<table>
<thead>
<tr>
<th>Title</th>
<th>Note on a class of convex functions (Study on Calculus Operators in Univalent Function Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Acu, Mugur; Owa, Shigeyoshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1538: 11-19</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2007-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/59050">http://hdl.handle.net/2433/59050</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Note on a class of convex functions
Mugur Acu¹, Shigeyoshi Owa²

ABSTRACT. In this paper we define a general class of convex functions, denoted by $SL_{\beta}(q)$, with respect to a convex domain $D$ ($q(z) \in \mathcal{H}_{u}(U)$, $q(0) = 1$, $q(U) = D$) contained in the right half plane by using the linear operator $D_{\lambda}^{\beta}$ defined by

$$D_{\lambda}^{\beta} : A \rightarrow A,$$

$$D_{\lambda}^{\beta} f(z) = z + \sum_{j=2}^{\infty} (1 + (j - 1)\lambda)^{\beta} a_{j} z^{j},$$

where $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_{j} z^{j}$. This operator generalizes the Sălăgean operator and the Al-Oboudi operator. Regarding the class $SL_{\beta}(q)$ we give an inclusion theorem, a preserving theorem (we use the Libera-Pascu integral operator) and many particular results.

2000 Mathematical Subject Classification: 30C45

Key words and phrases: Convex functions, Libera-Pascu integral operator, Briot-Bouquet differential subordination, generalized Sălăgean operator

1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U$, $A = \{ f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0 \}$, $\mathcal{H}_{u}(U) = \{ f \in \mathcal{H}(U) : f$ is univalent in $U \}$ and $S = \{ f \in A : f$ is univalent in $U \}$.

Let $D^{n}$ be the Sălăgean differential operator (see [13]) defined as:

$$D^{n} : A \rightarrow A,$$

$$D^{n} f(z) = Df(z) = zf'(z), \quad D^{n} f(z) = D(D^{n-1} f(z)).$$

**Remark 1.1** If $f \in S$, $f(z) = z + \sum_{j=2}^{\infty} a_{j} z^{j}$, $z \in U$ then $D^{n} f(z) = z + \sum_{j=2}^{\infty} j^{n} a_{j} z^{j}$.

Let $n \in \mathbb{N}$ and $\lambda \geq 0$. Let denote with $D_{\lambda}^{n}$ the Al-Oboudi operator (see [7]) defined by

$$D_{\lambda}^{n} : A \rightarrow A,$$
$D_{\lambda}^0 f(z) = f(z), \quad D_{\lambda}^1 f(z) = (1 - \lambda)f(z) + \lambda zf'(z) = D_{\lambda} f(z),
D_{\lambda}^n f(z) = D_{\lambda} (D_{\lambda}^{n-1} f(z)).$

We observe that $D_{\lambda}^n$ is a linear operator and for $f(z) = z + \sum_{j=2}^{\infty} a_{j}z^{j}$ we have

$$D_{\lambda}^n f(z) = z + \sum_{j=2}^{\infty} (1 + (j - 1)\lambda)^{n} a_{j}z^{j}.$$  

The aim of this paper is to define a general class of convex functions with respect to a convex domain $D$, contained in the right half plane, by using an operator which generalizes the Sălăgean operator and the Al-Oboudi operator and to obtain some properties of this class.

2 Preliminary results

We recall here the definition of the well-known class of convex functions

$$S^c = CV = K = \left\{ f \in H(U); f(0) = f'(0) - 1 = 0, \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \ z \in U \right\}.$$  

Remark 2.1 By using the subordination relation, we may define the class $S^c$ thus if $f(z) = z + a_2z^2 + \ldots, \ z \in U$, then $f \in S^c$ if and only if $\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \prec \frac{1+z}{1-z}, \ z \in U$, where by "$\prec$" we denote the subordination relation.

Let consider the Libera-Pascu integral operator $L_a : A \rightarrow A$ defined as:

$$f(z) = L_a F(z) = \frac{1+a}{z^a} \int_{0}^{z} F(t) \cdot t^{a-1} dt, \quad a \in \mathbb{C}, \quad \text{Re} \ a \geq 0.$$  

In the case $a = 1$ this operator was introduced by R.J.Libera and it was studied by many authors in different general cases. In this general form ($a \in \mathbb{C}, \quad \text{Re} \ a \geq 0$) was used first time by N.N. Pascu in [12].

Definition 2.1 [6] Let $\beta, \lambda \in \mathbb{R}, \ \beta \geq 0, \ \lambda \geq 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_{j}z^{j}$. We denote by $D_{\lambda}^{\beta}$ the linear operator defined by

$$D_{\lambda}^{\beta} : A \rightarrow A,$$

$$D_{\lambda}^{\beta} f(z) = z + \sum_{j=2}^{\infty} (1 + (j - 1)\lambda)^{\beta} a_{j}z^{j}.$$
Remark 2.2 It is easy to observe that for $\beta = n \in \mathbb{N}$ we obtain the Al-Oboudi operator and for $\beta = n \in \mathbb{N}, \lambda = 1$ we obtain the Sălăgean operator.

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [9], [10], [11]).

Theorem 2.1 Let $h$ convex in $U$ and $\text{Re}[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H(U)$ with $p(0) = h(0)$ and $p$ satisfied the Briot-Bouquet differential subordination
\[
p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \text{then } p(z) \prec h(z).
\]

3 Main results

Definition 3.1 Let $q(z) \in \mathcal{H}_u(U)$, with $q(0) = 1$ and $q(U) = D$, where $D$ is a convex domain contained in the right half plane, $\beta, \lambda \in \mathbb{R}, \beta \geq 0$ and $\lambda \geq 0$. We say that a function $f(z) \in A$ is in the class $SL_{\beta}^{c}(q)$ if
\[
\frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)} < q(z), \quad z \in U.
\]

Remark 3.1 Geometric interpretation: $f(z) \in SL_{\beta}^{c}(q)$ if and only if $\frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)}$ take all values in the convex domain $D$ contained in the right half-plane.

Remark 3.2 It is easy to observe that if we choose different function $q(z)$ we obtain variously classes of convex functions, such as (for example), for $\beta = n \in \mathbb{N}$ the class $SL_{n}^{c}(q)$ (see [3]), for $\lambda = 1$ and $\beta = 0$, the class of convex functions, the class of convex functions of order $\gamma$ (see [8]), the class of convex functions with respect to a hyperbola (see [5]) , and, for $\beta = n \in \mathbb{N}$ and $\lambda = 1$, the class of $n$-convex functions (see [2]), the class of $n$-convex functions with respect to a hyperbola (see [1]), the class of $n$-convex functions with respect to a convex domain contained in the right half-plane(see [2]), for $\beta \in \mathbb{R}$ and $\lambda = 1$, the class $SL_{\beta}(q)$ of the $\beta$-$q$-convex functions (see [4]).

Remark 3.3 For $q_1(z) \prec q_2(z)$ we have $SL_{\beta}^{c}(q_1) \subset SL_{\beta}^{c}(q_2)$. From the above we obtain $SL_{\beta}^{c}(q) \subset SL_{\beta}^{c}\left(\frac{1+z}{1-z}\right)$.

Theorem 3.1 Let $\beta, \lambda \in \mathbb{R}, \beta \geq 0$ and $\lambda > 0$. We have
\[
SL_{\beta+1}^{c}(q) \subset SL_{\beta}^{c}(q).
\]
Proof. Let $f(z) \in SL_{\beta+1}(q)$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$.

With notation

$$p(z) = \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta+1} f(z)}, \quad p(0) = 1,$$

we obtain

$$\frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+2} f(z)} = \frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+1} f(z)} \cdot \frac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta+2} f(z)} = \frac{1}{p(z)} \cdot \frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+1} f(z)}$$

We have

$$\frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+1} f(z)} = \frac{z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+3} a_j z^j}{z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} a_j z^j}$$

and

$$zp'(z) = \frac{z \left( D_{\lambda}^{\beta+2} f(z) \right)' - \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta+1} f(z)} \cdot \frac{z(D_{\lambda}^{\beta+1} f(z))'}{D_{\lambda}^{\beta+1} f(z)} \cdot \frac{z(D_{\lambda}^{\beta+1} f(z))'}{D_{\lambda}^{\beta+1} f(z)}}{z \left( z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+2} a_j z^{j-1} \right) - \frac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta+1} f(z)} \cdot \frac{z(D_{\lambda}^{\beta+1} f(z))'}{D_{\lambda}^{\beta+1} f(z)} \cdot \frac{z(D_{\lambda}^{\beta+1} f(z))'}{D_{\lambda}^{\beta+1} f(z)}}$$

or

$$zp'(z) = \frac{z \left( z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+2} a_j z^j \right)'}{z \left( z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} a_j z^j \right) - p(z) \cdot \frac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta+1} f(z)}}$$

Also, we have

$$z + \sum_{j=2}^{\infty} j (1 + (j-1)\lambda)^{\beta+1} a_j z^j = z + \sum_{j=2}^{\infty} ((j-1) + 1) (1 + (j-1)\lambda)^{\beta+1} a_j z^j$$

$$= z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} a_j z^j + \sum_{j=2}^{\infty} (j-1) (1 + (j-1)\lambda)^{\beta+1} a_j z^j$$
\[ = D_{\lambda}^{\beta+1}f(z) + \sum_{j=2}^{\infty} (j-1)(1+(j-1)\lambda) D_{\lambda}^{\beta+1}a_{j}z^{j} \]

\[ = D_{\lambda}^{\beta+1}f(z) + \frac{1}{\lambda} \sum_{j=2}^{\infty} (1+(j-1)\lambda) D_{\lambda}^{\beta+1}a_{j}z^{j} \]

Similarly we have
\[ z + \sum_{j=2}^{\infty} j(1+(j-1)\lambda) D_{\lambda}^{\beta+1}a_{j}z^{j} = \frac{1}{\lambda} ((\lambda-1)D_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+2}f(z)) \]

From (3) we obtain
\[ zp'(z) = \frac{1}{\lambda} \left( \frac{D_{\lambda}^{\beta+2}f(z) + D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+1}f(z)} - p(z) \frac{(\lambda-1)D_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)} \right) \]

\[ = \frac{1}{\lambda} \left( (\lambda-1)p(z) + \frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+1}f(z)} - p(z) ((\lambda-1) + p(z)) \right) \]

Thus
\[ \lambda zp'(z) = \frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+1}f(z)} - p(z)^2 \]

or
\[ \frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+1}f(z)} = p(z)^2 + \lambda zp'(z) \]

From (2) we obtain
\[ \frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+2}f(z)} = \frac{1}{p(z)} \left( p(z)^2 + \lambda zp'(z) \right) = p(z) + \frac{zp'(z)}{p(z)}, \]
where $\beta, \lambda \in \mathbb{R}, \beta \geq 0, \lambda > 0$.

From $f(z) \in SL_{\beta+1}^{c}(q)$ we have

$$p(z) + \lambda \frac{zp'(z)}{p(z)} < q(z),$$

with $p(0) = q(0) = 1$, $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda > 0$. In this conditions from Theorem 2.1, we obtain

$$p(z) < q(z)$$

or

$$\frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)} < q(z).$$

This means $f(z) \in SL_{\beta}^{c}(q)$.

**Corollary 3.1** For every $\beta \in \mathbb{N}^{*}$ we have $SL_{\beta}^{c}(q) \subset SL_{0}^{c}(q) \subset S^{c}$.

**Theorem 3.2** Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda \geq 1$. If $F(z) \in SL_{\beta}^{c}(q)$ then $f(z) = L_{a}F(z) \in SL_{\beta}^{c}(q)$, where $L_{a}$ is the Libera-Pascu integral operator defined by (1).

**Proof.** From (1) we have

$$(1+a)F(z) = af(z) + zf'(z)$$

and, by using the linear operator $D_{\lambda}^{\beta+1}$, we obtain

$$(1+a)D_{\lambda}^{\beta+1}F(z) = aD_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+1}\left(z + \sum_{j=2}^{\infty}j a_{j}z^{j}\right)$$

$$= aD_{\lambda}^{\beta+1}f(z) + z + \sum_{j=2}^{\infty}(1 + (j-1)\lambda)^{\beta+1}ja_{j}z^{j}$$

We have (see the proof of the above theorem)

$$z + \sum_{j=2}^{\infty}j (1 + (j-1)\lambda)^{\beta+1}a_{j}z^{j} = \frac{1}{\lambda} \left((\lambda-1)D_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+2}f(z)\right)$$

Thus

$$(1+a)D_{\lambda}^{\beta+1}F(z) = aD_{\lambda}^{\beta+1}f(z) + \frac{1}{\lambda} \left((\lambda-1)D_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+2}f(z)\right)$$

$$= \left(a + \frac{\lambda-1}{\lambda}\right)D_{\lambda}^{\beta+1}f(z) + \frac{1}{\lambda}D_{\lambda}^{\beta+2}f(z)$$

or

$$\lambda(1+a)D_{\lambda}^{\beta+1}F(z) = ((a+1)\lambda - 1)D_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+2}f(z).$$
Similarly, we obtain
\[ \lambda(1 + a)D^{\beta+2}_\lambda F(z) = ((a + 1)\lambda - 1) D^{\beta+2}_\lambda f(z) + D^{\beta+3}_\lambda f(z). \]

Then
\[ \frac{D^{\beta+2}_\lambda F(z)}{D^{\beta+1}_\lambda F(z)} = \frac{\frac{D^{\beta+3}_\lambda f(z)}{D^{\beta+1}_\lambda f(z)} + ((a + 1)\lambda - 1)}{\frac{D^{\beta+2}_\lambda f(z)}{D^{\beta+1}_\lambda f(z)} + ((a + 1)\lambda - 1)}. \]

With notation
\[ \frac{D^{\beta+2}_\lambda f(z)}{D^{\beta+1}_\lambda f(z)} = p(z), \quad p(0) = 1, \]
we obtain
\[ \frac{D^{\beta+2}_\lambda F(z)}{D^{\beta+1}_\lambda F(z)} = \frac{\frac{D^{\beta+3}_\lambda f(z)}{D^{\beta+1}_\lambda f(z)} + ((a + 1)\lambda - 1) \cdot p(z)}{p(z) + ((a + 1)\lambda - 1)}. \]  

We have (see the proof of the above theorem)
\[ \frac{D^{\beta+3}_\lambda f(z)}{D^{\beta+1}_\lambda f(z)} = p(z)^2 + \lambda z p'(z). \]

From (4), we obtain
\[ \frac{D^{\beta+2}_\lambda F(z)}{D^{\beta+1}_\lambda F(z)} = \frac{p(z)^2 + \lambda z p'(z) + ((a + 1)\lambda - 1) \cdot p(z)}{p(z) + ((a + 1)\lambda - 1)}, \]
where \( a \in \mathbb{C}, \ Re a \geq 0, \ \beta, \lambda \in \mathbb{R}, \ \beta \geq 0 \) and \( \lambda \geq 1 \). From \( F(z) \in SL_{\beta}^c(q) \) we have
\[ p(z) + \frac{zp'(z)}{\frac{1}{\lambda}(p(z) + ((a + 1)\lambda - 1))} < q(z), \]
where \( a \in \mathbb{C}, \ Re a \geq 0, \ \beta, \lambda \in \mathbb{R}, \ \beta \geq 0, \ \lambda \geq 1 \), and from her construction, we have \( Re q(z) > 0 \). In this conditions we have from Theorem 2.1 we obtain
\[ p(z) < q(z) \]
or
\[ \frac{D^{\beta+2}_\lambda f(z)}{D^{\beta+1}_\lambda f(z)} < q(z). \]
This means \( f(z) = L_{\beta} F(z) \in SL_{\beta}^c(q) \).

For \( \beta = n \in \mathbb{N} \) and \( \lambda = 1 \) we obtain
Corollary 3.2 If $F(z) \in CV_n(q)$ then $f(z) = L_a F(z) \in CV_n(q)$, where $L_a$ is the Libera-Pascu integral operator and by $CV_n(q)$ we denote the class of $n$-convex functions subordinate to the function $q(z)$ (see [2]).

For $\beta = n \in \mathbb{N}$ we obtain

Corollary 3.3 [3] Let $n \in \mathbb{N}$ and $\lambda \geq 1$. If $F(z) \in SL_n^\lambda(q)$ then $f(z) = L_a F(z) \in SL_n^\lambda(q)$, where $L_a$ is the Libera-Pascu integral operator defined by (1).

For $\beta \in \mathbb{R}$ and $\lambda = 1$ we obtain

Corollary 3.4 [4] If $F(z) \in S_\beta^\lambda(q)$ then $f(z) = L_a F(z) \in S_\beta^\lambda(q)$, where $L_a$ is the Libera-Pascu integral operator defined by (1).
References


---

1 University "Lucian Blaga" of Sibiu, Department of Mathematics, Str. Dr. I. Raţiu, No. 5-7, 550012 - Sibiu, Romania

2 Department of Mathematics, School of Science and Engineering, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan