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Random Julia sets that are Jordan curves but not quasicircles *

Hiroki Sumi
Department of Mathematics
Graduate School of Science
Osaka University
1-1, Machikaneyama, Toyonaka, Osaka, 560-0043, Japan
E-mail: sumi@math.sci.osaka-u.ac.jp
http://www.math.sci.osaka-u.ac.jp/~sumi/welcomeou-e.html

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Abstract

We consider the random dynamics of polynomials and the dynamics of polynomial semigroups (semigroups generated by a family of polynomial maps) on the Riemann sphere $\hat{\mathbb{C}}$. We show that under a certain condition, for almost every sequence $\gamma=(\gamma_1,\gamma_2,\dots)$ of polynomials, the random Julia set of γ is a Jordan curve but not a quasicircle and the basin A_γ of infinity is a John domain. Note that there exists no polynomial h such that the above holds. Furthermore, we give a classification of polynomial semigroups G such that G is generated by a compact family, the planar postcritical set of G is bounded, and G is (semi-) hyperbolic. Many phenomena of polynomial semigroups and random dynamics of polynomials that do not occur in the usual dynamics of polynomials are found and investigated.

1 Introduction

The theory of complex dynamical systems, which has its origin in the important work of Fatou and Julia in the 1910s, has been investigated by many

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people and discussed in depth. In particular, since D. Sullivan showed the famous "no wandering domain theorem" using Teichmüller theory in the 1980s, this subject has attracted many researchers from a wide area. For a general reference on complex dynamical systems, see Milnor's textbook [M].

There are several areas in which we deal with generalized notions of classical iteration theory of rational functions. One of them is the theory of dynamics of rational semigroups (semigroups generated by holomorphic maps on the Riemann sphere $\hat{\mathbb{C}}$), and another one is the theory of random dynamics of holomorphic maps on the Riemann sphere.

In this paper, we will discuss these subjects. A **rational semigroup** is a semigroup generated by non-constant rational maps on $\hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, with the semigroup operation being functional composition ([HM1]). A **polynomial semigroup** is a semigroup generated by non-constant polynomial maps. Research on the dynamics of rational semigroups was initiated by A. Hinkkanen and G. J. Martin ([HM1],[HM2]), who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group([ZR], [GR]), who studied such semigroups from the perspective of random dynamical systems. Moreover, the research on rational semigroups is related to that on "iterated function systems" in fractal geometry. In fact, the Julia set of a rational semigroup generated by a compact family has " backward self-similarity" (cf. Lemma 3.1-2). For other research on rational semigroups, see [Sta1], [Sta2], [Sta3], [SY], [SSS], [SS], [SU1], [SU2], and [S1]-[S11].

The research on the dynamics of rational semigroups is also directly related to that on the random dynamics of holomorphic maps. The first study in this direction was by Fornaess and Sibony ([FS]), and much research has followed. (See [Br],[Bu1],[Bu2], [BBR].)

We remark that the complex dynamical systems can be used to describe some mathematical models. For example, the behavior of the population of a certain species can be described as the dynamical system of a polynomial $f(z) = az(1-z)$ such that f preserves the unit interval and the postcritical set in the plane is bounded (cf. [D]). From this point of view, it is very important to consider the random dynamics of such polynomials (see also Example 1.4). For the random dynamics of polynomials on the unit interval, see [Steins].

We shall give some definitions for the dynamics of rational semigroups:

Definition 1.1 ([HM1], [GR]). Let G be a rational semigroup. We set

$$F(G) = \{z \in \hat{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\}, \quad J(G) = \hat{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the **Fatou set** of G and $J(G)$ is called the **Julia set** of G. We let $\langle h_1, h_2, \dots \rangle$ denote the rational semigroup generated by the family $\{h_i\}$. The Julia set of the semigroup generated by a single map g is denoted by $J(g)$.

Definition 1.2.

1. For each rational map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we set $CV(g) :=$ {all critical values of $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$}. Moreover, for each polynomial map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we set $CV^*(g) := CV(g) \setminus \{\infty\}$.

2. Let G be a rational semigroup. We set

$$P(G) := \overline{\bigcup_{g \in G} CV(g)} \ (\subset \hat{\mathbb{C}}).$$

This is called the **postcritical set** of G. Furthermore, for a polynomial semigroup G, we set $P^*(G) := P(G) \setminus \{\infty\}$. This is called the **planar postcritical set** (or **finite postcritical set**) of G. We say that a polynomial semigroup G is **postcritically bounded** if $P^*(G)$ is bounded in $\mathbb{C}$.

Remark 1.3. Let G be a rational semigroup generated by a family Λ of rational maps. Then, $P(G) = \overline{\cup_{g \in G}\, g(\cup_{h \in \Lambda} CV(h))}$ and $g(P(G)) \subset P(G)$ for each $g \in G$. From this formula, one can figure out how the set $P(G)$ (resp. $P^*(G)$) spreads in $\hat{\mathbb{C}}$ (resp. $\mathbb{C}$). In fact, in Section 2.3, using the above formula, we present a way to construct examples of postcritically bounded polynomial semigroups (with some additional properties).

Example 1.4. Let $\Lambda := \{h(z) = cz^a(1-z)^b \mid a, b \in \mathbb{N},\ c > 0,\ c(\frac{a}{a+b})^a(\frac{b}{a+b})^b \leq 1\}$ and let G be the polynomial semigroup generated by Λ. Since for each $h \in \Lambda$, $h([0,1]) \subset [0,1]$ and $CV^*(h) \subset [0,1]$, it follows that each subsemigroup H of G is postcritically bounded.

Remark 1.5. It is well-known that for a polynomial g with $\deg(g) \geq 2$, $P^*(\langle g \rangle)$ is bounded in $\mathbb{C}$ if and only if $J(g)$ is connected ([M], Theorem 9.5).

As mentioned in Remark 1.5, the planar postcritical set is one piece of important information regarding the dynamics of polynomials. Concerning the theory of iteration of quadratic polynomials, we have been investigating the famous "Mandelbrot set".

When investigating the dynamics of polynomial semigroups, it is natural for us to discuss the relationship between the planar postcritical set and the figure of the Julia set. The first question in this regard is:

Question 1.6. Let G be a polynomial semigroup such that each element $g \in G$ is of degree at least two. Is $J(G)$ necessarily connected when $P^*(G)$ is bounded in $\mathbb{C}$?

The answer is **NO.**

Example 1.7 ([SY]). Let $G = \langle z^3, \frac{z^2}{4} \rangle$. Then $P^*(G) = \{0\}$ (which is bounded in $\mathbb{C}$) and $J(G)$ is disconnected ($J(G)$ is a Cantor set of round circles). Furthermore, according to ([S5], Theorem 2.4.1), it can be shown that a small perturbation H of G still satisfies that $P^*(H)$ is bounded in $\mathbb{C}$ and that $J(H)$ is disconnected. ($J(H)$ is a Cantor set of quasi-circles with uniform dilatation.)

Question 1.8. What happens if $P^*(G)$ is bounded in $\mathbb{C}$ and $J(G)$ is disconnected?

Problem 1.9. Classify postcritically bounded polynomial semigroups.

In this paper, we investigate (semi-)hyperbolic, postcritically bounded, polynomial semigroups generated by a compact family Γ of polynomials. We show that if G is such a semigroup with disconnected Julia set, and if there exists an element $g \in G$ such that $J(g)$ is not a Jordan curve, then, for almost every sequence $\gamma \in \Gamma^{\mathbb{N}}$, the Julia set J_γ of γ is a Jordan curve but not a quasicircle, the basin of infinity A_γ is a John domain, and the bounded component U_γ of the Fatou set F_γ of γ is not a John domain (cf. Theorem 2.26). Moreover, we classify the semi-hyperbolic, postcritically bounded, polynomial semigroups generated by a compact family Γ of polynomials. We show that such a semigroup G satisfies either (I) every fiberwise Julia set is a quasicircle with uniform distortion, or (II) for almost every sequence $\gamma \in \Gamma^{\mathbb{N}}$, the Julia set J_γ is a Jordan curve but not a quasicircle, the basin of infinity A_γ is a John domain, and the bounded component U_γ of the Fatou set is not a John domain, or (III) for every $\alpha, \beta \in \Gamma^{\mathbb{N}}$, the intersection of the Julia sets J_α and J_β is not empty, and $J(G)$ is arcwise connected (cf. Theorem 2.30). Furthermore, we also classify the hyperbolic, postcritically bounded, polynomial semigroups generated by a compact family Γ of polynomials. We show that such a semigroup G satisfies either (I) above, or (II) above, or (III)' for every $\alpha, \beta \in \Gamma^{\mathbb{N}}$, the intersection of the Julia sets J_α and J_β is not empty, $J(G)$ is arcwise connected, and for every sequence $\gamma \in \Gamma^{\mathbb{N}}$, there exist infinitely many bounded components of the Fatou set F_γ (cf. Theorem 2.32). We give some examples of situation (II) above (cf. Example 2.27, Example 2.33 and Section 2.3). Note that situation (II) above is a special phenomenon of random dynamics of polynomials that does not occur in the usual dynamics of polynomials.

The key to investigating the dynamics of postcritically bounded polynomial semigroups is the density of repelling fixed points in the Julia set (cf. Theorem 3.2), which can be shown by an application of the Ahlfors five island theorem, and the lower semi-continuity of $\gamma \mapsto J_\gamma$ (Lemma 3.4-2), which is a consequence of potential theory. Moreover, one of the keys to investigating the fiberwise dynamics of skew products is, the observation of non-constant limit functions (cf. Lemma 3.12 and [S1]). The key to investigating the dynamics of semi-hyperbolic polynomial semigroups is, the continuity of the map $\gamma \mapsto J_\gamma$ (this is highly nontrivial; see [S1]) and the Johnness of the basin A_γ of infinity (cf. [S4]). Note that the continuity of the map $\gamma \mapsto J_\gamma$ does not hold in general, if we do not assume semi-hyperbolicity. Moreover, one of the original aspects of this paper is the idea of "combining both the theory of rational semigroups and that of random complex dynamics". It is quite natural to investigate both fields simultaneously. However, no study thus far has done so.

Furthermore, in Section 2.3, we provide a way of constructing examples of postcritically bounded polynomial semigroups with some additional properties (disconnectivity of Julia set, semi-hyperbolicity, hyperbolicity, etc.) (cf. Proposition 2.36, Theorem 2.39, Theorem 2.42). For example, by Proposition 2.36, there exists a 2-generator postcritically bounded polynomial semigroup $G = \langle h_1, h_2 \rangle$ with disconnected Julia set such that h_1 has a Siegel disk.

As wee see in Example 1.4 and Section 2.3, it is not difficult to construct many examples, it is not difficult to verify the hypothesis "postcritically bounded", and the class of postcritically bounded polynomial semigroups is very wide.

Throughout the paper, we will see many phenomena in polynomial semigroups or random dynamics of polynomials that do not occur in the usual dynamics of polynomials.

In Section 2, we present the main results of this paper. We give some tools in Section 3. The proofs of the main results are given in Section 4.

2 Main results

In order to state the main results, we give some notations and definitions.

Definition 2.1. We set Rat $: = \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h$ is a non-constant rational map$\}$ endowed with the topology induced by uniform convergence on $\hat{\mathbb{C}}$ with respect to the spherical distance. We set Poly $:= \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h$ is a non-constant polynomial map$\}$ endowed with the relative topology from Rat. Moreover, we set $\mathrm{Poly}_{\deg\geq 2} := \{g \in \mathrm{Poly} \mid \deg(g) \geq 2\}$ endowed with the relative topology from Rat.

Remark 2.2. Let $d \geq 1$, $\{p_n\}_{n\in\mathbb{N}}$ a sequence of polynomials of degree d, and p a polynomial. Then, $p_n \to p$ in Poly if and only if the coefficients converge appropriately and p is of degree d.

Definition 2.3. Let $\mathcal{G}$ be the set of all polynomial semigroups G with the following properties:

- each element of G is of degree at least two, and
- $P^*(G)$ is bounded in $\mathbb{C}$, i.e., G is postcritically bounded.

Furthermore, we set $\mathcal{G}_{con} = \{G \in \mathcal{G} \mid J(G) \text{ is connected}\}$ and $\mathcal{G}_{dis} = \{G \in \mathcal{G} \mid J(G) \text{ is disconnected}\}$.

Notation: For a polynomial semigroup G, we denote by $\mathcal{J} = \mathcal{J}_G$ the set of all connected components J of $J(G)$ such that $J \subset \mathbb{C}$. Moreover, we denote by $\hat{\mathcal{J}} = \hat{\mathcal{J}}_G$ the set of all connected components of $J(G)$.

Remark 2.4. If a polynomial semigroup G is generated by a compact set in $\mathrm{Poly}_{\deg\geq 2}$, then $\infty \in F(G)$ and thus $\mathcal{J} = \hat{\mathcal{J}}$.

Definition 2.5. For any connected sets K_1 and K_2 in $\mathbb{C}$, "$K_1 \leq K_2$" indicates that $K_1 = K_2$, or K_1 is included in a bounded component of $\mathbb{C} \setminus K_2$. Furthermore, "$K_1 < K_2$" indicates $K_1 \leq K_2$ and $K_1 \neq K_2$. Note that "$\leq$" is a partial order in the space of all non-empty compact connected sets in $\mathbb{C}$. This "$\leq$" is called the **surrounding order.**

Definition 2.6. For a polynomial semigroup G, we set

$$\hat{K}(G) := \{z \in \mathbb{C} \mid \bigcup_{g\in G}\{g(z)\} \text{ is bounded in } \mathbb{C}\}$$

and call $\hat{K}(G)$ the **smallest filled-in Julia set** of G. For a polynomial g, we set $K(g) := \hat{K}(\langle g\rangle)$.

Notation: For a set $A \subset \hat{\mathbb{C}}$, we denote by $\mathrm{int}(A)$ the set of all interior points of A.

Notation: For a polynomial semigroup G with $\infty \in F(G)$, we denote by $F_\infty(G)$ the connected component of $F(G)$ containing ∞. Moreover, for a polynomial g with $\deg(g) \geq 2$, we set $F_\infty(g) := F_\infty(\langle g\rangle)$.

In [S11], the following results (Theorem 2.7, Theorem 2.8 and Proposition 2.9) were shown. These results are used to present the main result of this paper.

Theorem 2.7 **([S11]).** *Let $G \in \mathcal{G}$ (possibly generated by a non-compact family). Then*

1. *$(\mathcal{J}, \leq)$ is totally ordered.*

2. *Each connected component of $F(G)$ is either simply or doubly connected.*

3. *For any $g \in G$ and any connected component J of $J(G)$, we have that $g^{-1}(J)$ is connected. Let $g^*(J)$ be the connected component of $J(G)$ containing $g^{-1}(J)$. If $J \in \mathcal{J}$, then $g^*(J) \in \mathcal{J}$. If $J_1, J_2 \in \mathcal{J}$ and $J_1 \leq J_2$, then $g^{-1}(J_1) \leq g^{-1}(J_2)$ and $g^*(J_1) \leq g^*(J_2)$.*

Theorem 2.8 **([S11]).** *Let $G \in \mathcal{G}_{dis}$. Under the above notation, we have the following.*

1. *We have that $\infty \in F(G)$ and the connected component $F_\infty(G)$ of $F(G)$ containing ∞ is simply connected. Furthermore, the element $J_{\max} = J_{\max}(G) \in \mathcal{J}$ containing $\partial F_\infty(G)$ is the unique element of $\mathcal{J}$ satisfying that $J \leq J_{\max}$ for each $J \in \mathcal{J}$.*

2. *There exists a unique element $J_{\min} = J_{\min}(G) \in \mathcal{J}$ such that $J_{\min} \leq J$ for each element $J \in \mathcal{J}$. Furthermore, let D be the unbounded component of $\mathbb{C} \setminus J_{\min}$. Then, $P^*(G) \subset \hat{K}(G) \subset \mathbb{C} \setminus D$ and $\partial \hat{K}(G) \subset J_{\min}$.*

3. *We have that $\mathrm{int}(\hat{K}(G)) \neq \emptyset$.*

Proposition 2.9 **([S11]).** *Let G be a polynomial semigroup generated by a compact subset Γ of $\mathrm{Poly}_{\deg \geq 2}$. Suppose that $G \in \mathcal{G}_{dis}$. Then, there exists an element $h_1 \in \Gamma$ with $J(h_1) \subset J_{\max}$ and there exists an element $h_2 \in \Gamma$ with $J(h_2) \subset J_{\min}$.*

Notation: We denote by d the spherical distance on $\hat{\mathbb{C}}$. Given $A \subset \hat{\mathbb{C}}$ and $z \in \hat{\mathbb{C}}$, we set $d(z, A) := \inf\{d(z, w) \mid w \in A\}$. Given $A \subset \hat{\mathbb{C}}$ and $\epsilon > 0$, we set $B(A, \epsilon) := \{a \in \hat{\mathbb{C}} \mid d(a, A) < \epsilon\}$. Furthermore, given $A \subset \mathbb{C}$, $z \in \mathbb{C}$, and $\epsilon > 0$, we set $d_e(z, A) := \inf\{|z - w| \mid w \in A\}$ and $D(A, \epsilon) := \{a \in \mathbb{C} \mid d_e(a, A) < \epsilon\}$.

Definition 2.10 **([S1],[S4]).**

1. Let X be a compact metric space, $g : X \to X$ a continuous map, and $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ a continuous map. We say that f is a rational skew product (or fibered rational map on trivial bundle $X \times \hat{\mathbb{C}}$) over $g : X \to X$, if $\pi \circ f = g \circ \pi$ where $\pi : X \times \hat{\mathbb{C}} \to X$ denotes the natural projection, and if for each $x \in X$, the restriction $f_x :=$

$f|_{\pi^{-1}\{x\}} : \pi^{-1}\{x\} \to \pi^{-1}\{g(x)\}$ of f is a non-constant rational map, under the canonical identification $\pi^{-1}\{x'\} \cong \hat{\mathbb{C}}$ for each $x' \in X$. Let $d(x) = \deg(f_x)$, for each $x \in X$. Let $f_{x,n}$ be the rational map defined by: $f_{x,n}(y) = \pi_{\hat{\mathbb{C}}}(f^n(x,y))$, for each $n \in \mathbb{N}, x \in X$ and $y \in \hat{\mathbb{C}}$, where $\pi_{\hat{\mathbb{C}}} : X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the projection map.

Moreover, if $f_{x,1}$ is a polynomial for each $x \in X$, then we say that $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ is a polynomial skew product over $g : X \to X$.

2. Let Γ be a compact subset of Rat. We set $\Gamma^{\mathbb{N}} := \{\gamma = (\gamma_1, \gamma_2, \dots) \mid \forall j, \gamma_j \in \Gamma\}$ endowed with the product topology. This is a compact metric space. Let $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ be the shift map, which is defined by $\sigma(\gamma_1, \gamma_2, \dots) := (\gamma_2, \gamma_3, \dots)$. Moreover, we define a map $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ by: $(\gamma, y) \mapsto (\sigma(\gamma), \gamma_1(y))$, where $\gamma = (\gamma_1, \gamma_2, \dots)$. This is called **the skew product associated with the family Γ of rational maps.** Note that $f_{\gamma,n}(y) = \gamma_n \circ \cdots \circ \gamma_1(y)$.

Remark 2.11. Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a rational skew product over $g : X \to X$. Then, the function $x \mapsto d(x)$ is continuous in X.

Definition 2.12 ([S1],[S4]). Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a rational skew product over $g : X \to X$. Then, we use the following notation.

1. For each $x \in X$ and $n \in \mathbb{N}$, we set $f_x^n := f^n|_{\pi^{-1}\{x\}} : \pi^{-1}\{x\} \to \pi^{-1}\{g^n(x)\} \subset X \times \hat{\mathbb{C}}$.

2. For each $x \in X$, we denote by $F_x(f)$ the set of points $y \in \hat{\mathbb{C}}$ which has a neighborhood U in $\hat{\mathbb{C}}$ such that $\{f_{x,n} : U \to \hat{\mathbb{C}}\}_{n \in \mathbb{N}}$ is normal. Moreover, we set $F^x(f) := \{x\} \times F_x(f)$ $(\subset X \times \hat{\mathbb{C}})$.

3. For each $x \in X$, we set $J_x(f) := \hat{\mathbb{C}} \setminus F_x(f)$. Moreover, we set $J^x(f) := \{x\} \times J_x(f)$ $(\subset X \times \hat{\mathbb{C}})$. These sets $J^x(f)$ and $J_x(f)$ are called the fiberwise Julia sets.

4. We set $\tilde{J}(f) := \overline{\bigcup_{x \in X} J^x(f)}$, where the closure is taken in the product space $X \times \hat{\mathbb{C}}$.

5. For each $x \in X$, we set $\hat{J}^x(f) := \pi^{-1}\{x\} \cap \tilde{J}(f)$. Moreover, we set $\hat{J}_x(f) := \pi_{\hat{\mathbb{C}}}(\hat{J}^x(f))$.

6. We set $\tilde{F}(f) := (X \times \hat{\mathbb{C}}) \setminus \tilde{J}(f)$.

Remark 2.13. We have $\hat{J}^x(f) \supset J^x(f)$ and $\hat{J}_x(f) \supset J_x(f)$. However, strict containment can occur. For example, let h_1 be a polynomial having a Siegel disk with center $z_1 \in \mathbb{C}$. Let h_2 be a polynomial such that z_1 is a repelling fixed point of h_2. Let $\Gamma = \{h_1, h_2\}$. Let $f : \Gamma \times \hat{\mathbb{C}} \to \Gamma \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Let $x = (h_1, h_1, h_1, \dots) \in \Gamma^{\mathbb{N}}$. Then, $(x, z_1) \in \hat{J}^x(f) \setminus J^x(f)$ and $z_1 \in \hat{J}_x(f) \setminus J_x(f)$.

Definition 2.14. Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$. Then for each $x \in X$, we set $K_x(f) := \{y \in \hat{\mathbb{C}} \mid \{f_{x,n}(y)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{C}\}$, and $A_x(f) := \{y \in \hat{\mathbb{C}} \mid f_{x,n}(y) \to \infty,\ n \to \infty\}$. Moreover, we set $K^x(f) := \{x\} \times K_x(f)$ $(\subset X \times \hat{\mathbb{C}})$ and $A^x(f) := \{x\} \times A_x(f)$ $(\subset X \times \hat{\mathbb{C}})$.

Definition 2.15. Let G be a polynomial semigroup generated by a subset Γ of $\text{Poly}_{\deg\geq 2}$. Suppose $G \in \mathcal{G}_{dis}$. Then we set

$$\Gamma_{\min} := \{h \in \Gamma \mid J(h) \subset J_{\min}\},$$

where $J_{\min}$ denotes the unique minimal element in $(\mathcal{J}, \leq)$ in Theorem 2.8-2. Furthermore, if $\Gamma_{\min} \neq \emptyset$, let $G_{\min,\Gamma}$ be the subsemigroup of G that is generated by $\Gamma_{\min}$.

Remark 2.16. Let G be a polynomial semigroup generated by a compact subset Γ of $\text{Poly}_{\deg\geq 2}$. Suppose $G \in \mathcal{G}_{dis}$. Then, by Proposition 2.9, we have $\Gamma_{\min} \neq \emptyset$ and $\Gamma \setminus \Gamma_{\min} \neq \emptyset$. Moreover, $\Gamma_{\min}$ is a compact subset of Γ. For, if $\{h_n\}_{n\in\mathbb{N}} \subset \Gamma_{\min}$ and $h_n \to h_\infty$ in Γ, then for a repelling periodic point $z_0 \in J(h_\infty)$ of h_∞, we have that $d(z_0, J(h_n)) \to 0$ as $n \to \infty$, which implies that $z_0 \in J_{\min}$ and thus $h_\infty \in \Gamma_{\min}$.

Notation: Let $\mathcal{F} := \{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of meromorphic functions in a domain V. We say that a meromorphic function ψ is a limit function of $\mathcal{F}$ if there exists a strictly increasing sequence $\{n_j\}_{j\in\mathbb{N}}$ of positive integers such that $\varphi_{n_j} \to \psi$ locally uniformly on V, as $j \to \infty$.

Definition 2.17. Let G be a rational semigroup.

1. We say that G is hyperbolic if $P(G) \subset F(G)$.

2. We say that G is semi-hyperbolic if there exists a number $\delta > 0$ and a number $N \in \mathbb{N}$ such that for each $y \in J(G)$ and each $g \in G$, we have $\deg(g : V \to B(y, \delta)) \leq N$ for each connected component V of $g^{-1}(B(y, \delta))$, where $B(y, \delta)$ denotes the ball of radius δ with center y with respect to the spherical distance, and $\deg(g : \cdot \to \cdot)$ denotes the degree of finite branched covering. (For background of semi-hyperbolicity, see [S1] and [S4].)

Definition 2.18. Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a rational skew product over $g : X \to X$. We set

$$C(f) := \{(x, y) \in X \times \hat{\mathbb{C}} \mid y \text{ is a critical point of } f_{x,1}\}.$$

Moreover, we set $P(f) := \overline{\cup_{n \in \mathbb{N}} f^n(C(f))}$, where the closure is taken in the product space $X \times \hat{\mathbb{C}}$. This $P(f)$ is called the **fiber-postcritical set** of f.

We say that f is hyperbolic (along fibers) if $P(f) \subset F(f)$.

Definition 2.19 ([S1]). Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a rational skew product over $g : X \to X$. Let $N \in \mathbb{N}$. We say that a point $(x_0, y_0) \in X \times \hat{\mathbb{C}}$ belongs to $SH_N(f)$ if there exists a neighborhood U of x_0 in X and a positive number δ such that for any $x \in U$, any $n \in \mathbb{N}$, any $x_n \in g^{-n}(x)$, and any connected component V of $(f_{x_n,n})^{-1}(B(y_0, \delta))$, $\deg(f_{x_n,n} : V \to B(y_0, \delta)) \leq N$. Moreover, we set $UH(f) := (X \times \hat{\mathbb{C}}) \setminus \cup_{N \in \mathbb{N}} SH_N(f)$. We say that f is semi-hyperbolic (along fibers) if $UH(f) \subset \tilde{F}(f)$.

Remark 2.20. Under the above notation, we have $UH(f) \subset P(f)$.

Remark 2.21. Let Γ be a compact subset of Rat and let $f : \Gamma^{\mathbf{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbf{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let G be the rational semigroup generated by Γ. Then, by Lemma 3.5-1, it is easy to see that f is semi-hyperbolic if and only if G is semi-hyperbolic. Similarly, it is easy to see that f is hyperbolic if and only if G is hyperbolic.

Definition 2.22. Let Γ and S be non-empty subsets of $\text{Poly}_{\deg \geq 2}$ with $S \subset \Gamma$. We set $R(\Gamma, S) := \left\{\gamma = (\gamma_1, \gamma_2, \dots) \in \Gamma^{\mathbf{N}} \mid \sharp(\{n \in \mathbb{N} \mid \gamma_n \in S\}) = \infty\right\}$.

2.1 Fiberwise Julia sets that are Jordan curves but not quasicircles

We present a result on a sufficient condition for a fiberwise Julia set $J_x(f)$ to be a Jordan curve but not a quasicircle. The proofs are given in Section 4.1.

Definition 2.23. Let $K \geq 1$. A Jordan curve ξ in $\mathbb{C}$ is said to be a K-quasicircle if there exists a K-quasiconformal map $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\xi = \varphi(\{z \in \mathbb{C} \mid |z| = 1\})$.

Definition 2.24. Let V be a subdomain of $\hat{\mathbb{C}}$ such that $\partial V \subset \mathbb{C}$. We say that V is a John domain if there exists a constant $c > 0$ and a point $z_0 \in V$ ($z_0 = \infty$ when $\infty \in V$) satisfying the following: for all $z_1 \in V$ there exists an arc $\xi \subset V$ connecting z_1 to z_0 such that for any $z \in \xi$, we have $\min\{|z - a| \mid a \in \partial V\} \geq c|z - z_1|$.

Remark 2.25. Let V be a simply connected domain in $\hat{\mathbb{C}}$ such that $\partial V \subset \mathbb{C}$. It is well-known that if V is a John domain, then ∂V is locally connected ([NV], page 26). Moreover, a Jordan curve $\xi \subset \mathbb{C}$ is a quasicircle if and only if both components of $\hat{\mathbb{C}} \setminus \xi$ are John domains ([NV],Theorem 9.3).

Theorem 2.26. *(Theorem A) Let G be a polynomial semigroup generated by a compact subset Γ of $\mathrm{Poly}_{\deg\geq 2}$. Suppose that $G \in \mathcal{G}_{dis}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ of polynomials. Let $m \in \mathbb{N}$ and suppose that there exists an element $(h_1, h_2, \dots, h_m) \in \Gamma^m$ such that $J(h_m \circ \cdots \circ h_1)$ is not a quasicircle. Let $\alpha = (\alpha_1, \alpha_2, \dots) \in \Gamma^{\mathbb{N}}$ be the element such that for each $k, l \in \mathbb{N} \cup \{0\}$ with $1 \leq l \leq m$, $\alpha_{km+l} = h_l$. Then, the following statements 1 and 2 hold.*

1. *Suppose that G is hyperbolic. Let $\gamma \in R(\Gamma, \Gamma \setminus \Gamma_{\min})$ be an element such that there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers satisfying that $\sigma^{n_k}(\gamma) \to \alpha$ as $k \to \infty$. Then, $J_\gamma(f)$ is a Jordan curve but not a quasicircle. Moreover, the unbounded component $A_\gamma(f)$ of $F_\gamma(f)$ is a John domain, but the unique bounded component U_γ of $F_\gamma(f)$ is not a John domain.*

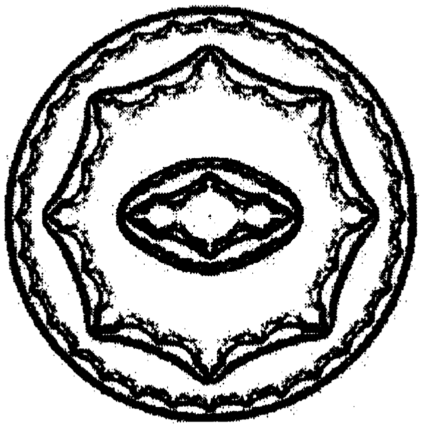
2. *Suppose that G is semi-hyperbolic. Let $\rho_0 \in \Gamma \setminus \Gamma_{\min}$ be any element and let $\beta := (\rho_0, \alpha_1, \alpha_2, \dots) \in \Gamma^{\mathbb{N}}$. Let $\gamma \in R(\Gamma, \Gamma \setminus \Gamma_{\min})$ be an element such that there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers satisfying that $\sigma^{n_k}(\gamma) \to \beta$ as $k \to \infty$. Then, $J_\gamma(f)$ is a Jordan curve but not a quasicircle. Moreover, the unbounded component $A_\gamma(f)$ of $F_\gamma(f)$ is a John domain, but the unique bounded component U_γ of $F_\gamma(f)$ is not a John domain.*

Example 2.27. Let $g_1(z) := z^2 - 1$ and $g_2 := \frac{z^2}{4}$. Let $\Gamma := \{g_1^2, g_2^2\}$. Moreover, let G be the polynomial semigroup generated by Γ. Then, it is easy to see that $G \in \mathcal{G}_{dis}$ and G is hyperbolic. Moreover, it is easy to see that $\Gamma_{\min} = \{g_1^2\}$. Since $J(g_1^2)$ is not a Jordan curve, we can apply Theorem 2.26. Setting $\alpha := (g_1^2, g_1^2, g_1^2, \dots) \in \Gamma^{\mathbb{N}}$, it follows that for any

$$\gamma \in \{\omega \in R(\Gamma, \Gamma \setminus \Gamma_{\min}) \mid \exists (n_k) \text{ with } \sigma^{n_k}(\omega) \to \alpha\},$$

$J_\gamma(f)$ is a Jordan curve but not a quasicircle, and $A_\gamma(f)$ is a John domain but the bounded component of $F_\gamma(f)$ is not a John domain. (See Figure 1: Julia set of G above. In this example, $\hat{J}_G = \{J_\gamma(f) \mid \gamma \in \Gamma^{\mathbb{N}}\}$ and if $\gamma \neq \omega$, $J_\gamma(f) \cap J_\omega(f) = \emptyset$.)

Figure 1: The Julia set of $G = \langle g_1^2, g_2^2 \rangle$.



2.2 Random dynamics of polynomials and classification of compactly generated, (semi-)hyperbolic, polynomial semigroups G in $\mathcal{G}$

In this section, we present some results on the random dynamics of polynomials. Moreover, we present some results on classification of compactly generated, (semi-) hyperbolic, polynomial semigroups G in $\mathcal{G}$. The proofs are given in Section 4.2.

Let τ be a Borel probability measure in $\text{Poly}_{\deg \geq 2}$. We consider the i.i.d. random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a polynomial map $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ according to the distribution τ. (Hence, this is a kind of Markov process on $\hat{\mathbb{C}}$.)

Notation: For a Borel probability measure τ in $\text{Poly}_{\deg \geq 2}$, we denote by Γ_τ the support of τ in $\text{Poly}_{\deg \geq 2}$. (Hence, Γ_τ is a closed set in $\text{Poly}_{\deg \geq 2}$.) Moreover, we set $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau$. This is a Borel probability measure in $\Gamma_\tau^{\mathbb{N}}$. Furthermore, we denote by G_τ the polynomial semigroup generated by Γ_τ.

Definition 2.28. Let X be a complete metric space. A subset A of X is said to be residual if A is a countable intersection of open dense subsets of X.

Theorem 2.29. (**Theorem B**) *Let Γ be a non-empty compact subset of* $\text{Poly}_{\deg \geq 2}$. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ of polynomials. Let G be the polynomial semigroup generated by Γ. Suppose $G \in \mathcal{G}_{dis}$. Then, there exists a residual subset $\mathcal{U}$ of $\Gamma^{\mathbb{N}}$ such that for each Borel probability measure τ in* $\text{Poly}_{\deg \geq 2}$ *with $\Gamma_\tau = \Gamma$, we have $\tilde{\tau}(\mathcal{U}) = 1$, and such that each $\gamma \in \mathcal{U}$ satisfies all of the following.*

1. *There exists exactly one bounded component U_γ of $F_\gamma(f)$. Furthermore, $\partial U_\gamma = J_\gamma(f)$.*

2. *Each limit function of $\{f_{\gamma,n}\}_n$ in U_γ is constant. Moreover, for each $y \in U_\gamma$, there exists a number $n \in \mathbb{N}$ such that $f_{\gamma,n}(y) \in \mathrm{int}(\hat{K}(G))$.*

3. *$\hat{J}_\gamma(f) = J_\gamma(f)$. Moreover, the map $\omega \mapsto J_\omega(f)$ defined on $\Gamma^{\mathbf{N}}$ is continuous at γ, with respect to the Hausdorff topology in the space of non-empty compact subsets of $\hat{\mathbb{C}}$.*

4. *The 2-dimensional Lebesgue measure of $\hat{J}_\gamma(f) = J_\gamma(f)$ is equal to zero.*

Next we present a result on compactly generated, semi-hyperbolic, polynomial semigroups in $\mathcal{G}$.

Theorem 2.30. (**Theorem C**) *Let Γ be a non-empty compact subset of $\mathrm{Poly}_{\mathbf{deg}\geq 2}$. Let $f : \Gamma^{\mathbf{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbf{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ of polynomials. Let G be the polynomial semigroup generated by Γ. Suppose that $G \in \mathcal{G}$ and that G is semi-hyperbolic. Then, exactly one of the following three statements 1, 2, and 3 holds.*

1. *G is hyperbolic. Moreover, there exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^{\mathbf{N}}$, $J_\gamma(f)$ is a K-quasicircle.*

2. *There exists a residual subset $\mathcal{U}$ of $\Gamma^{\mathbf{N}}$ such that for each Borel probability measure τ in $\mathrm{Poly}_{\mathbf{deg}\geq 2}$ with $\Gamma_\tau = \Gamma$, we have $\tilde{\tau}(\mathcal{U}) = 1$, and such that for each $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve but not a quasicircle, $A_\gamma(f)$ is a John domain, and the bounded component of $F_\gamma(f)$ is not a John domain. Moreover, there exists a dense subset $\mathcal{V}$ of $\Gamma^{\mathbf{N}}$ such that for each $\gamma \in \mathcal{V}$, $J_\gamma(f)$ is not a Jordan curve. Furthermore, there exist two elements $\alpha, \beta \in \Gamma^{\mathbf{N}}$ such that $J_\beta(f) < J_\alpha(f)$.*

3. *There exists a dense set $\mathcal{V}$ in $\Gamma^{\mathbf{N}}$ such that for each $\gamma \in \mathcal{V}$, $J_\gamma(f)$ is not a Jordan curve. Moreover, for each $\alpha, \beta \in \Gamma^{\mathbf{N}}$, $J_\alpha(f) \cap J_\beta(f) \neq \emptyset$. Furthermore, $J(G)$ is arcwise connected.*

Corollary 2.31. *Let Γ be a non-empty compact subset of $\mathrm{Poly}_{\mathbf{deg}\geq 2}$. Let $f : \Gamma^{\mathbf{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbf{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ of polynomials. Let G be the polynomial semigroup generated by Γ. Suppose that $G \in \mathcal{G}_{dis}$ and that G is semi-hyperbolic. Then, either statement 1 or statement 2 in Theorem 2.30 holds. In particular, for any Borel Probability measure τ in $\mathrm{Poly}_{\mathbf{deg}\geq 2}$ with $\Gamma_\tau = \Gamma$, for almost every $\gamma \in \Gamma_\tau^{\mathbf{N}}$ with respect to $\tilde{\tau}$, $J_\gamma(f)$ is a Jordan curve.*

We now classify compactly generated, hyperbolic, polynomial semigroups in $\mathcal{G}$.

Theorem 2.32. (**Theorem D**) *Let Γ be a non-empty compact subset of* $\text{Poly}_{\text{deg}\geq 2}$. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Let G be the polynomial semigroup generated by Γ. Suppose that $G \in \mathcal{G}$ and that G is hyperbolic. Then, exactly one of the following three statements 1, 2, 3 holds.*

1. *There exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is a K-quasicircle.*

2. *There exists a residual subset $\mathcal{U}$ of $\Gamma^{\mathbb{N}}$ such that for each Borel probability measure τ in* $\text{Poly}_{\text{deg}\geq 2}$ *with $\Gamma_\tau = \Gamma$, we have $\tilde{\tau}(\mathcal{U}) = 1$, and such that for each $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve but not a quasicircle, $A_\gamma(f)$ is a John domain, and the bounded component of $F_\gamma(f)$ is not a John domain. Moreover, there exists a dense subset $\mathcal{V}$ of $\Gamma^{\mathbb{N}}$ such that for each $\gamma \in \mathcal{V}$, $J_\gamma(f)$ is a quasicircle. Furthermore, there exists a dense subset $\mathcal{W}$ of $\Gamma^{\mathbb{N}}$ such that for each $\gamma \in \mathcal{W}$, there are infinitely many bounded connected components of $F_\gamma(f)$.*

3. *For each $\gamma \in \Gamma^{\mathbb{N}}$, there are infinitely many bounded connected components of $F_\gamma(f)$. Moreover, for each $\alpha, \beta \in \Gamma^{\mathbb{N}}$, $J_\alpha(f) \cap J_\beta(f) \neq \emptyset$. Furthermore, $J(G)$ is arcwise connected.*

Example 2.33. Let $h_1(z) := z^2 - 1$ and $h_2(z) := az^2$, where $a \in \mathbb{C}$ with $0 < |a| < 0.1$. Let $\Gamma := \{h_1, h_2\}$. Moreover, let $G := \langle h_1, h_2 \rangle$. Let $U := \{|z| < 0.2\}$. Then, it is easy to see that $h_2(U) \subset U$, $h_2(h_1(U)) \subset U$, and $h_1^2(U) \subset U$. Hence, $U \subset F(G)$. It follows that $P^*(G) \subset \text{int}(\hat{K}(G)) \subset F(G)$. Therefore, $G \in \mathcal{G}$ and G is hyperbolic. Since $J(h_1)$ is not a Jordan curve and $J(h_2)$ is a Jordan curve, Theorem 2.32 implies that there exists a residual subset $\mathcal{U}$ of $\Gamma^{\mathbb{N}}$ such that for each Borel probability measure τ in $\text{Poly}_{\text{deg}\geq 2}$ with $\Gamma_\tau = \Gamma$, we have $\tilde{\tau}(\mathcal{U}) = 1$, and such that for each $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve but not a quasicircle. Moreover, for each $\gamma \in \mathcal{U}$, $A_\gamma(f)$ is a John domain, but the bounded component of $F_\gamma(f)$ is not a John domain. Furthermore, by [S11], $J(G)$ is connected.

Remark 2.34. Let $h \in \text{Poly}_{\text{deg}\geq 2}$ be a polynomial. Suppose that $J(h)$ is a Jordan curve but not a quasicircle. Then, it is easy to see that there exists a parabolic fixed point of h in $\mathbb{C}$ and the bounded connected component of $F(h)$ is the immediate parabolic basin. Hence, $\langle h \rangle$ is not semi-hyperbolic. Moreover, by [CJY], $F_\infty(h)$ is not a John domain.

Thus what we see in statement 2 in Theorem 2.30 and statement 2 in Theorem 2.32, as illustrated in Example 2.27, Example 2.33, the following Section 2.3 and Proposition 2.40, is a special phenomenon which can hold in the *random* dynamics of a family of polynomials, but cannot hold in the usual iteration dynamics of a single polynomial. Namely, it can hold that for almost every $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is a Jordan curve and fails to be a quasicircle all while the basin of infinity $A_\gamma(f)$ is still a John domain. Whereas, if $J(h)$, for some polynomial h, is a Jordan curve which fails to be a quasicircle, then the basin of infinity $F_\infty(h)$ is necessarily **not** a John domain.

Pilgrim and Tan Lei ([PT]) showed that there exists a hyperbolic rational map h with *disconnected* Julia set such that "almost every" connected component of $J(h)$ is a Jordan curve but not a quasicircle.

Proposition 2.35. *Let* Γ *be a non-empty compact subset of* $\text{Poly}_{\deg \geq 2}$. *Let* $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ *be the skew product associated with the family* Γ. *Let* G *be the polynomial semigroup generated by* Γ. *Suppose that* $P^*(G)$ *is included in a connected component of* $\text{int}(\hat{K}(G))$. *Then, there exists a constant* $K \geq 1$ *such that for each* $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ *is a* K*-quasicircle.*

2.3 Construction of examples

We present a way to construct examples of semigroups G in $\mathcal{G}_{dis}$.

Proposition 2.36. *Let* G *be a polynomial semigroup generated by a compact subset* Γ *of* $\text{Poly}_{\deg \geq 2}$. *Suppose that* $G \in \mathcal{G}$ *and* $\text{int}(\hat{K}(G)) \neq \emptyset$. *Let* $b \in \text{int}(\hat{K}(G))$. *Moreover, let* $d \in \mathbb{N}$ *be any positive integer such that* $d \geq 2$, *and such that* $(d, \deg(h)) \neq (2, 2)$ *for each* $h \in \Gamma$. *Then, there exists a number* $c > 0$ *such that for each* $a \in \mathbb{C}$ *with* $0 < |a| < c$, *there exists a compact neighborhood* V *of* $g_a(z) = a(z-b)^d + b$ *in* $\text{Poly}_{\deg \geq 2}$ *satisfying that for any non-empty subset* V' *of* V, *the polynomial semigroup* $H_{\Gamma, V'}$ *generated by the family* $\Gamma \cup V'$ *belongs to* $\mathcal{G}_{dis}$, $\hat{K}(H_{\Gamma,V'}) = \hat{K}(G)$ *and* $(\Gamma \cup V')_{\min} = \Gamma_{\min}$. *Moreover, in addition to the assumption above, if* G *is semi-hyperbolic (resp. hyperbolic), then the above* $H_{\Gamma,V'}$ *is semi-hyperbolic (resp. hyperbolic).*

Remark 2.37. By Proposition 2.36, there exists a 2-generator polynomial semigroup $G = \langle h_1, h_2 \rangle$ in $\mathcal{G}_{dis}$ such that h_1 has a Siegel disk.

Definition 2.38. Let $d \in \mathbb{N}$ with $d \geq 2$. We set $\mathcal{Y}_d := \{h \in \text{Poly} \mid \deg(h) = d\}$ endowed with the relative topology from Poly.

Theorem 2.39. *Let* $m \geq 2$ *and let* $d_1, d_2, \ldots, d_m \in \mathbb{N}$ *such that* $d_j \geq 2$ *for each* $j = 1, \ldots, m$. *Let* $h_1 \in \mathcal{Y}_{d_1}$ *with* $\text{int}(K(h_1)) \neq \emptyset$ *such that* $\langle h_1 \rangle \in \mathcal{G}$. *Let* $b_2, b_3, \ldots, b_m \in \text{int}(K(h_1))$. *Then, all of the following statements hold.*

1. *Suppose that* $\langle h_1 \rangle$ *is semi-hyperbolic (resp. hyperbolic). Then, there exists a number* $c > 0$ *such that for each* $(a_2, a_3, \ldots, a_m) \in \mathbb{C}^{m-1}$ *with* $0 < |a_j| < c$ $(j = 2, \ldots, m)$, *setting* $h_j(z) = a_j(z - b_j)^{d_j} + b_j$ $(j =$

$2,\dots,m$), *the polynomial semigroup* $G = \langle h_1,\dots,h_m\rangle$ *satisfies that* $G \in \mathcal{G}$, $\hat{K}(G) = K(h_1)$ *and* G *is semi-hyperbolic (resp. hyperbolic).*

2. *Suppose that* $\langle h_1\rangle$ *is semi-hyperbolic (resp. hyperbolic). Suppose also that either (i) there exists a* $j \geq 2$ *with* $d_j \geq 3$, *or (ii)* $d_1 = 3$, $b_2 = \cdots = b_m$. *Then, there exist* $a_2, a_3, \dots, a_m > 0$ *such that setting* $h_j(z) = a_j(z-b_j)^{d_j} + b_j$ $(j = 2,\dots,m)$, *the polynomial semigroup* $G = \langle h_1, h_2, \dots, h_m\rangle$ *satisfies that* $G \in \mathcal{G}_{dis}$, $\hat{K}(G) = K(h_1)$ *and* G *is semi-hyperbolic (resp. hyperbolic).*

Proposition 2.40. *Let* h_1 *be a polynomial with* $\deg(h_1) \geq 2$ *such that* $\langle h_1\rangle$ *is semi-hyperbolic (resp. hyperbolic),* $P(\langle h_1\rangle)\setminus\{\infty\}$ *is bounded in* $\mathbb{C}$, $\operatorname{int}(K(h_1)) \neq \emptyset$, *and* $J(h_1)$ *is not a Jordan curve. Moreover, let* $d \in \mathbb{N}$ *with* $d \geq 2$ *and let* $b \in \operatorname{int}(K(h_1))$. *Then, there exists a number* $c > 0$ *such that for each* $a \in \mathbb{C}$ *with* $0 < |a| < c$, *setting* $h_2(z) := a(z-b)^d + b \in \mathcal{Y}_d$ *and setting* $\Gamma = \{h_1, h_2\}$, *statement 2 in Theorem C (resp. statement 2 in Theorem D) holds.*

Definition 2.41. Let $m \in \mathbb{N}$. We set

- $\mathcal{H}_m := \{(h_1,\dots,h_m) \in (\mathrm{Poly}_{\deg\geq 2})^m \mid \langle h_1,\dots,h_m\rangle$ is hyperbolic$\}$,
- $\mathcal{B}_m := \{(h_1,\dots,h_m) \in (\mathrm{Poly}_{\deg\geq 2})^m \mid \langle h_1,\dots,h_m\rangle \in \mathcal{G}\}$, and
- $\mathcal{D}_m := \{(h_1,\dots,h_m) \in (\mathrm{Poly}_{\deg\geq 2})^m \mid J(\langle h_1,\dots,h_m\rangle)$ is disconnected$\}$.

Moreover, let $\pi_1 : (\mathrm{Poly}_{\deg\geq 2})^m \to \mathrm{Poly}_{\deg\geq 2}$ be the projection defined by $\pi(h_1,\dots,h_m) = h_1$.

Theorem 2.42. *Under the above notation, all of the following statements hold.*

1. $\mathcal{H}_m, \mathcal{H}_m \cap \mathcal{B}_m, \mathcal{H}_m \cap \mathcal{D}_m$, *and* $\mathcal{H}_m \cap \mathcal{B}_m \cap \mathcal{D}_m$ *are open in* $(\mathrm{Poly}_{\deg\geq 2})^m$.
2. *Let* $d_1,\dots,d_m \in \mathbb{N}$ *such that* $d_j \geq 2$ *for each* $j = 1,\dots,m$. *Then,* $\pi_1 : \mathcal{H}_m \cap \mathcal{B}_m \cap (\mathcal{Y}_{d_1} \times \cdots \times \mathcal{Y}_{d_m}) \to \mathcal{H}_1 \cap \mathcal{B}_1 \cap \mathcal{Y}_{d_1}$ *is surjective.*
3. *Let* $d_1,\dots,d_m \in \mathbb{N}$ *such that* $d_j \geq 2$ *for each* $j = 1,\dots,m$ *and such that* $(d_1,\dots,d_m) \neq (2,2,\dots,2)$. *Then,* $\pi_1 : \mathcal{H}_m \cap \mathcal{B}_m \cap \mathcal{D}_m \cap (\mathcal{Y}_{d_1} \times \cdots \times \mathcal{Y}_{d_m}) \to \mathcal{H}_1 \cap \mathcal{B}_1 \cap \mathcal{Y}_{d_1}$ *is surjective.*

Remark 2.43. Combining Proposition 2.36, Theorem 2.39, and Theorem 2.42, we can construct many examples of semigroups G in $\mathcal{G}$ (or $\mathcal{G}_{dis}$) with some additional properties (semi-hyperbolicity, hyperbolicity, etc.).

3 Tools

To show the main results, we need some tools in this section.

3.1 Fundamental properties of rational semigroups

Notation: For a rational semigroup G, we set $E(G) := \{z \in \hat{\mathbb{C}} \mid \sharp(\cup_{g\in G} g^{-1}\{z\}) < \infty\}$. This is called the exceptional set of G.

Lemma 3.1 ([HM1],[GR],[S1]). *Let G be a rational semigroup.*

1. *For each $h \in G$, we have $h(F(G)) \subset F(G)$ and $h^{-1}(J(G)) \subset J(G)$. Note that we do not have that the equality holds in general.*
2. *If $G = \langle h_1, \dots , h_m \rangle$, then $J(G) = h_1^{-1}(J(G)) \cup \cdots \cup h_m^{-1}(J(G))$. More generally, if G is generated by a compact subset Γ of* Rat, *then $J(G) = \cup_{h\in\Gamma} h^{-1}(J(G))$. (We call this property of the Julia set of a compactly generated rational semigroup "backward self-similarity.")*
3. *If $\sharp(J(G)) \geq 3$, then $J(G)$ is a perfect set.*
4. *If $\sharp(J(G)) \geq 3$, then $\sharp E(G) \leq 2$.*
5. *If a point z is not in $E(G)$, then $J(G) \subset \overline{\cup_{g\in G} g^{-1}(\{z\})}$. In particular if a point z belongs to $J(G) \setminus E(G)$, then $\overline{\cup_{g\in G} g^{-1}(\{z\})} = J(G)$.*
6. *If $\sharp(J(G)) \geq 3$, then $J(G)$ is the smallest closed backward invariant set containing at least three points. Here we say that a set A is backward invariant under G if for each $g \in G$, $g^{-1}(A) \subset A$.*

Theorem 3.2 ([HM1],[GR],[S1]). *Let G be a rational semigroup. If $\sharp(J(G)) \geq 3$, then $J(G) = \overline{\{z \in \hat{\mathbb{C}} \mid \exists g \in G,\ g(z) = z,\ |g'(z)| > 1\}}$. In particular, $J(G) = \overline{\bigcup_{g\in G} J(g)}$.*

Remark 3.3. If a rational semigroup G contains an element g with $\deg(g) \geq 2$, then $\sharp(J(g)) \geq 3$, which implies that $\sharp(J(G)) \geq 3$.

3.2 Fundamental properties of fibered rational maps

Lemma 3.4. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a rational skew product over $g : X \to X$. Then, we have the following.*

1. **(Lemma 2.4 in [S1])** *For each* $x \in X$, $(f_{x,1})^{-1}(J_{g(x)}(f)) = J_x(f)$. *Furthermore, we have* $\hat{J}_x(f) \supset J_x(f)$. *Note that* **equality** $\hat{J}_x(f) = J_x(f)$ **does not hold in general.**

   If $g : X \to X$ *is a surjective and open map, then* $f^{-1}(\tilde{J}(f)) = \tilde{J}(f) = f(\tilde{J}(f))$, *and for each* $x \in X$, $(f_{x,1})^{-1}(\hat{J}_{g(x)}(f)) = \hat{J}_x(f)$.

2. *(*[J], [S1]*) If* $d(x) \geq 2$ *for each* $x \in X$, *then for each* $x \in X$, $J_x(f)$ *is a non-empty perfect set with* $\sharp(J_x(f)) \geq 3$. *Furthermore, the map* $x \mapsto J_x(f)$ *is lower semicontinuous; i.e., for any point* $(x, y) \in X \times \hat{\mathbb{C}}$ *with* $y \in J_x(f)$ *and any sequence* $\{x^n\}_{n \in \mathbb{N}}$ *in* X *with* $x^n \to x$, *there exists a sequence* $\{y^n\}_{n \in \mathbb{N}}$ *in* $\hat{\mathbb{C}}$ *with* $y^n \in J_{x^n}(f)$ *for each* $n \in \mathbb{N}$ *such that* $y^n \to y$. *However,* $x \mapsto J_x(f)$ *is* **NOT** *continuous with respect to the Hausdorff topology in general.*

3. *If* $d(x) \geq 2$ *for each* $x \in X$, *then* $\inf_{x \in X} diam_S J_x(f) > 0$, *where* $diam_S$ *denotes the diameter with respect to the spherical distance.*

4. *If* $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ *is a polynomial skew product and* $d(x) \geq 2$ *for each* $x \in X$, *then we have that there exists a ball* B *around* ∞ *such that for each* $x \in X$, $B \subset A_x(f) \subset F_x(f)$, *and that for each* $x \in X$, $J_x(f) = \partial(K_x(f)) = \partial(A_x(f))$. *Moreover, for each* $x \in X$, $A_x(f)$ *is connected.*

5. *If* $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ *is a polynomial skew product and* $d(x) \geq 2$ *for each* $x \in X$, *and if* $\omega \in X$ *is a point such that* $\text{int}(K_\omega(f))$ *is a non-empty set, then* $\overline{\text{int}(K_\omega(f))} = K_\omega(f)$ *and* $\partial(\text{int}(K_\omega(f))) = J_\omega(f)$.

Lemma 3.5. *Let* $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ *be a skew product associated with a compact subset* Γ *of* Rat. *Let* G *be a rational semigroup generated by* Γ. *Suppose that* $\sharp(J(G)) \geq 3$. *Then, we have the following.*

1. $\pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) = J(G)$.

2. *For each* $\gamma = (\gamma_1, \gamma_2, \ldots,) \in \Gamma^{\mathbb{N}}$, $\hat{J}_\gamma(f) = \cap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G))$.

Lemma 3.6. *Let* $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ *be a polynomial skew product over* $g : X \to X$ *such that for each* $x \in X$, $d(x) \geq 2$. *Then, the following are equivalent.*

1. $\pi_{\hat{\mathbb{C}}}(P(f)) \setminus \{\infty\}$ *is bounded in* $\mathbb{C}$.

2. *For each* $x \in X$, $J_x(f)$ *is connected.*

3. *For each* $x \in X$, $\hat{J}_x(f)$ *is connected.*

Corollary 3.7. *Let $G = \langle h_1, h_2 \rangle \in \mathcal{G}$. Then, $h_1^{-1}(J(h_2))$ is connected.*

Lemma 3.8. *Let G be a polynomial semigroup generated by a compact subset Γ of $\text{Poly}_{\deg \geq 2}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Suppose that $G \in \mathcal{G}$. Then for each $\gamma = (\gamma_1, \gamma_2, \dots,) \in \Gamma^{\mathbb{N}}$, the sets $J_\gamma(f)$, $\hat{J}_\gamma(f)$, and $\cap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G))$ are connected.*

Lemma 3.9. *Under the same assumption as that in Lemma 3.8, let $\gamma, \rho \in \Gamma^{\mathbb{N}}$ be two elements with $J_\gamma(f) \cap J_\rho(f) = \emptyset$. Then, either $J_\gamma(f) < J_\rho(f)$ or $J_\rho(f) < J_\gamma(f)$.*

Definition 3.10. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$. Let $p \in \mathbb{C}$ and $\epsilon > 0$. We set
$\mathcal{F}_{f,p,\epsilon} := \{\alpha : D(p, \epsilon) \to \mathbb{C} \mid \alpha$ is a well-defined inverse branch of $(f_{x,n})^{-1}, x \in X, n \in \mathbb{N}\}$.

Lemma 3.11. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Let $R > 0, \epsilon > 0$, and $\mathcal{F} := \{\alpha \circ \beta : D(0,1) \to \mathbb{C} \mid \beta : D(0,1) \cong D(p, \epsilon),\ \alpha : D(p, \epsilon) \to \mathbb{C},\ \alpha \in \mathcal{F}_{f,p,\epsilon},\ p \in D(0, R)\}$. Then, $\mathcal{F}$ is normal on $D(0, 1)$.*

The following three results are the keys to prove the main results. In fact, these are non-trivial and difficult to show.

Lemma 3.12 ([S1], [S11]). *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $\omega \in X$, $\deg(f_\omega) \geq 2$. Let $x \in X$ be a point and $y_0 \in F_x(f)$ a point. Suppose that there exists a strictly increasing sequence $\{n_j\}_{j \in \mathbb{N}}$ of positive integers such that the sequence $\{f_{x,n_j}\}_{j \in \mathbb{N}}$ converges to a non-constant map around y_0, and such that $\lim_{j \to \infty} f^{n_j}(x, y_0)$ exists. We set $(x_\infty, y_\infty) := \lim_{j \to \infty} f^{n_j}(x, y_0)$. Then, there exists a non-empty bounded open set V in $\mathbb{C}$, a point x_∞ in X, and a number $k \in \mathbb{N}$ such that $\{x_\infty\} \times \partial V \subset \tilde{J}(f) \cap UH(f) \subset \tilde{J}(f) \cap P(f)$, and such that for each j with $j \geq k$, $f_{x,n_j}(y_0) \in V$.*

Theorem 3.13 ([S1], Theorem 2.14). *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a semi-hyperbolic polynomial skew product over $g : X \to X$. Suppose that for each $x \in X$, $d(x) \geq 2$. Then, the map $x \mapsto J_x(f)$ defined for all $x \in X$ is continuous, with respect to the Hausdorff topology in the space of non-empty compact subsets of $\mathbb{C}$.*

Theorem 3.14 ([S4], Theorem 1.12). *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a semi-hyperbolic polynomial skew product over $g : X \to X$. Suppose that for each $x \in X$, $d(x) \geq 2$, and that $\pi_{\hat{\mathbb{C}}}(P(f)) \cap \mathbb{C}$ is bounded in $\mathbb{C}$. Then, for each $x \in X$, $A_x(f)$ is a John domain and $J_x(f)$ is locally connected.*

Using the above result, we can show the following.

Proposition 3.15. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a semi-hyperbolic polynomial skew product over $g : X \to X$. Suppose that for each $x \in X$, $d(x) \geq 2$, and that $\pi_{\hat{\mathbb{C}}}(P(f)) \cap \mathbb{C}$ is bounded in $\mathbb{C}$. Let $\omega \in X$ be a point. If* $\mathrm{int}(K_\omega(f))$ *is a non-empty connected set, then $J_\omega(f)$ is a Jordan curve.*

Proof. By ([S4], Theorem 1.12) and Lemma 3.6, we get that the unbounded component $A_\omega(f)$ of $F_\omega(f)$ is a John domain and $J_\omega(f) = \partial(A_\omega(f))$ (cf. Lemma 3.4) is locally connected. Moreover, by Lemma 3.4-5, we have $\partial(\mathrm{int}(K_\omega(f))) = J_\omega(f)$. Hence, we see that $\hat{\mathbb{C}} \setminus J_\omega(f)$ has exactly two connected components $A_\gamma(f)$ and $\mathrm{int}(K_\omega(f))$, and that $J_\omega(f)$ is locally connected. By Lemma 5.1 in [PT], it follows that $J_\gamma(f)$ is a Jordan curve. Thus, we have proved Proposition 3.15. □

4 Proofs of the main results

In this section, we demonstrate the main results.

4.1 Proofs of results in 2.1

In this section, we demonstrate Theorem 2.26. We need the following notations and lemmas.

Definition 4.1. Let h be a polynomial with $\deg(h) \geq 2$. Suppose that $J(h)$ is connected. Let ψ be a biholomorphic map $\hat{\mathbb{C}} \setminus \overline{D(0,1)} \to F_\infty(h)$ with $\psi(\infty) = \infty$ such that $\psi^{-1} \circ h \circ \psi(z) = z^{\deg(h)}$, for each $z \in \hat{\mathbb{C}} \setminus \overline{D(0,1)}$. (For the existence of the biholomorphic map ψ, see [M], Theorem 9.5.) For each $\theta \in \partial D(0,1)$, we set $T(\theta) := \psi(\{r\theta \mid 1 < r \leq \infty\})$. This is called the external ray (for $K(h)$) with angle θ.

Lemma 4.2. *Let h be a polynomial with* $\deg(h) \geq 2$. *Suppose that $J(h)$ is connected and locally connected and $J(h)$ is not a Jordan curve. Moreover, suppose that there exists an attracting periodic point of h in $K(h)$. Then, for any $\epsilon > 0$, there exist a point $p \in J(h)$ and elements $\theta_1, \theta_2 \in \partial D(0,1)$ with $\theta_1 \neq \theta_2$, such that all of the following hold.*

1. *For each $i = 1, 2$, the external ray $T(\theta_i)$ lands at the point p.*
2. *Let V_1 and V_2 be the two connected components of $\hat{\mathbb{C}} \setminus (T(\theta_1) \cup T(\theta_2) \cup \{p\})$. Then, for each $i = 1, 2$, $V_i \cap J(h) \neq \emptyset$. Moreover, there exists an i such that diam $(V_i \cap K(h)) \leq \epsilon$.*

Lemma 4.3. *Let G be a polynomial semigroup generated by a compact subset Γ of $\text{Poly}_{\deg\geq 2}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Suppose $G \in \mathcal{G}_{dis}$. Let $m \in \mathbb{N}$ and suppose that there exists an element $(h_1, \dots, h_m) \in \Gamma^m$ such that setting $h = h_m \circ \cdots \circ h_1$, $J(h)$ is connected and locally connected, and $J(h)$ is not a Jordan curve. Moreover, suppose that there exists an attracting periodic point of h in $K(h)$. Let $\alpha = (\alpha_1, \alpha_2, \dots) \in \Gamma^{\mathbb{N}}$ be the element such that for each $k, l \in \mathbb{N} \cup \{0\}$ with $1 \leq l \leq m$, $\alpha_{km+l} = h_l$. Let $\rho_0 \in \Gamma \setminus \Gamma_{\min}$ be an element and let $\beta = (\rho_0, \alpha_1, \alpha_2, \dots) \in \Gamma^{\mathbb{N}}$. Moreover, let $\psi_\beta : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \to A_\beta(f)$ be a biholomorphic map with $\psi_\beta(\infty) = \infty$. Furthermore, for each $\theta \in \partial D(0,1)$, let $T_\beta(\theta) = \psi_\beta(\{r\theta \mid 1 < r \leq \infty\})$. Then, for any $\epsilon > 0$, there exist a point $p \in J_\beta(f)$ and elements $\theta_1, \theta_2 \in \partial D(0,1)$ with $\theta_1 \neq \theta_2$, such that all of the following statements 1 and 2 hold.*

1. *For each $i = 1, 2$, $T_\beta(\theta_i)$ lands at p.*

2. *Let V_1 and V_2 be the two connected components of $\hat{\mathbb{C}} \setminus (T_\beta(\theta_1) \cup T_\beta(\theta_2) \cup \{p\})$. Then, for each $i = 1, 2$, $V_i \cap J_\beta(f) \neq \emptyset$. Moreover, there exists an i such that diam $(V_i \cap K_\beta(f)) \leq \epsilon$ and such that $V_i \cap J_\beta(f) \subset \rho_0^{-1}(J(G)) \subset \mathbb{C} \setminus P(G)$.*

Lemma 4.4. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Let $\gamma \in X$ be a point. Suppose that $J_\gamma(f)$ is a Jordan curve. Then, for each $n \in \mathbb{N}$, $J_{g^n(\gamma)}(f)$ is a Jordan curve. Moreover, for each $n \in \mathbb{N}$, there exists no critical value of $f_{\gamma,n}$ in $J_{g^n(\gamma)}(f)$.*

Lemma 4.5. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Let $\mu > 0$ be a number. Then, there exists a number $\delta > 0$ such that the following statement holds.*

- *Let $\omega \in X$ be any point and $p \in J_\omega(f)$ any point with $\min\{|p - b| \mid b \in \pi_{\hat{\mathbb{C}}}(P(f)) \cap \mathbb{C}\} > \mu$. Suppose that $J_\omega(f)$ is connected. Let $\psi : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \to A_\omega(f)$ be a biholomorphic map with $\psi(\infty) = \infty$. For each $\theta \in \partial D(0,1)$, let $T(\theta) = \psi(\{r\theta \mid 1 < r \leq \infty\})$. Suppose that there exist two elements $\theta_1, \theta_2 \in \partial D(0,1)$ with $\theta_1 \neq \theta_2$ such that for each $i = 1, 2,$, $T(\theta_i)$ lands at p. Moreover, suppose that a connected component V of $\hat{\mathbb{C}} \setminus (T(\theta_1) \cup T(\theta_2) \cup \{p\})$ satisfies that diam $(V \cap K_\omega(f)) \leq \delta$. Furthermore, let $\gamma \in X$ be any point and suppose that there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that $g^{n_k}(\gamma) \to \omega$ as $k \to \infty$. Then, $J_\gamma(f)$ is not a quasicircle.*

Proof. Let $\mu > 0$. Let $R > 0$ with $\pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) \subset D(0, R)$. Combining Lemma 3.11 and Lemma 3.4-3, we see that there exists a $\delta_0 > 0$ with $0 < \delta_0 < \frac{1}{20}\min\{\inf_{x\in X} \text{diam } J_x(f), \mu\}$ such that the following statement holds:

- Let $x \in X$ be any point and $n \in \mathbb{N}$ any element. Let $p \in D(0, R)$ be any point with $\min\{|p - b| \mid b \in \pi_{\hat{\mathbb{C}}}(P(f)) \cap \mathbb{C}\} > \mu$. Let $\phi : D(p, \mu) \to \mathbb{C}$ be any well-defined inverse branch of $(f_{x,n})^{-1}$ on $D(p, \mu)$. Let A be any subset of $D(p, \frac{\mu}{2})$ with diam $A \leq \delta_0$. Then,

$$\text{diam } \phi(A) \leq \frac{1}{10} \inf_{x\in X} \text{diam } J_x(f). \tag{1}$$

We set $\delta := \frac{1}{10}\delta_0$. Let $\omega \in X$ and $p \in J_\omega(f)$ with $\min\{|p-b| \mid b \in \pi_{\hat{\mathbb{C}}}(P(f)) \cap \mathbb{C}\} > \mu$. Suppose that $J_\omega(f)$ is connected and let $\psi : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \to A_\omega(f)$ be a biholomorphic map with $\psi(\infty) = \infty$. Setting $T(\theta) := \psi(\{r\theta \mid 1 < r \leq \infty\})$ for each $\theta \in \partial D(0, 1)$, suppose that there exist two elements $\theta_1, \theta_2 \in \partial D(0, 1)$ with $\theta_1 \neq \theta_2$ such that for each $i = 1, 2$, $T(\theta_i)$ lands at p. Moreover, suppose that a connected component V of $\hat{\mathbb{C}} \setminus (T(\theta_1) \cup T(\theta_2) \cup \{p\})$ satisfies that

$$\text{diam}(V \cap K_\omega(f)) \leq \delta. \tag{2}$$

Furthermore, let $\gamma \in X$ and suppose that there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that $g^{n_k}(\gamma) \to \omega$ as $k \to \infty$. We now suppose that $J_\gamma(f)$ is a quasicircle, and we will deduce a contradiction. Since $g^{n_k}(\gamma) \to \omega$ as $k \to \infty$, we obtain

$$\max\{d_e(b, K_\omega(f)) \mid b \in J_{g^{n_k}(\gamma)}(f)\} \to 0 \text{ as } k \to \infty. \tag{3}$$

We take a point $a \in V \cap J_\omega(f)$ and fix it. By Lemma 3.4-2, there exists a number $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$, there exists a point y_k satisfying that

$$y_k \in J_{g^{n_k}(\gamma)}(f) \cap D(a, \frac{|a-p|}{10k}). \tag{4}$$

Let V' be the connected component of $\hat{\mathbb{C}} \setminus (T(\theta_1) \cup T(\theta_2) \cup \{p\})$ with $V' \neq V$. Then, by Lemma 17.5 in [M],

$$V' \cap J_\omega(f) \neq \emptyset. \tag{5}$$

Combining (5) and Lemma 3.4-2, we see that there exists a $k_1(\geq k_0) \in \mathbb{N}$ such that for each $k \geq k_1$,

$$V' \cap J_{g^{n_k}(\gamma)}(f) \neq \emptyset. \tag{6}$$

By assumption and Lemma 4.4, for each $k \geq k_1$, $J_{g^{n_k}(\gamma)}(f)$ is a Jordan curve. Combining it with (4) and (6), there exists a $k_2(\geq k_1) \in \mathbb{N}$ satisfying that for each $k \geq k_2$, there exists a smallest closed subarc ξ_k of $J_{g^{n_k}(\gamma)}(f) \cong S^1$ such that $y_k \in \xi_k$, $\xi_k \subset \overline{V}$, $\sharp(\xi_k \cap (T(\theta_1) \cup T(\theta_2) \cup \{p\})) = 2$, and such that $\xi_k \neq J_{g^{n_k}(\gamma)}(f)$. For each $k \geq k_2$, let $y_{k,1}$ and $y_{k,2}$ be the two points such that $\{y_{k,1}, y_{k,2}\} = \xi_k \cap (T(\theta_1) \cup T(\theta_2) \cup \{p\})$. Then, (3) implies that

$$y_{k,i} \to p \text{ as } k \to \infty, \text{ for each } i = 1, 2. \tag{7}$$

Combining that $\xi_k \subset V \cup \{y_{k,1}, y_{k,2}\}$, (3), and (2), we get that there exists a $k_3(\geq k_2) \in \mathbb{N}$ such that for each $k \geq k_3$,

$$\text{diam } \xi_k \leq \frac{\delta_0}{2}. \tag{8}$$

Moreover, combining (4) and (7), we see that there exists a constant $C > 0$ such that

$$\text{diam } \xi_k > C. \tag{9}$$

Combining (7), (8), and (9), we may assume that there exists a constant $C > 0$ such that for each $k \in \mathbb{N}$,

$$C < \text{ diam } \xi_k \leq \frac{\delta_0}{2} \text{ and } \xi_k \subset D(p, \delta_0) \subset \mathbb{C} \setminus \pi_{\hat{\mathbb{C}}}(P(f)). \tag{10}$$

By Lemma 4.4, each connected component v of $(f_{\gamma,n_k})^{-1}(\xi_k)$ is a subarc of $J_\gamma(f) \cong S^1$ and $f_{\gamma,n_k} : v \to \xi_k$ is a homeomorphism. For each $k \in \mathbb{N}$, let λ_k be a connected component of $(f_{\gamma,n_k})^{-1}(\xi_k)$, and let $z_{k,1}, z_{k,2} \in \lambda_k$ be the two endpoints of λ_k such that $f_{\gamma,n_k}(z_{k,1}) = y_{k,1}$ and $f_{\gamma,n_k}(z_{k,2}) = y_{k,2}$. Then, combining (1) and (10), we obtain

$$\text{diam}\lambda_k < \text{ diam } (J_\gamma(f) \setminus \lambda_k), \text{ for each } k \in \mathbb{N}. \tag{11}$$

Moreover, combining (7), (10), and Koebe distortion theorem, it follows that

$$\frac{\text{diam } \lambda_k}{|z_{k,1} - z_{k,2}|} \to \infty \text{ as } k \to \infty. \tag{12}$$

Combining (11) and (12), we conclude that $J_\gamma(f)$ can not be a quasicircle, since we have the following well-known fact:
Fact([LV], Section 2): Let ξ be a Jordan curve in $\mathbb{C}$. Then, ξ is a quasicircle if and only if there exists a constant $K > 0$ such that for each $z_1, z_2 \in \xi$, we have $\frac{\text{diam } \lambda(z_1,z_2)}{|z_1 - z_2|} \leq K$, where $\lambda(z_1, z_2)$ denotes the smallest closed subarc of ξ such that $z_1, z_2 \in \lambda(z_1, z_2)$ and such that $\text{diam } \lambda(z_1, z_2) < \text{diam } (\xi \setminus \lambda(z_1, z_2))$.

Hence, we have proved Lemma 4.5. $\square$

In [S11], the following fact was shown.

Theorem 4.6 **([S11]).** *Let G be a polynomial semigroup generated by a compact subset Γ of* $\text{Poly}_{\deg\geq 2}$. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Suppose that $G \in \mathcal{G}_{dis}$ and that G is semi-hyperbolic. Let $\gamma \in R(\Gamma, \Gamma \setminus \Gamma_{\min})$ be any element. Then, $\hat{J}_\gamma(f) = J_\gamma(f)$ and $J_\gamma(f)$ is a Jordan curve. Moreover, for each point $y_0 \in \text{int}(K_\gamma(f))$, there exists an $n \in \mathbb{N}$ such that $f_{\gamma,n}(y_0) \in \text{int}(\hat{K}(G))$.*

We now demonstrate Theorem 2.26-1.

Proof of Theorem 2.26-1: Let γ be as in Theorem 2.26-1. Then, by Theorem 4.6, $J_\gamma(f)$ is a Jordan curve. Moreover, setting $h = h_m \circ \cdots \circ h_1$, since h is hyperbolic and $J(h)$ is not a quasicircle, $J(h)$ is not a Jordan curve. Combining it with Lemma 4.5 and Lemma 4.2, it follows that $J_\gamma(f)$ is not a quasicircle. Moreover, $A_\gamma(f)$ is a John domain(cf. [S4], Theorem 1.12). Combining the above arguments with ([NV], Theorem 9.3), we conclude that the bounded component U_γ of $F_\gamma(f)$ is not a John domain.

Thus, we have proved Theorem 2.26-1.

□

We now demonstrate Theorem 2.26-2.

Proof of Theorem 2.26-2: Let ρ_0, β, γ be as in Theorem 2.26-2. By Theorem 4.6, $J_\gamma(f)$ is a Jordan curve. By Theorem 2.8-3, we have $\emptyset \neq \text{int}(\hat{K}(G)) \subset \text{int}(K(h))$. Moreover, h is semi-hyperbolic. Hence, h has an attracting periodic point in $K(h)$. Combining Lemma 4.5 and Lemma 4.3, we get that $J_\gamma(f)$ is not a quasicircle. Combining it with the argument in the proof of Theorem 2.26-1, it follows that $A_\gamma(f)$ is a John domain, but the bounded component U_γ of $F_\gamma(f)$ is not a John domain.

Thus, we have proved Theorem 2.26-2.

□

4.2 Proofs of results in 2.2

In this subsection, we will demonstrate results in Section 2.2. To demonstrate Theorem 2.30, we need several lemmas.

Lemma 4.7. *Let Γ be a compact set in* $\text{Poly}_{\deg\geq 2}$. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Let G be the polynomial semigroup generated by Γ. Suppose that $G \in \mathcal{G}$ and that G is semi-hyperbolic. Moreover, suppose that there exist two elements $\alpha, \beta \in \Gamma^{\mathbb{N}}$ such that $J_\beta(f) < J_\alpha(f)$. Let $\gamma \in \Gamma^{\mathbb{N}}$ and suppose that there exists a sequence (n_k) in $\mathbb{N}$ such that $\sigma^{n_k}(\gamma) \to \alpha$ as $k \to \infty$. Then, $J_\gamma(f)$ is a Jordan curve.*

Lemma 4.8. *Let Γ be a non-empty compact subset of* $\text{Poly}_{\deg\geq 2}$. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ of polynomials. Let G be the polynomial semigroup generated by Γ. Let $\alpha, \rho \in \Gamma^{\mathbb{N}}$ be two elements. Suppose that $G \in \mathcal{G}$, that G is semi-hyperbolic, that α is a periodic point of $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$, that $J_\alpha(f)$ is a quasicircle, and that $J_\rho(f)$ is not a Jordan curve. Then, for each $\epsilon > 0$, there exist $n \in \mathbb{N}$ and two elements $\theta_1, \theta_2 \in \partial D(0,1)$ with $\theta_1 \neq \theta_2$ satisfying all of the following.*

1. *Let $\omega = (\alpha_1, \dots, \alpha_n, \rho_1, \rho_2, \dots) \in \Gamma^{\mathbb{N}}$ and let $\psi : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \cong A_\omega(f)$ be a biholomorphic map with $\psi(\infty) = \infty$. Moreover, for each $i = 1, 2$, let $T(\theta_i) := \psi(\{r\theta_i \mid 1 < r \leq \infty\})$. Then, there exists a point $p \in J_\omega(f)$ such that for each $i = 1, 2$, $T(\theta_i)$ lands at p.*
2. *Let V_1 and V_2 be the two connected components of $\hat{\mathbb{C}} \setminus (T(\theta_1) \cup T(\theta_2) \cup \{p\})$. Then, for each $i = 1, 2$, $V_i \cap J_\omega(f) \neq \emptyset$. Moreover, there exists an $i \in \{1, 2\}$ such that* $\text{diam}\,(V_i \cap K_\omega(f)) \leq \epsilon$, *and such that $V_i \cap J_\omega(f) \subset D(J_\alpha(f), \epsilon)$.*

Lemma 4.9. *Let Γ be a non-empty compact subset of* $\text{Poly}_{\deg\geq 2}$. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ of polynomials. Let G be the polynomial semigroup generated by Γ. Let $\alpha, \beta, \rho \in \Gamma^{\mathbb{N}}$ be three elements. Suppose that $G \in \mathcal{G}$, that G is semi-hyperbolic, that α is a periodic point of $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$, that $J_\beta(f) < J_\alpha(f)$, and that $J_\rho(f)$ is not a Jordan curve. Then, there exists an $n \in \mathbb{N}$ such that setting $\omega := (\alpha_1, \dots, \alpha_n, \rho_1, \rho_2, \dots) \in \Gamma^{\mathbb{N}}$ and $\mathcal{U} := \{\gamma \in \Gamma^{\mathbb{N}} \mid \exists\{m_j\}_{j\in\mathbb{N}}, \exists\{n_k\}_{k\in\mathbb{N}}, \sigma^{m_j}(\gamma) \to \alpha, \sigma^{n_k}(\gamma) \to \omega\}$, we have that for each $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve but not a quasicircle, $A_\gamma(f)$ is a John domain, and the bounded component U_γ of $F_\gamma(f)$ is not a John domain.*

In [S11], the following result was shown.

Theorem 4.10 ([S11]). *(***Uniform fiberwise quasiconformal surgery***) Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Suppose that f is hyperbolic and that $\pi_{\hat{\mathbb{C}}}(P(f)) \setminus \{\infty\}$ is bounded in $\mathbb{C}$. Moreover, suppose that for each $x \in X$,* $\text{int}(K_x(f))$ *is connected. Then, there exists a constant K such that for each $x \in X$, $J_x(f)$ is a K-quasicircle.*

We now demonstrate Theorem 2.30.

Proof of Theorem 2.30: We suppose the assumption of Theorem 2.30. We will consider several cases. First, we show the following claim.

Claim 1: If $J_\gamma(f)$ is a Jordan curve for each $\gamma \in \Gamma^{\mathbb{N}}$, then statement 1 in Theorem 2.30 holds.

To show this claim, Lemma 4.4 implies that for each $\gamma \in X$, any critical point $v \in \pi^{-1}\{\gamma\}$ of $f_\gamma : \pi^{-1}\{\gamma\} \to \pi^{-1}\{\sigma(\gamma)\}$ (under the canonical identification $\pi^{-1}\{\gamma\} \cong \pi^{-1}\{\sigma(\gamma)\} \cong \hat{\mathbb{C}}$) belongs to $F^\gamma(f)$. Moreover, by ([S1], Theorem 2.14-(2)), $\tilde{J}(f) = \cup_{\gamma \in \Gamma^{\mathbb{N}}} J^\gamma(f)$. Hence, it follows that $C(f) \subset \tilde{F}(f)$. Therefore, $C(f)$ is a compact subset of $\tilde{F}(f)$. Since f is semi-hyperbolic, ([S1], Theorem 2.14-(5)) implies that $P(f) = \overline{\bigcup_{n\in\mathbb{N}} f^n(C(f))} \subset \tilde{F}(f)$. Hence, $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ is hyperbolic. Combining it with Remark 2.21, we conclude that G is hyperbolic. Moreover, Theorem 4.10 implies that there exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is a K-quasicircle. Hence, we have proved Claim 1.

Next, we will show the following claim.

Claim 2: If $J_\alpha(f) \cap J_\beta(f) \neq \emptyset$ for each $(\alpha, \beta) \in \Gamma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$, then $J(G)$ is arcwise connected.

To show this claim, since G is semi-hyperbolic, combining ([S4],Theorem 1.12), Lemma 3.6, and ([NV], page 26), we get that for each $\gamma \in \Gamma^{\mathbb{N}}$, $A_\gamma(f)$ is a John domain and $J_\gamma(f)$ is locally connected. In particular, for each $\gamma \in \Gamma^{\mathbb{N}}$,

$$J_\gamma(f) \text{ is arcwise connected.} \tag{13}$$

Moreover, by Theorem 2.14(2) in [S1], we have

$$\tilde{J}(f) = \cup_{\gamma \in \Gamma^{\mathbb{N}}} J^\gamma(f). \tag{14}$$

Combining (13), (14) and Lemma 3.5-1, we conclude that $J(G)$ is arcwise connected. Hence, we have proved Claim 2.

Next, we will show the following claim.

Claim 3: If $J_\alpha(f) \cap J_\beta(f) \neq \emptyset$ for each $(\alpha, \beta) \in \Gamma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$, and if there exists an element $\rho \in \Gamma^{\mathbb{N}}$ such that $J_\rho(f)$ is not a Jordan curve, then statement 3 in Theorem 2.30 holds.

To show this claim, let $\mathcal{V} := \cup_{n \in \mathbb{N}} (\sigma^n)^{-1}(\{\rho\})$. Then, $\mathcal{V}$ is a dense subset of $\Gamma^{\mathbb{N}}$. By Lemma 4.4, it follows that for each $\gamma \in \mathcal{V}$, $J_\gamma(f)$ is not a Jordan curve. Combining this result with Claim 2, we conclude that statement 3 in Theorem 2.30 holds. Hence, we have proved Claim 3.

We now show the following claim.

Claim 4: If there exist two elements $\alpha, \beta \in \Gamma^{\mathbb{N}}$ such that $J_\alpha(f) \cap J_\beta(f) = \emptyset$, and if there exists an element $\rho \in \Gamma^{\mathbb{N}}$ such that $J_\rho(f)$ is not a Jordan curve, then statement 2 in Theorem 2.30 holds.

To show this claim, using Lemma 3.9, We may assume that $J_\beta(f) < J_\alpha(f)$. Combining this, Lemma 3.9, ([S1], Theorem 2.14-(4)), and that the set of all periodic points of σ in $\Gamma^{\mathbb{N}}$ is dense in $\Gamma^{\mathbb{N}}$, we may assume further that α is a periodic point of σ. Applying Lemma 4.9 to $(\Gamma, \alpha, \beta, \rho)$ above, let $n \in \mathbb{N}$ be the element in the statement of Lemma 4.9, and we set $\omega =$

$(\alpha_1, \dots, \alpha_n, \rho_1, \rho_2, \dots) \in \Gamma^{\mathbb{N}}$ and $\mathcal{U} := \{\gamma \in \Gamma^{\mathbb{N}} \mid \exists (m_j), \exists (n_k), \sigma^{m_j}(\gamma) \to \alpha, \sigma^{n_k}(\gamma) \to \omega\}$. Then, by the statement of Lemma 4.9, we have that for each $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve but not a quasicircle, $A_\gamma(f)$ is a John domain, and the bounded component U_γ of $F_\gamma(f)$ is not a John domain. Moreover, $\mathcal{U}$ is residual in $\Gamma^{\mathbb{N}}$, and for any Borel probability measure τ in $\text{Poly}_{\text{deg}\geq 2}$ with $\Gamma_\tau = \Gamma$, we have $\tilde{\tau}(\mathcal{U}) = 1$. Furthermore, let $\mathcal{V} := \cup_{n \in \mathbb{N}} (\sigma^n)^{-1}(\{\rho\})$. Then, $\mathcal{V}$ is a dense subset of $\Gamma^{\mathbb{N}}$, and the argument in the proof of Claim 3 implies that for each $\gamma \in \mathcal{V}$, $J_\gamma(f)$ is not a Jordan curve. Hence, we have proved Claim 4.

Combining Claims 1,2,3 and 4, Theorem 2.30 follows.

$\square$

To demonstrate Theorem 2.32, we need several lemmas.

Notation: For a subset A of $\hat{\mathbb{C}}$, we denote by $\mathcal{C}(A)$ the set of all connected components of A.

Lemma 4.11. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Let $\alpha \in X$ be a point. Suppose that* $2 \leq \sharp\,(\mathcal{C}(\text{int}(K_\alpha(f)))) < \infty$. *Then,* $\sharp\,(\mathcal{C}(\text{int}(K_{g(\alpha)}(f)))) < \sharp\,(\mathcal{C}(\text{int}(K_\alpha(f))))$. *In particular, there exists an $n \in \mathbb{N}$ such that* $\text{int}(K_{g^n(\alpha)}(f))$ *is a non-empty connected set.*

Lemma 4.12. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Let $\omega \in X$ be a point. Suppose that f is hyperbolic, that $\pi_{\hat{\mathbb{C}}}(P(f)) \cap \mathbb{C}$ is bounded in $\mathbb{C}$, and that* $\text{int}(K_\omega(f))$ *is not connected. Then, there exist infinitely many connected components of* $\text{int}(K_\omega(f))$.

Lemma 4.13. *Let Γ be a non-empty compact subset of* $\text{Poly}_{\text{deg}\geq 2}$. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the polynomial skew product associated with the family Γ. Let G be the polynomial semigroup generated by Γ. Let $\alpha \in \Gamma^{\mathbb{N}}$ be an element. Suppose that $G \in \mathcal{G}$, that G is hyperbolic, and that* $\text{int}(K_\alpha(f)))$ *is connected. Then, there exists a neighborhood $\mathcal{U}_0$ of α in $\Gamma^{\mathbb{N}}$ satisfying the following.*

- *Let $\gamma \in \Gamma^{\mathbb{N}}$ and suppose that there exists a sequence $\{m_j\}_{j \in \mathbb{N}} \subset \mathbb{N}, m_j \to \infty$ such that for each $j \in \mathbb{N}$, $\sigma^{m_j}(\gamma) \in \mathcal{U}_0$. Then, $J_\gamma(f)$ is a Jordan curve.*

We now demonstrate Theorem 2.32.

Proof of Theorem 2.32: We suppose the assumption of Theorem 2.32. We consider the following three cases.

Case 1: For each $\gamma \in \Gamma^{\mathbb{N}}$, $\text{int}(K_\gamma(f))$ is connected.

Case 2: For each $\gamma \in \Gamma^{\mathbb{N}}$, $\text{int}(K_\gamma(f))$ is disconnected.

Case 3: There exist two elements $\alpha \in \Gamma^{\mathbb{N}}$ and $\beta \in \Gamma^{\mathbb{N}}$ such that $\text{int}(K_\alpha(f))$ is connected and such that $\text{int}(K_\beta(f))$ is disconnected.

Suppose that we have Case 1. Then, by Theorem 4.10, there exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is a K-quasicircle.

Suppose that we have Case 2. Then, by Lemma 4.12, we get that for each $\gamma \in \Gamma^{\mathbb{N}}$, there exist infinitely many connected components of $\text{int}(K_\gamma(f))$. Moreover, by Theorem 2.30, we see that statement 3 in Theorem 2.30 holds. Hence, statement 3 in Theorem 2.32 holds.

Suppose that we have Case 3. By Lemma 4.12, there exist infinitely many connected components of $\text{int}(K_\beta(f))$. Let $\mathcal{W} := \cup_{n \in \mathbb{N}}(\sigma^n)^{-1}(\{\beta\})$. Then, for each $\gamma \in \mathcal{W}$, there exist infinitely many connected components of $\text{int}(K_\gamma(f))$. Moreover, $\mathcal{W}$ is dense in $\Gamma^{\mathbb{N}}$.

Next, combining Lemma 4.13 and that the set of all periodic points of $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ is dense in $\Gamma^{\mathbb{N}}$, we may assume that the above α is a periodic point of σ. Then, $J_\alpha(f)$ is a quasicircle. We set $\mathcal{V} := \cup_{n \in \mathbb{N}}(\sigma^n)^{-1}(\{\alpha\})$. Then $\mathcal{V}$ is dense in $\Gamma^{\mathbb{N}}$. Let $\gamma \in \mathcal{V}$ be an element. Then there exists an $n \in \mathbb{N}$ such that $\sigma^n(\gamma) = \alpha$. Since $(f_{\gamma,n})^{-1}(K_\alpha(f)) = K_\gamma(f)$, it follows that $\sharp(\mathcal{C}(\text{int}(K_\gamma(f)))) < \infty$. Combining it with Lemma 4.12 and Proposition 3.15, we get that $J_\gamma(f)$ is a Jordan curve. Combining it with that $J_\alpha(f)$ is a quasicircle, it follows that $J_\gamma(f)$ is a quasicircle.

Next, let $\mu := \frac{1}{3}\min\{|b-c| \mid b \in J(G),\ c \in P^*(G)\} (> 0)$. Applying Lemma 4.5 to (f, μ) above, let δ be the number in the statement of Lemma 4.5. We set $\epsilon := \min\{\delta, \mu\}$ and $\rho := \beta$. Applying Lemma 4.8 to $(\Gamma, \alpha, \rho, \epsilon)$ above, let $(n, \theta_1, \theta_2, \omega)$ be the element in the statement of Lemma 4.8. Let $\mathcal{U} := \{\gamma \in \Gamma^{\mathbb{N}} \mid \exists \{m_j\}_{j \in \mathbb{N}}, \exists \{n_k\}_{k \in \mathbb{N}}, m_j \to \infty, \sigma^{m_j}(\gamma) \to \alpha, \sigma^{n_k}(\gamma) \to \omega\}$. Then, combining the statement of Lemma 4.5 and that of Lemma 4.8, it follows that for any $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is not a quasicircle. Moreover, by Lemma 4.13, we get that for any $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve. Combining the above argument, ([S4], Theorem 1.12), Lemma 3.6, and ([NV], Theorem 9.3), we see that for any $\gamma \in \mathcal{U}$, $A_\gamma(f)$ is a John domain, and the bounded component U_γ of $F_\gamma(f)$ is not a John domain. Furthermore, it is easy to see that $\mathcal{U}$ is residual in $\Gamma^{\mathbb{N}}$, and that for any Borel probability measure τ in $\text{Poly}_{\deg \geq 2}$ with $\Gamma_\tau = \Gamma$, $\tilde{\tau}(\mathcal{U}) = 1$. Thus, we have proved Theorem 2.32.

□

We now demonstrate Proposition 2.35.

Proof of Proposition 2.35: Since $P^*(G) \subset \text{int}(\hat{K}(G)) \subset F(G)$, G is hyperbolic. Let $\gamma \in \Gamma^{\mathbb{N}}$ be any element. We will show the following claim.

Claim: $\text{int}(K_\gamma(f))$ is a non-empty connected set.

To show this claim, since G is hyperbolic, $\text{int}(K_\gamma(f))$ is non-empty. Suppose that there exist two distinct connected components W_1 and W_2 of

$\mathrm{int}(K_\gamma(f))$. Since $P^*(G)$ is included in a connected component U of $\mathrm{int}(\hat{K}(G)) \subset F(G)$, ([S4], Corollary 2.7) implies that there exists an $n \in \mathbb{N}$ such that $P^*(G) \subset f_{\gamma,n}(W_1) = f_{\gamma,n}(W_2)$. Let $W := f_{\gamma,n}(W_1) = f_{\gamma,n}(W_2)$. Then, any critical value of $f_{\gamma,n}$ in $\mathbb{C}$ is included in W. Using the method in the proof of Lemma 4.13, we se that $(f_{\gamma,n})^{-1}(W)$ is connected. However, this is a contradiction, since $W_1 \neq W_2$. Hence, we have proved the above claim.

By Claim above and Theorem 4.10, it follows that there exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is a K-quasicircle.

Hence, we have proved Proposition 2.35. □

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