数理解析研究所講究録
1538巻 2007年 20-29

On The N-Fractional-Calculus of Some Composite Functions

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Abstract

In this article N-fractional calculus of composite functions

$$((z-b)^{\beta}-c)^{\alpha} \qquad ((z-b)^{\beta}-c \neq 0)$$

and

$$\log((z-b)^{\beta}-c) \qquad ((z-b)^{\beta}-c \neq 0, 1)$$

are discussed.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D=\{D_-, D_+\}$, $C=\{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty+i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_-, D_+ be a domain surrounded by C_+.

(Here D contains the points over the curve C).

Moreover, let $f=f(z)$ be a regular function in $D(z\in D)$,

$$f_\nu(z)=(f)_\nu={}_C(f)_\nu=\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \qquad (\nu\notin \mathbf{Z}^-), \tag{1}$$

$$(f)_{-m}=\lim_{\nu\to -m}(f)_\nu \qquad (m\in \mathbf{Z}^+), \tag{2}$$

where $-\pi\le\arg(\zeta-z)\le\pi$ for C_-, $0\le\arg(\zeta-z)\le 2\pi$ for C_+,

$\zeta\neq z$, $z\in \mathbf{C}$, $\nu\in \mathbf{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$), with respect to z, of the function f, if $|(f)_\nu|<\infty$.

(II) On the fractional calculus operator N^{ν} [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^{ν} be*

$$N^{\nu}=\left(\frac{\Gamma(\nu+1)}{2\pi i}\int_{C}\frac{d\zeta}{(\zeta-z)^{\nu+1}}\right)\qquad(\nu\notin Z^{-}),\quad [\,Refer\ to\,(1)\,] \tag{3}$$

with

$$N^{-m}=\lim_{\nu\to -m}N^{\nu}\qquad(m\in Z^{+}), \tag{4}$$

and define the binary operation $\circ$ as

$$N^{\beta}\circ N^{\alpha}f=N^{\beta}N^{\alpha}f=N^{\beta}(N^{\alpha}f)\qquad(\alpha,\beta\in R), \tag{5}$$

then the set

$$\{N^{\nu}\}=\{N^{\nu}|\ \nu\in R\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^{\nu})^{-1}=N^{-\nu}$ to the fractional calculus operator N^{ν} , for the function f such that $f\in F=\{f;\ 0\neq|f_{\nu}|<\infty,\nu\in R\}$, where $f=f(z)$ and $z\in C$.
(vis. $-\infty<\nu<\infty$).

(For our convenience, we call $N^{\beta}\circ N^{\alpha}$ as product of N^{β} and N^{α}.)

Theorem B. " F.O.G. $\{N^{\nu}\}$ " *is an " Action product group which has continuous index ν " for the set of F . (F.O.G.* ; Fractional calculus operator group)

Theorem C. *Let*

$$S:=\{\pm N^{\nu}\}\cup\{0\}=\{N^{\nu}\}\cup\{-N^{\nu}\}\cup\{0\}\quad(\nu\in R). \tag{7}$$

Then the set S is a commutative ring for the function $f\in F$, when the identity

$$N^{\alpha}+N^{\beta}=N^{\gamma}\quad(N^{\alpha},N^{\beta},N^{\gamma}\in S) \tag{8}$$

holds. [5]

(III) Lemma. We have [1]

(i) $$((z-c)^{\beta})_{\alpha}=e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\alpha}\qquad\left(\left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right|<\infty\right),$$

(ii) $$(\log(z-c))_{\alpha}=-e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha}\qquad(|\Gamma(\alpha)|<\infty),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha}=-e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c)\qquad(|\Gamma(\alpha)|<\infty),$$

where $z-c\neq 0$ in (i), and $z-c\neq 0,1$ in (ii) and (iii). (Γ; Gamma function),

(iv) $$(u\cdot v)_{\alpha}:=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}u_{\alpha-k}v_{k}\qquad\left(\begin{array}{l}u=u(z),\\ v=v(z)\end{array}\right).$$

§ 1. N- Fractional Calculus of A Power Function

Theorem 1. *We have*

(i) $$(((z-b)^{\beta}-c)^{\alpha})_{\gamma}=e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{[-\alpha]_k}{k!}\cdot\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}\left(\frac{c}{(z-b)^{\beta}}\right)^k,\quad\left(\left|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}\right|<\infty\right)\quad(1)$$

and

(ii) $$(((z-b)^{\beta}-c)^{\alpha})_{m}=(-1)^m(z-b)^{\alpha\beta-m}\sum_{k=0}^{\infty}\frac{[-\alpha]_k\,[\beta k-\alpha\beta]_m}{k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k,\quad(2)$$

$(m\in\mathbf{Z}_0^{+})$,

where $|c/(z-b)^{\beta}|<1$,

and

$$[\lambda]_k=\lambda(\lambda+1)\cdots(\lambda+k-1)=\Gamma(\lambda+k)/\Gamma(\lambda)\quad \textit{with}\quad [\lambda]_0=1,$$

(Notation of Pochhammer).

Proof of (i). We have

$$((z-b)^{\beta}-c)^{\alpha}=\sum_{k=0}^{\infty}\frac{c^k[-\alpha]_k}{k!}(z-b)^{\beta(\alpha-k)},\quad(3)$$

using the identity

$$\Gamma(\alpha+1-k)=(-1)^{-k}\frac{\Gamma(\alpha+1)\Gamma(-\alpha)}{\Gamma(k-\alpha)}.\quad(4)$$

Since

$$((z-b)^{\beta}-c)^{\alpha}=(z-b)^{\alpha\beta}\left(1-\frac{c}{(z-b)^{\beta}}\right)^{\alpha}\quad(5)$$

$$=(z-b)^{\alpha\beta}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}\left(\frac{-c}{(z-b)^{\beta}}\right)^k,\quad\left(\left|\frac{-c}{(z-b)^{\beta}}\right|<1\right).\quad(6)$$

Operate N- fractional calculus operator N^{γ} to the both sides of (3), we have then

$$N^{\gamma}((z-b)^{\beta}-c)^{\alpha}=\sum_{k=0}^{\infty}\frac{c^k[-\alpha]_k}{k!}N^{\gamma}(z-b)^{\beta(\alpha-k)},\quad(7)$$

that is, we have

$$\left(((z-b)^{\beta}-c)^{\alpha}\right)_{\gamma}=\sum_{k=0}^{\infty}\frac{c^k[-\alpha]_k}{k!}((z-b)^{\beta(\alpha-k)})_{\gamma}\quad(8)$$

$$=e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}\sum_{k=0}^{\infty}\frac{[-\alpha]_k}{k!}\cdot\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}\left(\frac{c}{(z-b)^{\beta}}\right)^k,\left(\left|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}\right|<\infty\right)$$

(1)

because we have

$$((z-c)^{\beta(\alpha-k)})_\gamma = e^{-i\pi\gamma}\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}(z-b)^{\beta(\alpha-k)-\gamma} \tag{9}$$

by Lemma (i), under the conditions.

Proof of (i i). Set $\gamma = m \in \mathbf{Z}_0^+$ in (i).

Corollary 1. *We have*

(i) $$((z^\beta - c)^\alpha)_\gamma = e^{-i\pi\gamma} z^{\alpha\beta-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k}{k!}\cdot\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}\left(\frac{c}{z^\beta}\right)^k, \quad \left(\left|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}\right| < \infty\right) \tag{1}$$

and

(i i) $$((z^\beta - c)^\alpha)_m = (-1)^m z^{\alpha\beta-m}\sum_{k=0}^{\infty}\frac{[-\alpha]_k\,[\beta k-\alpha\beta]_m}{k!}\left(\frac{c}{z^\beta}\right)^k, \quad (m \in \mathbf{Z}_0^+), \tag{2}$$

where

$$|c/z^\beta| < 1 .$$

Proof . Set $b = 0$ in Theorem 1.

§ 2. Some Special Cases of § 1. (1)

[I] When $\beta = 1$, we obtain

$$\left((z-b-c)^\alpha\right)_\gamma = e^{-i\pi\gamma}(z-b)^{\alpha-\gamma}\sum_{k=0}^{\infty}\frac{[-\alpha]_k}{k!}\cdot\frac{\Gamma(k-\alpha+\gamma)}{\Gamma(k-\alpha)}\left(\frac{c}{z-b}\right)^k, \tag{1}$$

$$\left(\left|\frac{\Gamma(k-\alpha+\gamma)}{\Gamma(k-\alpha)}\right| < \infty, \quad \left|\frac{c}{z-b}\right| < 1\right)$$

from **§ 1.** (1).

Now we have

$$\Gamma(k-\alpha) = \Gamma(-\alpha)[-\alpha]_k \tag{2}$$

and

$$\Gamma(k-\alpha+\gamma) = \Gamma(-\alpha+\gamma)[-\alpha+\gamma]_k \ . \tag{3}$$

Hence we obtain

$$\left((z-b-c)^\alpha\right)_\gamma = e^{-i\pi\gamma}(z-b)^{\alpha-\gamma}\frac{\Gamma(-\alpha+\gamma)}{\Gamma(-\alpha)}\sum_{k=0}^{\infty}\frac{[-\alpha+\gamma]_k}{k!}\left(\frac{c}{z-b}\right)^k, \tag{4}$$

from (1) , (2) and (3).

Now we have

$$\sum_{k=0}^{\infty}\frac{[-\alpha+\gamma]_k}{k!}\left(\frac{c}{z-b}\right)^k=\left(1-\frac{c}{z-b}\right)^{\alpha-\gamma}=\left(\frac{z-b-c}{z-b}\right)^{\alpha-\gamma}, \tag{5}$$

since

$$\sum_{k=0}^{\infty}\frac{[\lambda]_k}{k!}z^k=(1-z)^{-\lambda}. \tag{6}$$

Therefore, we obtain

$$\left((z-b-c)^{\alpha}\right)_{\gamma}=e^{-i\pi\gamma}\,\frac{\Gamma(-\alpha+\gamma)}{\Gamma(-\alpha)}(z-b-c)^{\alpha-\gamma}\qquad\left(\left|\frac{\Gamma(-\alpha+\gamma)}{\Gamma(-\alpha)}\right|<\infty\right), \tag{7}$$

from (4) and (5).

That is, we obtain (7) from (1). The result (7) is same as the one obtained from Lemma (i).

[I I] When $\gamma=m=1$, we obtain

$$\left(((z-b)^{\beta}-c)^{\alpha}\right)_1=-(z-b)^{\alpha\beta-1}\sum_{k=0}^{\infty}\frac{[-\alpha]_k}{k!}\cdot\frac{\Gamma(\beta k-\alpha\beta+1)}{\Gamma(\beta k-\alpha\beta)}\left(\frac{c}{(z-b)^{\beta}}\right)^k \tag{8}$$

$$=-\beta(z-b)^{\alpha\beta-1}\sum_{k=0}^{\infty}\frac{[-\alpha]_k}{k!}\cdot k\left(\frac{c}{(z-b)^{\beta}}\right)^k+\alpha\beta(z-b)^{\alpha\beta-1}\sum_{k=0}^{\infty}\frac{[-\alpha]_k}{k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k \tag{9}$$

from § 1. (2).

Now we have

$$\sum_{k=0}^{\infty}\frac{[-\alpha]_k}{k!}\cdot k\left(\frac{c}{(z-b)^{\beta}}\right)^k=(-\alpha)\sum_{k=1}^{\infty}\frac{\Gamma(k-\alpha)}{(k-1)!\,(-\alpha)\Gamma(-\alpha)}\left(\frac{c}{(z-b)^{\beta}}\right)^k \tag{10}$$

$$=-\frac{\alpha c}{(z-b)^{\beta}}\sum_{k=0}^{\infty}\frac{[1-\alpha]_k}{k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k \tag{11}$$

$$=-\frac{\alpha c}{(z-b)^{\beta}}\left(1-\frac{c}{(z-b)^{\beta}}\right)^{\alpha-1} \tag{12}$$

and

$$\sum_{k=0}^{\infty}\frac{[-\alpha]_k}{k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k=\left(1-\frac{c}{(z-b)^{\beta}}\right)^{\alpha}, \tag{13}$$

using the identity (6).

Therefore, we have

$$\left(((z-b)^{\beta}-c)^{\alpha}\right)_1=\alpha\beta c\,(z-b)^{\alpha\beta-\beta-1}\left(1-\frac{c}{(z-b)^{\beta}}\right)^{\alpha-1}$$

$$+\alpha\beta(z-b)^{\alpha\beta-1}\left(1-\frac{c}{(z-b)^{\beta}}\right)^{\alpha} \qquad (14)$$

$$=\alpha\beta(z-b)^{\alpha\beta-1}\left(1-\frac{c}{(z-b)^{\beta}}\right)^{\alpha}\left\{c(z-b)^{-\beta}\left(1-\frac{c}{(z-b)^{\beta}}\right)^{-1}+1\right\} \qquad (15)$$

$$=\alpha\beta(z-b)^{\beta-1}\left((z-b)^{\beta}-c\right)^{\alpha-1}. \qquad (16)$$

Note. We have

$$\frac{\Gamma(k-\alpha)}{(-\alpha)\Gamma(-\alpha)}=[1-\alpha]_{k-1}\ , \qquad (17)$$

and

$$\sum_{k=1}^{\infty}\frac{[1-\alpha]_{k-1}}{(k-1)!}\left(\frac{c}{(z-b)^{\beta}}\right)^{k} \qquad \left(\begin{matrix}\text{replace } k \text{ by } k+1,\\ \text{we have then}\end{matrix}\right)$$

$$=\sum_{k=0}^{\infty}\frac{[1-\alpha]_{k}}{k!}\left(\frac{c}{(z-b)^{\beta}}\right)^{k+1}. \qquad (18)$$

[I I I] When $\beta=2,\gamma=m=1$, we obtain

$$\left(((z-b)^{2}-c)^{\alpha}\right)_1=2\alpha\,(z-b)\left((z-b)^{2}-c\right)^{\alpha-1} \qquad (19)$$

from (16) clearly.

§ 3. N- Fractional Calculus of A Logarithmic Fumction

Theorem 2. We have

(i) $$(\log((z-b)^{\beta}-c))_{\gamma}=-e^{-i\pi\gamma}\beta(z-b)^{-\gamma}\,\Gamma(\gamma)$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\beta k+\gamma)}{\Gamma(\gamma)\Gamma(\beta k+1)}\left(\frac{c}{(z-b)^{\beta}}\right)^{k},\ \left(\,|\,\Gamma(\gamma)\,|,\ \left|\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k)}\right|<\infty\right),(1)$$

and

(i i) $$(\log((z-b)^{\beta}-c))_{m}=(-1)^{m+1}\beta(z-b)^{-m}\,\Gamma(m)$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\beta k+m)}{\Gamma(m)\Gamma(\beta k+1)}\left(\frac{c}{(z-b)^{\beta}}\right)^{k},\qquad (m\in\mathbf{Z}^{+}), \qquad (2)$$

where

$$(z-b)^{\beta}-c\neq 0,1\ \ \textit{and}\ \ |\,c/(z-b)^{\beta}\,|<1\ .$$

Proof of (i). We have

$$\log((z-b)^{\beta}-c)=\log(z-b)^{\beta}+\log\left(1-\frac{c}{(z-b)^{\beta}}\right), \qquad (3)$$

$$=\beta\log(z-b)-\sum_{k=1}^{\infty}\frac{c^k}{k}(z-b)^{-\beta k} \qquad \left(\left|\frac{c}{(z-b)^{\beta}}\right|<1\right). \qquad (4)$$

Operate N^{γ} to the both sides of (4), we have then

$$(\log((z-b)^{\beta}-c))_{\gamma}=\beta(\log(z-b))_{\gamma}-\sum_{k=1}^{\infty}\frac{c^k}{k}((z-b)^{-\beta k})_{\gamma}. \qquad (5)$$

Next we have

$$(\log(z-b))_{\gamma}=-e^{-i\pi\gamma}\Gamma(\gamma)(z-b)^{-\gamma} \qquad (\,|\Gamma(\gamma)|<\infty) \qquad (6)$$

and

$$((z-b)^{-\beta k})_{\gamma}=e^{-i\pi\gamma}\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k)}(z-b)^{-\beta k-\gamma} \qquad \left(\left|\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k)}\right|<\infty\right), \qquad (7)$$

from Lemmas (i i i) and (i), respectively.

Therefore, substituting (6) and (7) into (5), we obtain

$$(\log((z-b)^{\beta}-c))_{\gamma}=-e^{-i\pi\gamma}\beta(z-b)^{-\gamma}\Gamma(\gamma)$$

$$-e^{-i\pi\gamma}(z-b)^{-\gamma}\sum_{k=1}^{\infty}\frac{\Gamma(\beta k+\gamma)}{k\,\Gamma(\beta k)}\left(\frac{c}{(z-b)^{\beta}}\right)^{k} \qquad (8)$$

$$=-e^{-i\pi\gamma}\beta(z-b)^{-\gamma}\left\{\Gamma(\gamma)+\sum_{k=1}^{\infty}\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k+1)}\left(\frac{c}{(z-b)^{\beta}}\right)^{k}\right\}. \qquad (9)$$

We have then (1) from (9) clearly, under the conditions.

Proof of (i i). Set $\gamma=m\in\mathbf{Z}^{+}$ in (i).

Corollary 2. We have

(i) $$(\log(z^{\beta}-c))_{\gamma}=-e^{-i\pi\gamma}\beta z^{-\gamma}\Gamma(\gamma)\sum_{k=0}^{\infty}\frac{\Gamma(\beta k+\gamma)}{\Gamma(\gamma)\Gamma(\beta k+1)}\left(\frac{c}{z^{\beta}}\right)^{k}, \qquad (10)$$

$$\left(\,|\Gamma(\gamma)|,\ \left|\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k)}\right|<\infty\right)$$

and

(i i) $$(\log(z^{\beta}-c))_{m}=(-1)^{m+1}\beta z^{-m}\Gamma(m)\sum_{k=0}^{\infty}\frac{\Gamma(\beta k+m)}{\Gamma(m)\Gamma(\beta k+1)}\left(\frac{c}{z^{\beta}}\right)^{k}, \qquad (11)$$

$(m\in\mathbf{Z}^{+})$,

where

$$z^{\beta}-c \neq 0,1 \quad and \quad |c/z^{\beta}|<1 \ .$$

Proof . Set $b=0$ in Theorem 2.

§ 4. Some Special Cases of § 3. (1)

[I] When $\beta=1$, we obtain

$$(\log(z-b-c))_{\gamma}=-e^{-i\pi\gamma}(z-b)^{-\gamma}\,\Gamma(\gamma)\sum_{k=0}^{\infty}\frac{\Gamma(k+\gamma)}{k!\,\Gamma(\gamma)}\left(\frac{c}{z-b}\right)^{k} \qquad (1)$$

$$=-e^{-i\pi\gamma}\,\Gamma(\gamma)(z-b)^{-\gamma}\sum_{k=0}^{\infty}\frac{[\gamma]_k}{k!}\left(\frac{c}{z-b}\right)^{k} \qquad (|\Gamma(\gamma)|<\infty) \qquad (2)$$

from § 3. (1).

Now we have

$$\sum_{k=0}^{\infty}\frac{[\gamma]_k}{k!}\left(\frac{c}{z-b}\right)^{k}=\left(1-\frac{c}{z-b}\right)^{-\gamma} . \qquad (3)$$

Therefore, we have

$$(\log(z-b-c))_{\gamma}=-e^{-i\pi\gamma}\,\Gamma(\gamma)(z-b)^{-\gamma}\left(1-\frac{c}{z-b}\right)^{-\gamma}=-e^{-i\pi\gamma}\,\Gamma(\gamma)(z-b-c)^{-\gamma} , \qquad (4)$$

from (2) and (3).

This result is same as Lemma (i i).

[I I] When $\gamma=m=1$, we obtain

$$(\log((z-b)^{\beta}-c))_1=\beta(z-b)^{-1}\sum_{k=0}^{\infty}\left(\frac{c}{(z-b)^{\beta}}\right)^{k} \qquad (5)$$

$$=\beta(z-b)^{-1}\frac{1}{\left(1-\dfrac{c}{(z-b)^{\beta}}\right)}=\beta(z-b)^{\beta-1}((z-b)^{\beta}-c)^{-1} \qquad (6)$$

from § 3. (2).

[I I I] When $\beta=\gamma=1$, we obtain

$$(\log(z-b-c))_1=(z-b-c)^{-1} \qquad (7)$$

from (6), clearly.

§ 5. Commentary

(i) The result

$$(((z-b)^{\beta}-c)^{\alpha})_1 = \alpha\beta(z-b)^{\beta-1}((z-b)^{\beta}-c)^{\alpha-1} \qquad (\text{§ 2. } (16)) \qquad (1)$$

gives

$$\frac{d((z-b)^{\beta}-c)^{\alpha}}{dz} = \frac{du^{\alpha}}{du}\cdot\frac{du}{dz} \qquad (u=(z-b)^{\beta}-c)\ . \qquad (2)$$

(ii) The result

$$(\log((z-b)^{\beta}-c))_1 = \beta(z-b)^{\beta-1}((z-b)^{\beta}-c)^{-1} \qquad (\text{§ 4. } (6)) \qquad (3)$$

gives

$$\frac{d(\log((z-b)^{\beta}-c))}{dz} = \frac{d(\log u)}{du}\cdot\frac{du}{dz} \qquad (u=(z-b)^{\beta}-c)\ . \qquad (4)$$

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