A Note on Certain Analytic Functions

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Abstract

The object of the present paper is to obtain some interesting properties of analytic functions.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc $E = \{z||z|<1\}$. Sakaguchi [1] proved the following theorem.

**Theorem A.** If $f(z) \in \mathcal{A}$ satisfies the condition

$$\text{Re} \left( \frac{zf'(z)}{f(z) - f(-z)} \right) > 0 \quad \text{in} \quad E,$$

then $f(z)$ is univalent and starlike with respect to symmetrical points in $E$.

We call $f(z)$ a Sakaguchi functions which satisfies the condition (1).

In this paper, we need the following lemma.

**Lemma 1.** Let $f(z) \in \mathcal{A}$ and

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > K \quad \text{in} \quad E$$

where $K$ is a real and bounded constant, then we have

$$f(z) \neq 0 \quad \text{in} \quad 0 <|z|<1.$$

2 Results

**Theorem 1.** For arbitrary positive real number $\alpha$, $0 < \alpha \leq \pi$, if $f(z) \in \mathcal{A}$ satisfies the following condition

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\[ \text{Re}\left(\frac{z(e^{i\alpha}f'(ze^{i\alpha})-f'(z))}{f(ze^{i\alpha})-f(z)}\right) > 0 \text{ in } E, \] (1)

then \( f(z) \) is univalent in \( E \).

**Proof.** If there exists a \( r, 0 < r < 1 \) for which \( f(z) \) is univalent in \( |z| < r \) but \( f(z) \) is not univalent on \( |z| = r \), then there exists two points \( z_1, z_2 = z_1 e^{i\alpha}, 0 < \alpha \leq \pi \),

\[ f(z_1) = f(z_2) \] (2)

and \( f(z) \) is univalent on the arc \( C \) where

\[ C = \{ z | z = z_1 e^{i\theta}, 0 \leq \theta < \alpha \}. \] (3)

From the assumption of Theorem 1, we have

\[ \text{Re}\left(\frac{z(e^{i\alpha/2}f'(ze^{i\alpha/2})-f'(z))}{f(ze^{i\alpha/2})-f(z)}\right) > 0 \text{ in } E. \] (4)

This shows that \( f(ze^{i\alpha/2}) - f(z) \) is starlike with respect to the origin.

From (2) and (3), we get the following image of \( |z| = r \) under the mapping \( w = f(z) \),

where \( \beta \) is sufficiently small positive real number.

Then vectors \( f(z_1 e^{i\alpha/2}) - f(z_1) \) and \( f(z_1 e^{i(\alpha/2+\beta)}) - f(z_1 e^{i\beta}) \) move on the clockwise direction (the negative direction). This contradicts (4) and it completes the proof. \( \square \)

**Another proof of Theorem 1.** If there exists a \( r, 0 < r < 1 \) for which \( f(z) \) is univalent in \( |z| < r \) but \( f(z) \) is not univalent on \( |z| = r \), then there exists at least two points \( z_1, z_2 = z_1 e^{i\alpha}, 0 < \alpha \leq \pi \) and for which

\[ f(z_1) = f(z_2). \]

Applying Lemma 1 and form the hypothesis (1), we have

\[ f(ze^{i\alpha}) - f(z) \neq 0. \]

This is a contradiction and therefore, it completes the proof.
Remark. If $f(z) \in A$ satisfies the condition (1) only for the case $\alpha = \pi$, then $f(z)$ is a Sakaguchi function.

Theorem 2. If $f(z) \in A$ satisfy the following condition for sufficiently small and positive real number $\delta$ and arbitrary real number $\alpha$, $0 < |\alpha| < \delta$ for which

$$\Re \frac{z(e^{i\alpha} f'(ze^{i\alpha}) - f'(z))}{f(ze^{i\alpha}) - f(z)} > 0 \text{ in } \mathbb{E}. \quad (5)$$

Then $f(z)$ is convex in $\mathbb{E}$ or

$$1 + \Re \frac{zf''(z)}{f'(z)} > 0 \text{ in } \mathbb{E}.$$ 

Proof. From the hypothesis (5), all the tangent vector of $\mathbb{C}$ which is the image of $|z| = r$, $0 < r < 1$ under the mapping $w = f(z)$ move in the counterclockwise direction. Geometrically, this shows that $f(z)$ is convex in $\mathbb{E}$ or

$$1 + \Re \frac{zf''(z)}{f'(z)} > 0 \text{ in } \mathbb{E}.$$ 

References


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