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<tr>
<td>Author(s)</td>
<td>MAEGAWA, KAZUTOSHI</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2007, 1537: 33-36</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2007-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/59038">http://hdl.handle.net/2433/59038</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Some topics on Fatou maps
in higher dimensional complex dynamics

KAZUTOSHI MAEGAWA

This is the abstract of my talk in the conference held at RIMS, October 2-6 2006. The results obtained in [M] and recent related results will be explained.

We study Fatou maps for a holomorphic map in a compact complex manifold. Fatou maps were first introduced by Ueda in his research on dynamics in the complex projective space $\mathbb{P}^k$. (Fornæss & Sibony also considered such a notion in an implicit way.)

Let $M$ be a compact complex manifold of dimension $k \geq 1$ and let $f$ be a holomorphic self-map of $M$.

**Definition 0.1. (Fatou maps)** Let $N$ be a complex manifold and let $\psi : N \rightarrow M$ be a holomorphic map such that $\{f^n \circ \psi\}_{n \geq 0}$ is a normal family in $N$. We call such $\psi$ a Fatou map. Particularly, in case when $\psi$ is a holomorphic disc, we call it a Fatou disc. We say that a map $\phi : N \rightarrow M$ is a limit map of $\{f^n \circ \psi\}_{n \geq 0}$ if there is a subsequence of $\{f^n \circ \psi\}_{n \geq 0}$ which converges to $\phi$ locally uniformly in $N$.

We treat two topics on Fatou maps as follows.

1 **Stable dynamics in the whole space**

Let $M$ be a compact complex manifold of dimension $k \geq 1$ and let $f$ be a holomorphic self-map of $M$. Since $M$ is compact, the Remmert proper mapping theorem implies that $f^n(M)$ is an analytic subset of $M$ for all $n \geq 0$ and there exists a number $m_0 \geq 0$ such that

$$f^{m_0}(M) = f^{m_0+1}(M) = \cdots$$

We put $S := f^{m_0}(M)$ and call it the **minimal image**. Denoting by $g$ the restriction of $f$ on $S$, the map $g$ is a surjective holomorphic self-map of $S$.

In this section, we treat the case when $\{f^n\}_{n \geq 1}$ is a normal family in $M$, i.e. the case when the identity $\text{id}_M$ is a Fatou map. By using Bochner-Montgomery theorem, we can obtain the following criterion.
Theorem 1.1. $(M) \{f^n\}_{n \geq 1}$ is a normal family in $M$ if and only if $\{f^n\}_{n \geq 1}$ has at least one subsequence which converges uniformly in $M$.

By showing that $S$ is a holomorphic retract, the next proposition follows.

Proposition 1.2. $(M)$ Suppose that $\{f^n\}_{n \geq 1}$ is a normal family in $M$. Then, $S$ has no singular points, i.e. $S$ is a complex submanifold in $M$.

Next, we consider the number of periodic points of $f$. We denote by $\text{Fix}(f^n)$ the set of fixed points of $f^n$ and put

$$\text{Per}(f) := \bigcup_{n \geq 1} \text{Fix}(f^n).$$

The following theorem shows that the total number of periodic points of $f$ is independent of $f$ and it is regulated by the Euler characteristic $\chi(M)$.

Theorem 1.3. Let $f$ be a holomorphic automorphism of $M$. Suppose $\{f^n\}_{n \geq 1}$ is a normal family in $M$ and $\#	ext{Fix}(f^n) < +\infty$ for all $n \geq 1$. Then,

$$\#	ext{Per}(f) = \chi(M).$$

Example 1.4. We regard $f(x, y) = (e^{\beta}y, e^{\alpha}x)$ as a holomorphic self-map of $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose $\frac{\alpha + \beta}{2\pi i} \in \mathbb{R} \setminus \mathbb{Q}$. Then, $(0, 0), (\infty, \infty)$ are fixed points, $(0, \infty), (\infty, 0)$ are period 2 points and there are no other periodic points. Hence, $\#	ext{Per}(f) = 4 = \chi(\mathbb{P}^1 \times \mathbb{P}^1)$.

2 Semi-repellers outside the post-critical set

In this section, we describe semi-repelling property of forward invariant compact sets which are outside the closure of the post-critical set in terms of repelling points and non-contracting Fatou discs.

Let $M$ be a compact complex manifold of dimension $k \geq 1$ with a hermitian metric $| \cdot |$. Let $f : M \to M$ be a surjective holomorphic map. We denote by $C$ the set of critical points of $f$ and put

$$D := \bigcup_{n \geq 1} f^n(C).$$

Definition 2.1. (Non-contracting Fatou discs) Let $\psi$ be a Fatou disc for $f$. We say that $\psi$ is non-contracting if no limit map of $\{f^n \circ \psi\}_{n \geq 0}$ is constant.

Definition 2.2. (Repelling points) Let $p \in M$. Denote by $T_p$ the holomorphic tangent space at $p$. We say that $p$ is repelling for $f$ if $\min_{v \in T_p, |v| = 1} |D(f^j)(v)| \to +\infty$ as $j \to +\infty$. 

Let $\Delta$ denote the unit disc.

**Theorem 2.3.** ([M]) Let $E$ be a compact subset in $M$ such that $f(E) \subset E$ and $E \cap D = \emptyset$. Suppose that each connected component of $M \setminus D$ which meets $E$ is hyperbolically embedded in $M$. Then, there are two subsets $E^u, E^c \subset E$ which have the following properties;

(i) $E^u \cup E^c = E$, $E^u \cap E^c = \emptyset$;

(ii) $f(E^u) \subset E^u$, $f(E^c) \subset E^c$;

(iii) Each point in $E^u$ is repelling;

(iv) For each $p \in E^c$, there is a non-contracting Fatou disc $\psi : \Delta \rightarrow M$ such that $\psi$ is an embedding and $\psi(0) = p$.

Moreover, if $f(E) = E$ and $E^c = \emptyset$, then $E$ is a repeller with respect to the hermitian metric.

**Remark 2.4.** In Theorem 2.3, the hyperbolicity condition can not be removed.

In case when $f$ is a holomorphic self-map of $\mathbb{P}^k$ of degree at least 2, we can remove the hyperbolicity condition in Theorem 2.3, thanks to Ueda’s normality criterion.

**Theorem 2.5.** ([M]) Let $f$ be a holomorphic self-map of $\mathbb{P}^k$ of degree at least 2. Let $E$ be a compact subset in $M$ such that $f(E) \subset E$ and $E \cap D = \emptyset$. Then, there are two subsets $E^u, E^c \subset E$ which have the following properties;

(i) $E^u \cup E^c = E$, $E^u \cap E^c = \emptyset$;

(ii) $f(E^u) \subset E^u$, $f(E^c) \subset E^c$;

(iii) Each point in $E^u$ is repelling;

(iv) For each $p \in E^c$, there is a non-contracting Fatou disc $\psi : \Delta \rightarrow M$ such that $\psi$ is an embedding and $\psi(0) = p$.

Moreover, if $f(E) = E$ and $E^c = \emptyset$, then $E$ is a repeller with respect to the Fubini-Study metric.

Here we can find an interesting question.

**Question.** Let $f, E$ be the same as in Theorem 2.5. When $E$ is the support of the Green measure, $E^c$ is empty?

This is still unsolved at present.
References


DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KYOTO UNIVERSITY
606-8502, KYOTO, JAPAN

E-mail address: maegawa@math.kyoto-u.ac.jp