_{z \to \infty} \frac{\psi(z)}{z} = 1$. Moreover, the following holds:

$$f(\psi(z)) = \psi(z^d) \quad (\forall z \in \mathbb{C} \setminus \overline{\mathbb{D}}). \qquad (*)$$

We say $R_t \stackrel{\text{def}}{=} \{\psi(re^{2\pi it}) : 1 < r\}$ is the external ray with angle $t \in \mathbb{R}/\mathbb{Z}$. Now $(*)$ implies $f(R_t) = R_{dt}$. $\tau_j(K) = K$ implies $\tau_j \circ \psi = \psi \circ \tau_j$ and thus $\tau_j(R_t) = R_{t+\frac{j}{d}}$. A ray R_t lands at z if $\lim_{r \searrow 1} \psi(re^{2\pi it}) = z$. If two or more rays land at z then we say z is biaccessible. By a thorem of F. and M. Riesz [**Mi**], the point z is a cut point in the Julia set, namely $J \setminus \{z\}$ is disconnected.

2 Main result

Lemma 1 Suppose that K is connected. Let α be a fixed point of f. Suppose that z is a biaccessible point such that $\alpha \notin \{f^n(z)\}_{n\geq 0}$ and $0 \notin \{f^n(z)\}_{n\geq 0}$. Then there exists two distinct rays R_{t_1} and R_{t_2} with a common landing point w, such that $R_{t_1} \cup R_{t_2} \cup \{w\}$ separates α from 0.

Theorem 2 Let α be an indifferent fixed point of f. Let z be a biaccessible point. Then:

- in the parabolic case, $\alpha \in \{f^n(z)\}_{n\geq 0}$;
- in the Siegel case, $0 \in \{f^n(z)\}_{n\geq 0}$;
- in the Cremer case, $\alpha \in \{f^n(z)\}_{n\geq 0}$ or $0 \in \{f^n(z)\}_{n\geq 0}$.
  (If α has the small cycles property then $0 \notin \{f^n(z)\}_{n\geq 0}$ by [**Ki**, Th.1.1].)

Remark 3 The proof of Theorem 2 is based on Lemma 1. We can show Lemma 1 and Theorem 2 by arguments similar to those in [**Za**].

Theorem 4 ([Za] Th.5) Suppose that $f(z) = z^2 + c$ has an irrationally indifferent fixed point α. Let z be a biaccessible point. Then:

- in the Siegel case, $0 \in \{f^n(z)\}_{n\geq 0}$;
- in the Cremer case, $\alpha \in \{f^n(z)\}_{n\geq 0}$.

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