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Note on a class of starlike functions

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ABSTRACT. In this paper we define a general class of starlike functions, denoted by $SL_{\beta}^{*}(q)$, with respect to a convex domain D ($q(z)\in\mathcal{H}_{u}(U)$, $q(0)=1$, $q(U)=D$) contained in the right half plane by using the linear operator D_{λ}^{β} defined by

$$D_{\lambda}^{\beta}:A\rightarrow A,$$

$$D_{\lambda}^{\beta}f(z)=z+\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}a_{j}z^{j},$$

where $\beta,\lambda\in\mathbb{R}$, $\beta\geq 0$, $\lambda\geq 0$ and $f(z)=z+\sum_{j=2}^{\infty}a_{j}z^{j}$. This operator generalize the Sălăgean operator and the Al-Oboudi operator. Regarding the class $SL_{\beta}^{*}(q)$ we give a inclusion theorem, a preserving theorem (we use the Libera-Pascu integral operator) and many particular results.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A=\{f\in\mathcal{H}(U):f(0)=f'(0)-1=0\}$, $\mathcal{H}_{u}(U)=\{f\in\mathcal{H}(U):f$ is univalent in $U\}$ and $S=\{f\in A:f$ is univalent in $U\}$.

Let D^{n} be the Sălăgean differential operator (see [12]) defined as:

$$D^{n}:A\rightarrow A,\quad n\in\mathbb{N}\text{ and }D^{0}f(z)=f(z)$$

$$D^{1}f(z)=Df(z)=zf'(z),\quad D^{n}f(z)=D(D^{n-1}f(z)).$$

Remark 1.1 *If* $f\in S, f(z)=z+\sum_{j=2}^{\infty}a_{j}z^{j}$, $z\in U$ *then* $D^{n}f(z)=z+\sum_{j=2}^{\infty}j^{n}a_{j}z^{j}$.

Let $n\in\mathbb{N}$ and $\lambda\geq 0$. Let denote with D_{λ}^{n} the Al-Oboudi operator (see [4]) defined by

$$D_{\lambda}^{n}:A\rightarrow A,$$

$$D_\lambda^0 f(z) = f(z)\ ,\ D_\lambda^1 f(z) = (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z)\,,$$
$$D_\lambda^n f(z) = D_\lambda\left(D_\lambda^{n-1} f(z)\right)\ .$$

We observe that D_λ^n is a linear operator and for $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$ we have

$$D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^n a_j z^j\,.$$

The aim of this paper is to define a general class of starlike functions with respect to a convex domain D, contained in the right half plane, by using a operator which generalize the Sălăgean operator and the Al-Oboudi operator and to obtain some properties of this class.

2 Preliminary results

We recall here the definition of the well - known class of starlike functions

$$S^* = \left\{ f \in A : Re\frac{zf'(z)}{f(z)} > 0\ ,\ z \in U \right\}.$$

Remark 2.1 *By using the subordination relation, we may define the class S^* thus if $f(z) = z + a_2 z^2 + \ldots$, $z \in U$, then $f \in S^*$ if and only if $\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}$, $z \in U$, where by "$\prec$" we denote the subordination relation.*

Let consider the Libera-Pascu integral operator $L_a : A \to A$ defined as:

(1) $$f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt\ ,\quad a \in \mathbb{C}\ ,\quad Re\ a \geq 0.$$

In the case $a = 1$ this operator was introduced by R.J.Libera and it was studied by many authors in different general cases. In this general form ($a \in \mathbb{C}$, $Re\ a \geq 0$) was used first time by N.N. Pascu in [11].

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [8], [9], [10]).

Theorem 2.1 *Let h convex in U and $Re[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H(U)$ with $p(0) = h(0)$ and p satisfied the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \text{then}\ \ p(z) \prec h(z).$$

3 Main results

Definition 3.1 *Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. We denote by D_λ^β the linear operator defined by*

$$D_\lambda^\beta : A \to A\,,$$

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^\beta a_j z^j\,.$$

Remark 3.1 *It is easy to observe that for $\beta = n \in \mathbf{N}$ we obtain the Al-Oboudi operator and for $\beta = n \in \mathbf{N}$, $\lambda = 1$ we obtain the Sălăgean operator.*

Definition 3.2 *Let $q(z) \in \mathcal{H}_u(U)$, with $q(0) = 1$ and $q(U) = D$, where D is a convex domain contained in the right half plane, $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda \geq 0$. We say that a function $f(z) \in A$ is in the class $SL_\beta^*(q)$ if*

$$\frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} \prec q(z)\,,\ z \in U\,.$$

Remark 3.2 *Geometric interpretation: $f(z) \in SL_\beta^*(q)$ if and only if $\dfrac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)}$ take all values in the convex domain D contained in the right half-plane.*

Remark 3.3 *It is easy to observe that if we choose different function $q(z)$ we obtain variously classes of starlike functions, such as (for example), for $\beta = n \in \mathbf{N}$ the class $SL_n^*(q)$ (see [2]), for $\lambda = 1$ and $\beta = 0$, the class of starlike functions, the class of starlike functions of order γ (see [6]), the class of starlike functions with respect to a hyperbola (see [13]) , for $\beta = n \in \mathbf{N}$ and $\lambda = 1$, the class of n-starlike functions (see [12]), the class of n-starlike functions with respect to a hyperbola (see [1]), the class of n-uniformly starlike functions of order γ and type α(see [7]), and, for $\beta \in \mathbb{R}$ and $\lambda = 1$, the class $S_\beta^*(q)$ of the β-q-starlike functions (see [3]).*

Remark 3.4 *For $q_1(z) \prec q_2(z)$ we have $SL_\beta^*(q_1) \subset SL_\beta^*(q_2)$. From the above we obtain $SL_\beta^*(q) \subset SL_\beta^*\left(\dfrac{1+z}{1-z}\right)$.*

Theorem 3.1 *Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda > 0$. We have*

$$SL_{\beta+1}^*(q) \subset SL_\beta^*(q)\,.$$

Proof. Let $f(z) \in SL^*_{\beta+1}(q)$.

With notation

$$p(z) = \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^{\beta} f(z)}\,,\; p(0) = 1\,,$$

we obtain

$$\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} = \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta} f(z)} \cdot \frac{D_\lambda^{\beta} f(z)}{D_\lambda^{\beta+1} f(z)} = \frac{1}{p(z)} \cdot \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta} f(z)} \tag{2}$$

For $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we have

$$\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta} f(z)} = \frac{z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+2}\, a_j z^j}{z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta}\, a_j z^j}$$

and

$$zp'(z) = \frac{z\left(D_\lambda^{\beta+1} f(z)\right)'}{D_\lambda^{\beta} f(z)} - \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^{\beta} f(z)} \cdot \frac{z\left(D_\lambda^{\beta} f(z)\right)'}{D_\lambda^{\beta} f(z)}$$

$$= \frac{z\left(z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1}\, a_j z^j\right)'}{D_\lambda^{\beta} f(z)} - p(z) \cdot \frac{z\left(z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta}\, a_j z^j\right)'}{D_\lambda^{\beta} f(z)}$$

$$= \frac{z\left(1 + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1}\, j a_j z^{j-1}\right)}{D_\lambda^{\beta} f(z)} - p(z) \cdot \frac{z\left(1 + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta}\, j a_j z^{j-1}\right)}{D_\lambda^{\beta} f(z)}$$

or

$$zp'(z) = \frac{z + \sum_{j=2}^{\infty} j\,(1 + (j-1)\lambda)^{\beta+1}\, a_j z^j}{D_\lambda^{\beta} f(z)} - p(z) \cdot \frac{z + \sum_{j=2}^{\infty} j\,(1 + (j-1)\lambda)^{\beta}\, a_j z^j}{D_\lambda^{\beta} f(z)} \tag{3}$$

We have

$$z + \sum_{j=2}^{\infty} j\,(1 + (j-1)\lambda)^{\beta+1}\, a_j z^j = z + \sum_{j=2}^{\infty} ((j-1)+1)\,(1 + (j-1)\lambda)^{\beta+1}\, a_j z^j$$

$$= z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1}\, a_j z^j + \sum_{j=2}^{\infty} (j-1)\,(1 + (j-1)\lambda)^{\beta+1}\, a_j z^j$$

$$= D_\lambda^{\beta+1} f(z) + \sum_{j=2}^{\infty} (j-1)\,(1+(j-1)\lambda)^{\beta+1}\, a_j z^j$$

$$= D_\lambda^{\beta+1} f(z) + \frac{1}{\lambda}\sum_{j=2}^{\infty} ((j-1)\lambda)\,(1+(j-1)\lambda)^{\beta+1}\, a_j z^j$$

$$= D_\lambda^{\beta+1} f(z) + \frac{1}{\lambda}\sum_{j=2}^{\infty} (1+(j-1)\lambda-1)\,(1+(j-1)\lambda)^{\beta+1}\, a_j z^j$$

$$= D_\lambda^{\beta+1} f(z) - \frac{1}{\lambda}\sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+1}\, a_j z^j + \frac{1}{\lambda}\sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+2}\, a_j z^j$$

$$= D_\lambda^{\beta+1} f(z) - \frac{1}{\lambda}\left(D_\lambda^{\beta+1} f(z) - z\right) + \frac{1}{\lambda}\left(D_\lambda^{\beta+2} f(z) - z\right)$$

$$= D_\lambda^{\beta+1} f(z) - \frac{1}{\lambda} D_\lambda^{\beta+1} f(z) + \frac{z}{\lambda} + \frac{1}{\lambda} D_\lambda^{\beta+2} f(z) - \frac{z}{\lambda}$$

$$= \frac{\lambda-1}{\lambda} D_\lambda^{\beta+1} f(z) + \frac{1}{\lambda} D_\lambda^{\beta+2} f(z)$$

$$= \frac{1}{\lambda}\left((\lambda-1) D_\lambda^{\beta+1} f(z) + D_\lambda^{\beta+2} f(z)\right).$$

Similarly we have

$$z + \sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{\beta}\, a_j z^j = \frac{1}{\lambda}\left((\lambda-1) D_\lambda^{\beta} f(z) + D_\lambda^{\beta+1} f(z)\right).$$

From (3) we obtain

$$zp'(z) = \frac{1}{\lambda}\left(\frac{(\lambda-1) D_\lambda^{\beta+1} f(z) + D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta} f(z)} - p(z)\frac{(\lambda-1) D_\lambda^{\beta} f(z) + D_\lambda^{\beta+1} f(z)}{D_\lambda^{\beta} f(z)}\right)$$

$$= \frac{1}{\lambda}\left((\lambda-1)p(z) + \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta} f(z)} - p(z)\left((\lambda-1) + p(z)\right)\right)$$

$$= \frac{1}{\lambda}\left(\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta} f(z)} - p(z)^2\right)$$

Thus

$$\lambda z p'(z) = \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta} f(z)} - p(z)^2$$

or

$$\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta} f(z)} = p(z)^2 + \lambda z p'(z).$$

From (2) we obtain

$$\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} = \frac{1}{p(z)}\left(p(z)^2 + \lambda z p'(z)\right) = p(z) + \lambda\frac{zp'(z)}{p(z)},$$

where $\beta \geq 0$ and $\lambda > 0$.

From $f(z) \in SL^*_{\beta+1}(q)$ we have

$$p(z) + \lambda \frac{zp'(z)}{p(z)} \prec q(z),$$

with $p(0) = q(0) = 1$, $\beta \geq 0$ and $\lambda > 0$. In this conditions from Theorem 2.1, we obtain

$$p(z) \prec q(z)$$

or

$$\frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} \prec q(z).$$

This means $f(z) \in SL^*_\beta(q)$.

Corollary 3.1 *For every $\beta \in \mathbb{N}^*$ we have $SL^*_\beta(q) \subset SL^*_0(q) \subset S^*$.*

Theorem 3.2 *Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda \geq 1$. If $F(z) \in SL^*_\beta(q)$ then $f(z) = L_a F(z) \in SL^*_\beta(q)$, where L_a is the Libera-Pascu integral operator defined by (1).*

Proof. From (1) we have

$$(1+a)F(z) = af(z) + zf'(z)$$

and, by using the linear operator $D_\lambda^{\beta+1}$, for $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we obtain

$$(1+a)D_\lambda^{\beta+1}F(z) = aD_\lambda^{\beta+1}f(z) + D_\lambda^{\beta+1}\left(z + \sum_{j=2}^{\infty} j a_j z^j\right)$$

$$= aD_\lambda^{\beta+1}f(z) + z + \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1} j a_j z^j$$

We have (see the proof of the above theorem)

$$z + \sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{\beta+1} a_j z^j = \frac{1}{\lambda}\left((\lambda-1)D_\lambda^{\beta+1}f(z) + D_\lambda^{\beta+2}f(z)\right)$$

Thus

$$(1+a)D_\lambda^{\beta+1}F(z) = aD_\lambda^{\beta+1}f(z) + \frac{1}{\lambda}\left((\lambda-1)D_\lambda^{\beta+1}f(z) + D_\lambda^{\beta+2}f(z)\right)$$

$$= \left(a + \frac{\lambda-1}{\lambda}\right) D_\lambda^{\beta+1}f(z) + \frac{1}{\lambda}D_\lambda^{\beta+2}f(z)$$

or

$$\lambda(1+a)D_\lambda^{\beta+1}F(z) = ((a+1)\lambda - 1)\, D_\lambda^{\beta+1}f(z) + D_\lambda^{\beta+2}f(z)\,.$$

Similarly, we obtain

$$\lambda(1+a)D_\lambda^{\beta}F(z) = ((a+1)\lambda - 1)\, D_\lambda^{\beta}f(z) + D_\lambda^{\beta+1}f(z)\,.$$

Then

$$\frac{D_\lambda^{\beta+1}F(z)}{D_\lambda^{\beta}F(z)} = \frac{\dfrac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} \cdot \dfrac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} + ((a+1)\lambda - 1) \cdot \dfrac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)}}{\dfrac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} + ((a+1)\lambda - 1)}\,.$$

With notation

$$\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} = p(z)\,,\; p(0) = 1\,,$$

we obtain

$$\frac{D_\lambda^{\beta+1}F(z)}{D_\lambda^{\beta}F(z)} = \frac{\dfrac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} \cdot p(z) + ((a+1)\lambda - 1) \cdot p(z)}{p(z) + ((a+1)\lambda - 1)} \tag{4}$$

We have (see the proof of the above theorem)

$$\lambda z p'(z) = \frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} \cdot \frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} - p(z)^2$$

$$= \frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} \cdot p(z) - p(z)^2\,.$$

Thus

$$\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} = \frac{1}{p(z)} \cdot \left(p(z)^2 + \lambda z p'(z)\right).$$

Then, from (4), we obtain

$$\frac{D_\lambda^{\beta+1}F(z)}{D_\lambda^{\beta}F(z)} = \frac{p(z)^2 + \lambda z p'(z) + ((a+1)\lambda - 1)\, p(z)}{p(z) + ((a+1)\lambda - 1)} = p(z) + \lambda \frac{z p'(z)}{p(z) + ((a+1)\lambda - 1)}\,,$$

where $a \in \mathbb{C}$, $Re\, a \geq 0$, $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda \geq 1$. From $F(z) \in SL^*_\beta(q)$ we have

$$p(z) + \frac{z p'(z)}{\frac{1}{\lambda}\left(p(z) + ((a+1)\lambda - 1)\right)} \prec q(z)\,,$$

where $a \in \mathbb{C}$, $Re\, a \geq 0$, $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 1$, and from her construction, we have $Re\, q(z) > 0$. In this conditions we have from Theorem 2.1 we obtain

$$p(z) \prec q(z)$$

or

$$\frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} \prec q(z)\,.$$

This means $f(z) = L_a F(z) \in SL^*_\beta(q)$.

For $\beta = n \in \mathbb{N}$ and $\lambda = 1$ we obtain

Corollary 3.2 *If $F(z) \in S^*_n(q)$ then $f(z) = L_a F(z) \in S^*_n(q)$, where L_a is the Libera-Pascu integral operator and by $S^*_n(q)$ we denote the class of n-starlike functions subordinate to the function $q(z)$ (see [5]).*

For $\beta = n \in \mathbb{N}$ we obtain

Corollary 3.3 *[2] Let $n \in \mathbb{N}$ and $\lambda \geq 1$. If $F(z) \in SL^*_n(q)$ then $f(z) = L_a F(z) \in SL^*_n(q)$, where L_a is the Libera-Pascu integral operator defined by (1).*

For $\beta \in \mathbb{R}$ and $\lambda = 1$ we obtain

Corollary 3.4 *[3] If $F(z) \in S^*_\beta(q)$ then $f(z) = L_a F(z) \in S^*_\beta(q)$, where L_a is the Libera-Pascu integral operator defined by (1).*

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