Some properties of certain analytic functions

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Abstract
Defining the subclasses $\mathcal{M}D(\alpha,\beta)$ and $\mathcal{N}D(\alpha,\beta)$ of certain analytic functions $f(z)$ in the open unit disk $\mathbb{D}$, some properties for $f(z)$ belonging to the classes $\mathcal{M}D(\alpha,\beta)$ and $\mathcal{N}D(\alpha,\beta)$ are discussed. In this present paper, some coefficient estimates and some interesting applications of Jack's lemma for functions $f(z)$ in the classes $\mathcal{M}D(\alpha,\beta)$ and $\mathcal{N}D(\alpha,\beta)$ are given.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$. Shams, Kulkarni and Jahangiri [3] have considered the subclass $\mathcal{SD}(\alpha,\beta)$ of $\mathcal{A}$ consisting of $f(z)$ which satisfy

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{D})$$

for some $\alpha (\alpha \geq 0)$ and $\beta (0 \leq \beta < 1)$. The class $\mathcal{KD}(\alpha,\beta)$ is defined by the subclass of $\mathcal{A}$ consisting of $f(z)$ such that $zf'(z) \in \mathcal{SD}(\alpha,\beta)$. In view of the classes $\mathcal{SD}(\alpha,\beta)$ and $\mathcal{KD}(\alpha,\beta)$, we introduce the subclass $\mathcal{MD}(\alpha,\beta)$ of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{D})$$

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for some $\alpha (\alpha \leq 0)$ and $\beta (\beta > 1)$. The class $ND(\alpha, \beta)$ is also defined by $f(z) \in ND(\alpha, \beta)$ if and only if $zf'(z) \in MD(\alpha, \beta)$. The classes $MD(\alpha, \beta)$ and $ND(\alpha, \beta)$ were introduced by Nishiwaki and Owa [2]. We discuss some properties of functions $f(z)$ belonging to the classes $MD(\alpha, \beta)$ and $ND(\alpha, \beta)$.

We note if $f(z) \in MD(\alpha, \beta)$, then $\frac{zf'(z)}{f(z)} = u + iv$ maps $\mathbb{U}$ onto elliptic domain such that

$$\left(u - \frac{\alpha^2 - \beta}{\alpha^2 - 1}\right)^2 + \frac{\alpha^2}{\alpha^2 - 1}v^2 < \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2}$$

for $\alpha < -1$, the parabolic domain such that

$$u < -\frac{1}{2(\beta - 1)}v^2 + \frac{\beta + 1}{2}$$

for $\alpha = -1$, and the hyperbolic domain such that

$$\left(u - \frac{\alpha^2 - \beta}{\alpha^2 - 1}\right)^2 - \frac{\alpha^2}{1 - \alpha^2}v^2 > \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2}$$

for $-1 < \alpha < 0$.

### 2 Coefficient estimates for the classes $MD(\alpha, \beta)$ and $ND(\alpha, \beta)$

By definitions of $MD(\alpha, \beta)$ and $ND(\alpha, \beta)$, we derive

**Theorem 2.1.** If $f(z) \in MD(\alpha, \beta)$, then

$$f(z) \in MD(0, \frac{\beta - \alpha}{1 - \alpha}).$$

**Proof.** If $f(z) \in MD(\alpha, \beta)$,

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \leq \alpha \text{Re} \left( \frac{zf'(z)}{f(z)} - 1 \right) + \beta \quad (z \in \mathbb{U})$$

implies that

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \frac{\beta - \alpha}{1 - \alpha} \quad (\alpha \leq 0, \beta > 1).$$

Since $\frac{\beta - \alpha}{1 - \alpha} > 1$, we prove the Theorem. $\square$
**Corollary 2.1.** If \( f(z) \in \mathcal{ND}(\alpha, \beta) \), then

\[
 f(z) \in \mathcal{ND} \left( 0, \frac{\beta - \alpha}{1 - \alpha} \right).
\]

Our result for the coefficient estimates of \( \mathcal{MD}(\alpha, \beta) \) and \( \mathcal{ND}(\alpha, \beta) \) is contained in

**Theorem 2.2.** If \( f(z) \in \mathcal{MD}(\alpha, \beta) \), then

\[
 |a_2| \leq \frac{2(\beta - 1)}{1 - \alpha}
\]

and

\[
 |a_n| \leq \frac{2(\beta - 1)}{(n - 1)(1 - \alpha)} \prod_{j=1}^{n-2} \left( 1 + \frac{2(\beta - 1)}{j(1 - \alpha)} \right) \quad (n \geq 3).
\]

**Proof.** If \( f(z) \in \mathcal{MD}(\alpha, \beta) \), then

\[
 \beta - \alpha + (\alpha - 1)\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0
\]

from Theorem 2.1. And let us define the function \( p(z) \) by

\[
 p(z) = \frac{\beta - \alpha + (\alpha - 1)\frac{zf'(z)}{f(z)}}{\beta - 1}.
\]

Then \( p(z) \) is analytic in \( U \), \( p(0) = 1 \) and \( \text{Re} p(z) > 0 \) \((z \in U)\). Therefore, if we write

\[
 p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,
\]

then \( |p_n| \leq 2 \) \((n \geq 1)\). From (2.1) and (2.2), we obtain that

\[
 (\alpha - 1) \sum_{n=2}^{\infty} (n - 1)a_n z^n = (\beta - 1) \sum_{n=1}^{\infty} p_n z^n (z + \sum_{n=2}^{\infty} a_n z^n).
\]

Therefore we have

\[
 a_n = \frac{\beta - 1}{(n - 1)(\alpha - 1)} (p_{n-1} + p_{n-2}a_2 + \cdots + p_2a_{n-2} + p_1a_{n-1})
\]

for all \( n \geq 2 \). When \( n = 2 \),

\[
 |a_2| \leq \frac{\beta - 1}{1 - \alpha} |p_1| \leq \frac{2(\beta - 1)}{1 - \alpha}.
\]
And when \( n = 3 \),
\[
|a_3| \leq \frac{\beta - 1}{2(1 - \alpha)}(|p_2| + |p_1||a_2|)
= \frac{2(\beta - 1)}{2(1 - \alpha)} \left(1 + \frac{2(\beta - 1)}{1 - \alpha}\right).
\]

Let us suppose that
\[
(a) |a_k| \leq \frac{2(\beta - 1)}{(k - 1)(1 - \alpha)}(1 + |a_2| + \cdots + |a_{k-2}| + |a_{k-1}|)
\]
\[
\leq \frac{2(\beta - 1)}{(k - 1)(1 - \alpha)} \prod_{j=1}^{k-2} \left(1 + \frac{2(\beta - 1)}{j(1 - \alpha)}\right) \quad (k \geq 3).
\]
Then we see
\[
1 + |a_2| + \cdots + |a_{k-2}| + |a_{k-1}| \leq \prod_{j=1}^{k-2} \left(1 + \frac{2(\beta - 1)}{j(1 - \alpha)}\right).
\]
By using (2.3) and (2.4),
\[
|a_{k+1}| \leq \frac{2(\beta - 1)}{k(1 - \alpha)}(1 + |a_2| + \cdots + |a_{k-2}| + |a_{k-1}| + |a_k|)
\]
\[
\leq \left(1 + \frac{2(\beta - 1)}{(k - 1)(1 - \alpha)}\right) \frac{2(\beta - 1)}{k(1 - \alpha)} \prod_{j=1}^{k-2} \left(1 + \frac{2(\beta - 1)}{j(1 - \alpha)}\right)
\]
\[
\leq \frac{2(\beta - 1)}{k(1 - \alpha)} \prod_{j=1}^{k-1} \left(1 + \frac{2(\beta - 1)}{j(1 - \alpha)}\right).
\]

This completes the proof of the Theorem. \(\square\)

**Corollary 2.2.** If \( f(z) \in ND(\alpha, \beta) \), then
\[
|a_2| \leq \frac{2(\beta - 1)}{2(1 - \alpha)}
\]
and
\[
|a_n| \leq \frac{2(\beta - 1)}{n(n - 1)(1 - \alpha)} \prod_{j=1}^{n-2} \left(1 + \frac{2(\beta - 1)}{j(1 - \alpha)}\right) \quad (n \geq 3).
\]

**Proof.** From \( f(z) \in ND(\alpha, \beta) \) if and only if \( zf'(z) \in MD(\alpha, \beta) \), replacing \( a_n \) by \( na_n \) in Theorem 2.2, we have the corollary. \(\square\)
3 Applications of Jack’s lemma for the classes $\mathcal{M}\mathcal{D}(\alpha, \beta)$ and $\mathcal{N}\mathcal{D}(\alpha, \beta)$

In this section, some applications of Jack’s lemma for $f(z)$ belonging to the classes $\mathcal{M}\mathcal{D}(\alpha, \beta)$ and $\mathcal{N}\mathcal{D}(\alpha, \beta)$ are discussed. Next lemma was given by Jack [1].

**Lemma 3.1.** Let the function $w(z)$ be analytic in $U$ with $w(0) = 0$. If

$$\max_{|z|\leq |z_0|} |w(z)| = |w(z_0)|,$$

then

$$z_0 w'(z_0) = kw(z_0),$$

where $k$ is a real number and $k \geq 1$.

**Theorem 3.1.** If $f(z) \in \mathcal{M}\mathcal{D}(\alpha, \beta)$, then

$$\left| \left( \frac{f(z)}{z} \right)^{\frac{1+\delta(1-\alpha)}{(1+\delta)(\beta-1)}} - 1 \right| < 1 + \delta \quad (\delta \geq 0)$$

for some $\alpha(\alpha \leq 0)$ and $\beta(\beta > 1)$, or

$$\left| \left( \frac{f(z)}{z} \right)^{\frac{1+\delta(1+\alpha)}{(1+\delta)(\beta-1)}} - 1 \right| < 1 + \delta \quad (\delta \geq 0)$$

for some $\alpha(\alpha \leq -1)$ and $\beta(\beta > 1)$.

**Proof.** Let us define

$$\gamma = \frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)} > 0,$$

for $\alpha \leq 0$ and $\beta > 1$, and

$$\gamma = \frac{(1+\delta)(1+\alpha)}{(2+\delta)(\beta-1)} < 0$$

for $\alpha \leq -1$ and $\beta > 1$. Further, let the function $w(z)$ be defined by

$$w(z) = \left( \frac{f(z)}{z} \right)^{\gamma} - 1 \quad (\delta \geq 0)$$

which is equivalent to

$$zf'(z) = f(z) - 1 = \frac{(1+\delta)zw'(z)}{(1+\delta)w(z) + 1}.$$
Then we see that $w(z)$ is analytic in $\mathbb{U}$, and $w(0) = 0$. On the other hand, if $f(z) \in \mathcal{MD}(\alpha, \beta)$ ($\alpha \leq 0, \beta > 1$), then
\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| = 1 + \frac{1}{\gamma} \text{Re} \left( \frac{(1 + \delta)zw'(z)}{(1 + \delta)w(z) + 1} \right) - \frac{\alpha}{|\gamma|} \left| \frac{(1 + \delta)zw'(z)}{(1 + \delta)w(z) + 1} \right| < \beta.
\]
Furthermore, if there is a point $z_0$ ($z_0 \in \mathbb{U}$), which satisfies
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,
\]
then Lemma 3.1 gives us that
\[
1 + \frac{1}{\gamma} \text{Re} \left( \frac{(1 + \delta)zw'(z_0)}{(1 + \delta)w(z_0) + 1} \right) - \frac{\alpha}{|\gamma|} \left| \frac{(1 + \delta)zw'(z_0)}{(1 + \delta)w(z_0) + 1} \right| = 1 + \frac{k(1 + \delta)}{\gamma} \cdot \frac{1 + \delta + \cos \theta - \alpha \sqrt{(1 + \delta)^2 + 2(1 + \delta)\cos \theta + 1}}{(1 + \delta)^2 + 2(1 + \delta)\cos \theta + 1} = F(\theta).
\]
When $\gamma > 0$,
\[
F(\theta) \geq 1 + \frac{k(1 + \delta)(1 - \alpha)}{\gamma(2 + \delta)} \geq 1 + \frac{(1 + \delta)(1 - \alpha)}{\gamma(2 + \delta)} = \beta,
\]
because
\[
\gamma = \frac{(1 + \delta)(1 - \alpha)}{(2 + \delta)(\beta - 1)}.
\]
Further, when $\gamma < 0$,
\[
F(\theta) \geq 1 + \frac{k(1 + \delta)(1 + \alpha)}{\gamma(2 + \delta)} \geq 1 + \frac{(1 + \delta)(1 + \alpha)}{\gamma(2 + \delta)} = \beta.
\]
This contradicts our condition of the Theorem. Thus there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This completes the proof of the Theorem.

**Corollary 3.1.** If $f(z) \in \mathcal{ND}(\alpha, \beta)$, then
\[
\left| \left( f'(z) \right)^{\frac{(1+\delta)(1+\alpha)}{\delta(\beta-1)}} - 1 \right| < 1 + \delta \quad (\delta \geq 0)
\]
for some $\alpha (\alpha \leq 0)$ and $\beta (\beta > 1)$, or
\[
\left| \left( f'(z) \right)^{\frac{(1+\delta)(1+\alpha)}{\delta(\beta-1)}} - 1 \right| < 1 + \delta \quad (\delta \geq 0)
\]
for some $\alpha (\alpha \leq -1)$ and $\beta (\beta > 1)$.

**Proof.** Replacing $f(z)$ by $zf'(z)$ in Theorem 3.1, we have the corollary 3.1. □
References


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