Note on a class of convex functions
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ABSTRACT. In this paper we define a general class of convex functions, denoted by $\mathcal{SL}_\beta^\epsilon(q)$, with respect to a convex domain $D(q(z) \in \mathcal{H}_u(U), q(0) = 1, q(U) = D)$ contained in the right half plane by using the linear operator $D_\lambda^\beta$ defined by

$$D_\lambda^\beta : A \to A,$$

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^\infty (1 + (j - 1)\lambda)^\beta a_j z^j,$$

where $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum_{j=2}^\infty a_j z^j$. This operator generalize the Ságlagean operator and the Al-Oboudi operator. Regarding the class $\mathcal{SL}_\beta^\epsilon(q)$ we give an inclusion theorem, a preserving theorem (we use the Libera-Pascu integral operator) and many particular results.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U$, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f$ is univalent in $U\}$ and $S = \{f \in A : f$ is univalent in $U\}$.

Let $D^n$ be the Ságlagean differential operator (see [13]) defined as:

$$D^n : A \to A, \quad n \in \mathbb{N} \text{ and } D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z), \quad D^n f(z) = D(D^{n-1} f(z)).$$

Remark 1.1 If $f \in S, f(z) = z + \sum_{j=2}^\infty a_j z^j, z \in U$ then $D^n f(z) = z + \sum_{j=2}^\infty j^n a_j z^j$.

Let $n \in \mathbb{N}$ and $\lambda \geq 0$. Let denote with $D_\lambda^n$ the Al-Oboudi operator (see [7]) defined by

$$D_\lambda^n : A \to A,$$
\[ D_0^\lambda f(z) = f(z), \quad D_1^\lambda f(z) = (1 - \lambda)f(z) + \lambda zf'(z) = D_\lambda f(z), \]
\[ D_n^\lambda f(z) = D_\lambda (D_{n-1}^\lambda f(z)) . \]

We observe that \( D_n^\lambda \) is a linear operator and for \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \) we have
\[ D_n^\lambda f(z) = z + \sum_{j=2}^{\infty} (1 + (j - 1)\lambda)^n a_j z^j . \]

The aim of this paper is to define a general class of convex functions with respect to a convex domain \( D \), contained in the right half plane, by using an operator which generalize the Sălăgean operator and the Al-Oboudi operator and to obtain some properties of this class.

2 Preliminary results

We recall here the definition of the well-known class of convex functions
\[ S^c = CV = K = \left\{ f \in H(U); \ f(0) = f'(0) - 1 = 0, \ Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \ z \in U \right\} . \]

Remark 2.1 By using the subordination relation, we may define the class \( S^c \) thus
if \( f(z) = z + a_2 z^2 + \ldots, \ z \in U \), then \( f \in S^c \) if and only if \( \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \frac{1+z}{1-z}, \ z \in U \),
where by \( "<" \) we denote the subordination relation.

Let consider the Libera-Pascu integral operator \( L_a : A \to A \) defined as:
\[ f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt , \quad a \in \mathbb{C}, \quad Re \ a \geq 0. \]

In the case \( a = 1 \) this operator was introduced by R.J.Libera and it was studied by many authors in different general cases. In this general form \( (a \in \mathbb{C}, \ Re \ a \geq 0) \) was used first time by N.N. Pascu in [12].

Definition 2.1 [6] Let \( \beta, \lambda \in \mathbb{R}, \ \beta \geq 0, \ \lambda \geq 0 \) and \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \). We denote by \( D_\lambda^\beta \) the linear operator defined by
\[ D_\lambda^\beta : A \to A, \]
\[ D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} (1 + (j - 1)\lambda)^\beta a_j z^j . \]
Remark 2.2 It is easy to observe that for $\beta = n \in \mathbb{N}$ we obtain the Al-Oboudi operator and for $\beta = n \in \mathbb{N}, \lambda = 1$ we obtain the Sălăgean operator.

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [9], [10], [11]).

Theorem 2.1 Let $h$ convex in $U$ and $\Re[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H(U)$ with $p(0) = h(0)$ and $p$ satisfied the Briot-Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z),$$

then $p(z) \prec h(z)$.

3 Main results

Definition 3.1 Let $q(z) \in \mathcal{H}_u(U)$, with $q(0) = 1$ and $q(U) = D$, where $D$ is a convex domain contained in the right half plane, $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda \geq 0$. We say that a function $f(z) \in A$ is in the class $SL_\beta^c(q)$ if

$$\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} \prec q(z), \quad z \in U.$$

Remark 3.1 Geometric interpretation: $f(z) \in SL_\beta^c(q)$ if and only if $\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}$ take all values in the convex domain $D$ contained in the right half-plane.

Remark 3.2 It is easy to observe that if we choose different function $q(z)$ we obtain variously classes of convex functions, such as (for example), for $\beta = n \in \mathbb{N}$ the class $SL_\beta^c(q)$ (see [3]), for $\lambda = 1$ and $\beta = 0$, the class of convex functions, the class of convex functions of order $\gamma$ (see [8]), the class of convex functions with respect to a hyperbola (see [5]), and, for $\beta = n \in \mathbb{N}$ and $\lambda = 1$, the class of $n$-convex functions (see [2]), the class of $n$-convex functions with respect to a hyperbola (see [1]), the class of $n$-convex functions with respect to a convex domain contained in the right half-plane(see [2]), for $\beta \in \mathbb{R}$ and $\lambda = 1$, the class $S_\beta^c(q)$ of the $\beta$-$q$-convex functions (see [4]).

Remark 3.3 For $q_1(z) \prec q_2(z)$ we have $SL_\beta^c(q_1) \subset SL_\beta^c(q_2)$. From the above we obtain $SL_\beta^c(q) \subset SL_\beta^c \left(\frac{1+z}{1-z}\right)$.

Theorem 3.1 Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda > 0$. We have

$$SL_{\beta+1}^c(q) \subset SL_\beta^c(q).$$
Proof. Let \( f(z) \in SL^{\varepsilon+1}(q) \), \( f(z) = z + \sum_{j=2}^{\infty} a_{j}z^{j} \).

With notation
\[
p(z) = \frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)}, \quad p(0) = 1,
\]
we obtain
\[
\frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+2}f(z)} = \frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)} \cdot \frac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta+2}f(z)} = \frac{1}{p(z)} \cdot \frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+2}f(z)}
\]
(2)

We have
\[
\frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+1}f(z)} = \frac{z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)\beta+3 a_{j}z^{j}}{z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)\beta+1 a_{j}z^{j}}
\]
and
\[
zp'(z) = \frac{z \left( D_{\lambda}^{\beta+2}f(z) \right)'}{D_{\lambda}^{\beta+1}f(z)} - \frac{z \left( D_{\lambda}^{\beta+1}f(z) \right)'}{D_{\lambda}^{\beta+1}f(z)}
\]
\[
= \frac{z \left( D_{\lambda}^{\beta+2}f(z) \right)'}{D_{\lambda}^{\beta+1}f(z)} - p(z) \cdot \frac{z \left( D_{\lambda}^{\beta+1}f(z) \right)'}{D_{\lambda}^{\beta+1}f(z)}
\]
\[
z \left( z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)\beta+2 a_{j}z^{j-1} \right) = \frac{z \left( z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)\beta+1 a_{j}z^{j} \right)}{D_{\lambda}^{\beta+1}f(z)} - p(z) \cdot \frac{z \left( z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)\beta+1 a_{j}z^{j} \right)}{D_{\lambda}^{\beta+1}f(z)}
\]
or
\[
zp'(z) = \frac{z + \sum_{j=2}^{\infty} j (1 + (j-1)\lambda)\beta+2 a_{j}z^{j}}{D_{\lambda}^{\beta+1}f(z)} - p(z) \cdot \frac{z + \sum_{j=2}^{\infty} j (1 + (j-1)\lambda)\beta+1 a_{j}z^{j}}{D_{\lambda}^{\beta+1}f(z)}
\]
(3)

Also, we have
\[
z + \sum_{j=2}^{\infty} j (1 + (j-1)\lambda)\beta+1 a_{j}z^{j} = z + \sum_{j=2}^{\infty} ((j-1) + 1) (1 + (j-1)\lambda)\beta+1 a_{j}z^{j}
\]
\[
= z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)\beta+1 a_{j}z^{j} + \sum_{j=2}^{\infty} (j-1) (1 + (j-1)\lambda)\beta+1 a_{j}z^{j}
\]
\[
D_{\lambda}^{\beta+1}f(z) + \frac{1}{\lambda}\sum_{j=2}^{\infty}((j-1)\lambda)\lambda^{\beta+1}a_{j}z^{j}

= D_{\lambda}^{\beta+1}f(z) + \frac{1}{\lambda}\sum_{j=2}^{\infty}(1+(j-1)\lambda)\lambda^{\beta+1}a_{j}z^{j}

= D_{\lambda}^{\beta+1}f(z) + \frac{1}{\lambda}\sum_{j=2}^{\infty}(1+(j-1)\lambda-1)(1+(j-1)\lambda)^{\beta+2}a_{j}z^{j}

= D_{\lambda}^{\beta+1}f(z) - \frac{1}{\lambda}D_{\lambda}^{\beta+1}f(z) + \frac{1}{\lambda}D_{\lambda}^{\beta+2}f(z)

\]

Similarly we have
\[
z + \sum_{j=2}^{\infty}j(1+(j-1)\lambda)^{\beta+2}a_{j}z^{j}
\]

From (3) we obtain
\[
zp'(z) = \frac{1}{\lambda}\left(\frac{(\lambda-1)D_{\lambda}^{\beta+2}f(z) + D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+1}f(z)} - p(z)\frac{(\lambda-1)D_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)}\right)
\]

Thus
\[
\lambdazp'(z) = \frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+1}f(z)} - p(z)^{2}
\]
or
\[
\frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+1}f(z)} = p(z)^{2} + \lambdazp'(z).
\]

From (2) we obtain
\[
\frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+2}f(z)} = \frac{1}{p(z)} (p(z)^{2} + \lambdazp'(z)) = p(z) + \lambda\frac{zp'(z)}{p(z)},
\]
where $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda > 0$.

From $f(z) \in SL_{\beta+1}^c(q)$ we have

$$p(z) + \lambda \frac{zp'(z)}{p(z)} \prec q(z),$$

with $p(0) = q(0) = 1$, $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda > 0$. In this conditions from Theorem 2.1, we obtain

$$p(z) \prec q(z)$$

or

$$\frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)} \prec q(z).$$

This means $f(z) \in SL_{\beta}^c(q)$.

**Corollary 3.1** For every $\beta \in \mathbb{N}^*$ we have $SL_{\beta}^c(q) \subset SL_{\delta}^c(q) \subset S^c$.

**Theorem 3.2** Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda \geq 1$. If $F(z) \in SL_{\beta}^c(q)$ then $f(z) = L_\alpha F(z) \in SL_{\beta}^c(q)$, where $L_\alpha$ is the Libera-Pascu integral operator defined by (1).

**Proof.** From (1) we have

$$(1 + a)F(z) = af(z) + zf'(z)$$

and, by using the linear operator $D_{\lambda}^{\beta+1}$, we obtain

$$(1 + a)D_{\lambda}^{\beta+1}F(z) = aD_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+1} \left( z + \sum_{j=2}^{\infty} ja_j z^j \right)$$

$$= aD_{\lambda}^{\beta+1}f(z) + z + \sum_{j=2}^{\infty} (1 + (j - 1)\lambda)^{\beta+1} ja_j z^j$$

We have (see the proof of the above theorem)

$$z + \sum_{j=2}^{\infty} j (1 + (j - 1)\lambda)^{\beta+1} a_j z^j = \frac{1}{\lambda} \left( (\lambda - 1)D_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+2}f(z) \right)$$

Thus

$$(1 + a)D_{\lambda}^{\beta+1}F(z) = aD_{\lambda}^{\beta+1}f(z) + \frac{1}{\lambda} \left( (\lambda - 1)D_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+2}f(z) \right)$$

$$= \left( a + \frac{\lambda - 1}{\lambda} \right) D_{\lambda}^{\beta+1}f(z) + \frac{1}{\lambda} D_{\lambda}^{\beta+2}f(z)$$

or

$$\lambda(1 + a)D_{\lambda}^{\beta+1}F(z) = ((a + 1)\lambda - 1)D_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+2}f(z).$$
Similarly, we obtain

$$\lambda(1 + a)D_{\lambda}^{\beta+2}F(z) = ((a + 1)\lambda - 1) D_{\lambda}^{\beta+2}f(z) + D_{\lambda}^{\beta+3}f(z).$$

Then

$$\frac{D_{\lambda}^{\beta+2}F(z)}{D_{\lambda}^{\beta+1}F(z)} = \frac{D_{\lambda}^{\beta+3}f(z) + ((a + 1)\lambda - 1)\cdot D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta+1}f(z) + ((a + 1)\lambda - 1)}.$$

With notation

$$\frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)} = p(z), \quad p(0) = 1,$$

we obtain

$$\frac{D_{\lambda}^{\beta+2}F(z)}{D_{\lambda}^{\beta+1}F(z)} = \frac{D_{\lambda}^{\beta+3}f(z) + ((a + 1)\lambda - 1)\cdot p(z)}{p(z) + ((a + 1)\lambda - 1)}.$$  

(4)

We have (see the proof of the above theorem)

$$\frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+1}f(z)} = p(z)^2 + \lambda z p'(z).$$

From (4), we obtain

$$\frac{D_{\lambda}^{\beta+2}F(z)}{D_{\lambda}^{\beta+1}F(z)} = \frac{p(z)^2 + \lambda z p'(z) + ((a + 1)\lambda - 1) p(z)}{p(z) + ((a + 1)\lambda - 1)} = p(z) + \lambda \frac{zp'(z)}{p(z) + ((a + 1)\lambda - 1)},$$

where $a \in \mathbb{C}, \Re a \geq 0, \beta, \lambda \in \mathbb{R}, \beta \geq 0$ and $\lambda \geq 1$. From $F(z) \in SL_{\beta}^c(q)$ we have

$$p(z) + \frac{zp'(z)}{\frac{1}{\lambda}(p(z) + ((a + 1)\lambda - 1))} < q(z),$$

where $a \in \mathbb{C}, \Re a \geq 0, \beta, \lambda \in \mathbb{R}, \beta \geq 0, \lambda \geq 1$, and from her construction, we have $Re q(z) > 0$. In this conditions we have from Theorem 2.1 we obtain

$$p(z) < q(z)$$

or

$$\frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)} < q(z).$$

This means $f(z) = L_{a} F(z) \in SL_{\beta}^c(q)$.

For $\beta = n \in \mathbb{N}$ and $\lambda = 1$ we obtain
Corollary 3.2 If $F(z) \in CV_n(q)$ then $f(z) = L_a F(z) \in CV_n(q)$, where $L_a$ is the Libera-Pascu integral operator and by $CV_n(q)$ we denote the class of $n$-convex functions subordinate to the function $q(z)$ (see [2]).

For $\beta = n \in \mathbb{N}$ we obtain

Corollary 3.3 [3] Let $n \in \mathbb{N}$ and $\lambda \geq 1$. If $F(z) \in SL_n^\lambda(q)$ then $f(z) = L_a F(z) \in SL_n^\lambda(q)$, where $L_a$ is the Libera-Pascu integral operator defined by (1).

For $\beta \in \mathbb{R}$ and $\lambda = 1$ we obtain

Corollary 3.4 [4] If $F(z) \in S_\beta^q(q)$ then $f(z) = L_a F(z) \in S_\beta^q(q)$, where $L_a$ is the Libera-Pascu integral operator defined by (1).
References


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