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Kyoto University
On Rogosinski theorem

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1 Introduction

Let $F(z)$ be analytic and univalent in the unit disc $E = \{ z \mid |z| < 1 \}$ and let $D = F(E)$ be the image of $E$ under the mapping $w = f(z)$. Let $f(z)$ be analytic in $E$, but not necessarily univalent, and $f(E) \subset D$. Then $f(z)$ is said to be subordinate to $F(z)$ in $E$, denoted by $f(z) \prec F(z)$. It is well known that if $f(z) \prec F(z)$ in $E$, then there exists a function $w(z)$, analytic in $E$ and with $|w(z)| < 1$, such that

$$f(z) = F(w(z)), \quad z \in E.$$ If $f(0) = F(0)$, then $w(0) = 0$ and $|w(z)| \leq |z|$ in $E$.

Rogosinski[1] proved the following theorem.

Theorem A. Let $f(z) \prec F(z)$ in $E$. Then

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |F(re^{i\theta})|^p d\theta$$

where $0 < p$ and $0 \leq r < 1$.

2 Obtained results

Theorem 1. Let $f(z) \prec F(z)$ in $E$ and $F(z) \neq 0$ in $E$. Then

$$\int_0^{2\pi} \frac{1}{|f(re^{i\theta})|^p} d\theta \leq \int_0^{2\pi} \frac{1}{|F(re^{i\theta})|^p} d\theta$$

where $0 < p$ and $0 \leq r < 1$. 


Proof. From the assumption of the Theorem, $f(z)^{-p}$ and $F(z)^{-p}$ are analytic in $E$ and so, from the Poisson integral form of harmonic function theory, we have
\[
\frac{1}{f(z)^p} = \frac{1}{F(w(z))^p} = \frac{1}{2\pi} \int_{|\zeta|=R} \frac{1}{|F(\zeta)|^p} \left( \frac{\text{Re} \frac{\zeta + w(z)}{\zeta - w(z)}}{\zeta - w(z)} \right) d\zeta
\]
where $z = re^{i\theta}$, $\zeta = Re^{i\varphi}$, $|z| = r < |\zeta| = R < 1$, and $|w(z)| \leq |z|$. Since
\[
\text{Re} \left( \frac{\zeta + w(z)}{\zeta - w(z)} \right) > 0 \text{ in } E,
\]
it follows that
\[
\int_0^{2\pi} \frac{1}{|f(re^{i\theta})|^p} d\theta \leq \int_0^{2\pi} \frac{1}{|F(re^{i\theta})|^p} \left( \frac{\text{Re} \frac{\zeta + w(z)}{\zeta - w(z)}}{\zeta - w(z)} \right) d\varphi d\theta
\]
\[
= \frac{1}{2\pi} \int_{|\zeta|=R} \int_0^{2\pi} \frac{1}{|F(\zeta)|^p} \left( \frac{\text{Re} \frac{\zeta + w(z)}{\zeta - w(z)}}{\zeta - w(z)} \right) d\theta d\varphi
\]
\[
= \frac{1}{2\pi} \int_{|\zeta|=R} \left\{ \frac{1}{|F(Re^{i\varphi})|^p} \int_{|z|=r} \left( \frac{\text{Re} \frac{\zeta + w(z)}{\zeta - w(z)}}{iz} \right) \frac{dz}{iz} \right\} d\varphi
\]
Putting $R \to r$, we have
\[
\int_0^{2\pi} \frac{1}{|f(re^{i\theta})|^p} d\theta \leq \int_0^{2\pi} \frac{1}{|F(re^{i\theta})|^p} d\theta.
\]
Prof. Owa (Kinki Univ.) pointed out another proof as the following: if $f(z) \prec F(z)$ in $E$ and $F(z) \neq 0$ in $E$, then $\frac{1}{f(z)} \prec \frac{1}{F(z)}$ and applying Theorem A, we can obtain a proof of Theorem 1.
From Theorem A and Theorem 1, we obtain the following theorem.

Theorem 1'. Let $f(z) \prec F(z)$ in $E$ and $F(z) \neq 0$ in $E$.
Then
\[
\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |F(re^{i\theta})|^p d\theta
\]
where $p$ is arbitrary real number and $0 \leq r < 1$.

Theorem 2. Let $f(z) \prec F(z) = z^m(a_m + a_{m+1}z + a_{m+2}z^2 + \ldots)$ in $E$ and let $z_k$, $k = 1, 2, 3, \ldots, n$, $0 < |z_1| \leq |z_2| \leq |z_3| \leq \cdots \leq |z_n|$, are the zeros of $F(z)$ in $E$ which are to
be written with their multiplicities.

Then, if \( F(x) \neq 0 \) on certain circle \(|z| = r < 1\), \( z = re^{i\theta}\), we have

\[
\int_0^{2\pi} \frac{1}{|f(re^{i\theta})|^p} d\theta \geq \frac{2\pi}{r^{m+n}} \prod_{k=1}^{n} |z_k|
\]

where \( 0 < p \).

**Proof.** Without generalization, we can choose \( R, 0 < R < 1 \) in such a manner that \( F(z) \neq 0 \) on the circle \(|z| = R\). Let us construct a function \( B(z) \) which has the same zeros with the same multiplicities in \(|z| < R < 1\) as \( F(z) \) has, and so, we choose

\[
B(z) = \left(\frac{z}{R}\right)^m \prod_{k=1}^{l} \frac{R(z-z_k)}{R^2-z_k^2}, \quad l \leq n.
\]

Putting

\[
g(z) = \left(\frac{B(z)}{F(z)}\right)^p, \quad 0 < p \quad \text{and} \quad z = re^{i\theta},
\]

then \( g(z) \) is analytic in \(|z| < R\) and \( g(z) \neq 0 \) in \(|z| < R\). From the Poisson integral form of harmonic functions, we have

\[
g(z) = \frac{1}{2\pi} \int_{|\zeta|=R} g(\zeta) \text{Re} \left( \frac{\zeta+z}{\zeta-z} \right) d\varphi
\]

where \(|z| = r < |\zeta| = R < 1\) and \( \zeta = Re^{i\varphi} \).

Then, we have

\[
\left( \frac{B(w(z))}{F(w(z))} \right)^p = \left( \frac{B(w(z))}{f(z)} \right)^p
\]

\[
= \frac{1}{2\pi} \int_{|\zeta|=R} \left( \frac{B(\zeta)}{F(\zeta)} \right)^p \text{Re} \left( \frac{\zeta+w(z)}{\zeta-w(z)} \right) d\varphi.
\]

Here, we have

\[
\text{Re} \left( \frac{\zeta+w(z)}{\zeta-w(z)} \right) > 0 \quad \text{in} \quad |z| < R,
\]

\[
|B(w(z))| < 1 \quad \text{on} \quad |z| = r < R < 1,
\]

and

\[
|B(\zeta)| = 1 \quad \text{on} \quad |\zeta| = R.
\]

Then, it follows that

\[
\frac{1}{|f(re^{i\theta})|^p} > \frac{|B(w(re^{i\theta})))|^p}{|f(re^{i\theta})|^p}
\]

\[
= \left| \frac{1}{2\pi} \int_{|\zeta|=R} \left( \frac{B(\zeta)}{F(\zeta)} \right)^p \text{Re} \left( \frac{\zeta+w(z)}{\zeta-w(z)} \right) d\varphi \right|.
\]
Therefore, we have

\[
\int_0^{2\pi} \frac{1}{|f(r e^{i\theta})|^p} d\theta > \int_0^{2\pi} \left| \frac{1}{2\pi} \int_{|\zeta|=R} \left( \frac{B(\zeta)}{F(\zeta)} \right)^p \text{Re} \left( \frac{\zeta + w(z)}{\zeta - w(z)} \right) d\varphi \right| d\theta
\]

\[
= \left\{ \frac{1}{2\pi} \int_{|\zeta|=R} \left( \frac{B(\zeta)}{F(\zeta)} \right)^p \int_0^{2\pi} \text{Re} \left( \frac{\zeta + w(z)}{\zeta - w(z)} \right) d\theta d\varphi \right\}
\]

\[
= \left| \int_{|\zeta|=R} \left( \frac{B(\zeta)}{F(\zeta)} \right)^p d\varphi \right|
\]

\[
= \left| \int_{|\zeta|=R} \left( \frac{B(\zeta)}{F(\zeta)} \right)^p d\zeta \right| \quad i\zeta
\]

\[
= 2\pi \left( \frac{B(0)}{F(0)} \right)^p
\]

\[
= \prod_{k=1}^{l} \frac{|z_k|}{R^{m+l}} \prod_{k=1}^{n} \frac{|z_k|}{R^{m+n}}
\]

Putting \( R \to r \), we have

\[
\int_0^{2\pi} \frac{1}{|f(r e^{i\theta})|^p} d\theta > 2\pi \frac{\prod_{k=1}^{l} |z_k|}{|z|^{m+n}}
\]

This completes the proof of Theorem 2.

\[\square\]

References