<table>
<thead>
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<th>Title</th>
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</thead>
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Some topics on Fatou maps
in higher dimensional complex dynamics

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This is the abstract of my talk in the conference held at RIMS, October 2-6 2006. The results obtained in [M] and recent related results will be explained.

We study Fatou maps for a holomorphic map in a compact complex manifold. Fatou maps were first introduced by Ueda in his research on dynamics in the complex projective space \( \mathbb{P}^k \). (Fornaess & Sibony also considered such a notion in an implicit way.)

Let \( M \) be a compact complex manifold of dimension \( k \geq 1 \) and let \( f \) be a holomorphic self-map of \( M \).

**Definition 0.1.** (Fatou maps) Let \( N \) be a complex manifold and let \( \psi : N \to M \) be a holomorphic map such that \( \{ f^n \circ \psi \}_{n \geq 0} \) is a normal family in \( N \). We call such \( \psi \) a Fatou map. Particularly, in case when \( \psi \) is a holomorphic disc, we call it a Fatou disc. We say that a map \( \phi : N \to M \) is a limit map of \( \{ f^n \circ \psi \}_{n \geq 0} \) if there is a subsequence of \( \{ f^n \circ \psi \}_{n \geq 0} \) which converges to \( \phi \) locally uniformly in \( N \).

We treat two topics on Fatou maps as follows.

1 \hspace{1em} **Stable dynamics in the whole space**

Let \( M \) be a compact complex manifold of dimension \( k \geq 1 \) and let \( f \) be a holomorphic self-map of \( M \). Since \( M \) is compact, the Remmert proper mapping theorem implies that \( f^n(M) \) is an analytic subset of \( M \) for all \( n \geq 0 \) and there exists a number \( m_0 \geq 0 \) such that

\[
f^{m_0}(M) = f^{m_0+1}(M) = \cdots
\]

We put \( S := f^{m_0}(M) \) and call it the **minimal image**. Denoting by \( g \) the restriction of \( f \) on \( S \), the map \( g \) is a surjective holomorphic self-map of \( S \).

In this section, we treat the case when \( \{ f^n \}_{n \geq 1} \) is a normal family in \( M \), i.e. the case when the identity \( \text{id}_M \) is a Fatou map. By using Bochner-Montgomery theorem, we can obtain the following criterion.
Theorem 1.1. \((M)\) \(\{f^n\}_{n \geq 1}\) is a normal family in \(M\) if and only if \(\{f^n\}_{n \geq 1}\) has at least one subsequence which converges uniformly in \(M\).

By showing that \(S\) is a holomorphic retract, the next proposition follows.

Proposition 1.2. \((M)\) Suppose that \(\{f^n\}_{n \geq 1}\) is a normal family in \(M\). Then, \(S\) has no singular points, i.e. \(S\) is a complex submanifold in \(M\).

Next, we consider the number of periodic points of \(f\). We denote by \(\text{Fix}(f^n)\) the set of fixed points of \(f^n\) and put

\[
\text{Per}(f) := \bigcup_{n \geq 1} \text{Fix}(f^n).
\]

The following theorem shows that the total number of periodic points of \(f\) is independent of \(f\) and it is regulated by the Euler characteristic \(\chi(M)\).

Theorem 1.3. Let \(f\) be a holomorphic automorphism of \(M\). Suppose \(\{f^n\}_{n \geq 1}\) is a normal family in \(M\) and \(\#\text{Fix}(f^n) < +\infty\) for all \(n \geq 1\). Then,

\[
\#\text{Per}(f) = \chi(M).
\]

Example 1.4. We regard \(f(x, y) = (e^{\beta}y, e^{\alpha}x)\) as a holomorphic self-map of \(\mathbb{P}^1 \times \mathbb{P}^1\). Suppose \(\frac{\alpha + \beta}{2\pi i} \in \mathbb{R} \setminus \mathbb{Q}\). Then, \((0, 0), (\infty, \infty)\) are fixed points, \((0, \infty), (\infty, 0)\) are period 2 points and there are no other periodic points. Hence, \(\#\text{Per}(f) = 4 = \chi(\mathbb{P}^1 \times \mathbb{P}^1)\).

2 Semi-repellers outside the post-critical set

In this section, we describe semi-repelling property of forward invariant compact sets which are outside the closure of the post-critical set in terms of repelling points and non-contracting Fatou discs.

Let \(M\) be a compact complex manifold of dimension \(k \geq 1\) with a hermitian metric \(|\cdot|\). Let \(f : M \to M\) be a surjective holomorphic map. We denote by \(C\) the set of critical points of \(f\) and put

\[
D := \bigcup_{n \geq 1} f^n(C).
\]

Definition 2.1. (Non-contracting Fatou discs) Let \(\psi\) be a Fatou disc for \(f\). We say that \(\psi\) is non-contracting if no limit map of \(\{f^n \circ \psi\}_{n \geq 0}\) is constant.

Definition 2.2. (Repelling points) Let \(p \in M\). Denote by \(T_p\) the holomorphic tangent space at \(p\). We say that \(p\) is repelling for \(f\) if \(\min_{v \in T_p, |v| = 1} |D(f^j)(v)| \to +\infty\) as \(j \to +\infty\).
Let $\Delta$ denote the unit disc.

**Theorem 2.3.** ([M]) Let $E$ be a compact subset in $M$ such that $f(E) \subset E$ and $E \cap D = \emptyset$. Suppose that each connected component of $M \setminus D$ which meets $E$ is hyperbolically embedded in $M$. Then, there are two subsets $E^u, E^c \subset E$ which have the following properties;

(i) $E^u \cup E^c = E, \ E^u \cap E^c = \emptyset$;

(ii) $f(E^u) \subset E^u, \ f(E^c) \subset E^c$;

(iii) Each point in $E^u$ is repelling;

(iv) For each $p \in E^c$, there is a non-contracting Fatou disc $\psi : \Delta \to M$ such that $\psi$ is an embedding and $\psi(0) = p$.

Moreover, if $f(E) = E$ and $E^c = \emptyset$, then $E$ is a repeller with respect to the hermitian metric.

**Remark 2.4.** In Theorem 2.3, the hyperbolicity condition cannot be removed.

In case when $f$ is a holomorphic self-map of $\mathbb{P}^k$ of degree at least 2, we can remove the hyperbolicity condition in Theorem 2.3, thanks to Ueda's normality criterion.

**Theorem 2.5.** ([M]) Let $f$ be a holomorphic self-map of $\mathbb{P}^k$ of degree at least 2. Let $E$ be a compact subset in $M$ such that $f(E) \subset E$ and $E \cap D = \emptyset$. Then, there are two subsets $E^u, E^c \subset E$ which have the following properties;

(i) $E^u \cup E^c = E, \ E^u \cap E^c = \emptyset$;

(ii) $f(E^u) \subset E^u, \ f(E^c) \subset E^c$;

(iii) Each point in $E^u$ is repelling;

(iv) For each $p \in E^c$, there is a non-contracting Fatou disc $\psi : \Delta \to M$ such that $\psi$ is an embedding and $\psi(0) = p$.

Moreover, if $f(E) = E$ and $E^c = \emptyset$, then $E$ is a repeller with respect to the Fubini-Study metric.

Here we can find an interesting question.

**Question.** Let $f, E$ be the same as in Theorem 2.5. When $E$ is the support of the Green measure, $E^c$ is empty?

This is still unsolved at present.
References


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