# Squaring operations in the Hermitian symmetric spaces

By

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## § 0. Introduction

In this paper we calculate the squaring operations in the mod 2 cohomology of the irreducible Hermitian symmetric spaces of compact type. Each of them is obtained as a quotient of an appropriate compact simple Lie group by the centralizer of an appropriate 1-dimensional torus, and they are divided into six classes:

| | | | | |
|---|---|---|---|---|
| AIII | $W(m,n)$ | $=$ | $U(m+n)/(U(m)\times U(n))$ | $(m,n\geq 1)$ |
| BDI | $Q_n$ | $=$ | $SO(n+2)/(SO(2)\times SO(n))$ | $(n\geq 3)$ |
| CI | | | $Sp(n)/U(n)$ | $(n\geq 3)$ |
| DIII | | | $SO(2n)/U(n)$ | $(n\geq 4)$ |
| EIII | | $=$ | $E_6/(Spin(10)\cdot T^1)$ | $(Spin(10)\cap T^1\cong \mathbf{Z}_4)$ |
| EVII | | $=$ | $E_7/(E_6\cdot T^1)$ | $(E_6\cap T^1\cong \mathbf{Z}_3)$ |

Their cohomology rings have been obtained by several authors:

$$H^*(W(m,n);\ \mathbf{Z})=\mathbf{Z}[a_1,\cdots,a_m,b_1,\cdots,b_n]\Big/\Big(\sum_{i+j=k}a_ib_j\,;\ k\geq 1\Big);$$

$$H^*(Q_n;\ \mathbf{Z})=\begin{cases}\mathbf{Z}[t,e]/(t^m-2e,\ e^2) & (n=2m-1),\\ \mathbf{Z}[t,s]/(t^{m+1}-2st,\ s^2-\delta_m st^m) & (n=2m);\end{cases}$$

$$H^*(Sp(n)/U(n);\ \mathbf{Z})=\mathbf{Z}[c_1,\cdots,c_n]\Big/\Big(\sum_{i+j=2k}(-1)^ic_ic_j\,;\ k\geq 1\Big);$$

$$H^*(SO(2n)/U(n);\ \mathbf{Z})=\mathbf{Z}[e_2,e_4,\cdots,e_{2n-2}]\Big/\Big(e_{4k}+\sum_{i=1}^{2k-1}(-1)^ie_{2i}e_{4k-2i}\Big)$$

(it should be understood that $e_{2j}=0$ if $j\geq n$);

$$H^*(\mathrm{EIII};\ \mathbf{Z})=\mathbf{Z}[t,w]/(t^9-3w^2t,\ w^3+15w^2t^4-9wt^8);$$

$$H^*(\mathrm{EVII};\ \mathbf{Z})=\mathbf{Z}[u,v,w]/(v^2-2wu,\ u^{14}-2A,\ w^2-2B),$$

where $\delta_m=\dfrac{1+(-1)^m}{2}$, $A$ and $B$ are appropriate integral cohomology classes, and

$$|a_i|=|b_i|=|c_i|=|e_{2i}|=2i,\quad |t|=2,\quad |s|=|e|=2m$$

$$|u|=2,\quad |v|=10\quad \text{and}\quad |w|=8\ \text{for EIII};\ =18\ \text{for EVII}.$$

For details see [1], [6], [7], [9] and § 1.4 in this paper.

For AIII and CI the cohomology rings are generated by Chern classes and for DIII by the suspension images of Stiefel-Whitney classes, whence the squaring operations are obtained by the Wu formula.

For BDI ($n = 2m$) we calculate in $H^*\left(\dfrac{SO(2m+2)}{U(1)\times U(m)}\right)$ through the homomorphism induced by the projection $\dfrac{SO(2m+2)}{U(1)\times U(m)} \to Q_{2m}$, and obtain

**Theorem 1.4.** $\mathrm{Sq}^{2i} s = \binom{m+1}{i} st^i \qquad (i \geq 0).$

**Corollary 1.5.** $\mathrm{Sq}^{2i} e = \binom{m+1}{i} et^i \qquad (i \geq 0).$

For the exceptional types EIII and EVII we calculate in $G/T$ ($T$ a maximal torus of $G$) using the fibration $G/T \xrightarrow{\widetilde{i}} B\widetilde{T} \to B\widetilde{G}$, where $B\widetilde{G}$ is the 4-connective cover of $BG$ and $B\widetilde{T}$ is defined in a similar way (for details see [3]). The results are

**Theorem 2.5.** (i) *In* EIII *we have*

$$\mathrm{Sq}^2 w' = w't, \quad \mathrm{Sq}^4 w' = t^6, \quad \mathrm{Sq}^8 w' = w'^2 \qquad (\textit{where } w' = w + t^4).$$

(ii) *In* EVII

$$\mathrm{Sq}^2 v = 0, \qquad \mathrm{Sq}^4 v = vu^2 + u^7, \qquad \mathrm{Sq}^8 v = w + vu^4 + u^9\,;$$
$$\mathrm{Sq}^2 w = u^{10}, \quad \mathrm{Sq}^4 w = vu^6 + u^{11}, \quad \mathrm{Sq}^8 w = vu^8 + u^{13}, \quad \mathrm{Sq}^{16} w = vu^{12}.$$

As an application we give in the final section the Stiefel-Whitney classes of EIII and EVII by use of the Wu classes.

Throughout the paper $H^*(\ \ )$ denotes exclusively the mod 2 cohomology (integral cohomology is always denoted by $H^*(\ \ ;\ \mathbf{Z})$). $\mathbf{F}_2$ denotes the prime field of characteristic 2. For an integral element $x$ its mod 2 reduction is denoted by $\rho(x)$, or simply by $x$ unless there is danger of confusion. $\sigma_i(x_1, \cdots, x_n)$ denotes the $i$-th elementary symmetric polynomial in $x_1, \cdots, x_n$ ($i \geq 0$). $\Delta(a_1, \cdots, a_n)$ denotes an algebra with simple system of generators $a_1, \cdots, a_n$.

## § 1. Classical types

First recall the Wu formula. Let $x_1, \cdots, x_n$ be elements of degree $d$ with $\mathrm{Sq}^i x_j = 0$ $(0 < i < d)$, and put $c_j = \sigma_j(x_1, \cdots, x_n)$ ($j \geq 0$). Then we have

$$\mathrm{Sq}^{di} c_j = \sum_{0 \leq r \leq i} \binom{j-i+r-1}{r} c_{j+r} c_{i-r}. \tag{1.1}$$

**1.1.** The Grassmannian $W(m,n)$. We have the fibration

$$W(m,n) \xrightarrow{\ \iota\ } BU(m) \times BU(n) \longrightarrow BU(m+n).$$

Put $a_i = \iota^*(c_i \times 1)$ and $b_i = \iota^*(1 \times c_i)$, where $c_i$ is the $i$-th universal Chern class in $BU(m)$ or $BU(n)$ ($i \geq 0$). Then

$$H^*(W(m,n)\,;\ \mathbf{Z}) = \mathbf{Z}[a_1, \cdots, a_m, b_1, \cdots, b_n] \Big/ \Big( \sum_{i+j=k} a_i b_j\,;\ \ k \geq 1 \Big),$$

whence by the naturality the operation of $\mathrm{Sq}^i$ is obtained from the Wu formula (1.1).

**1.2.** The space $Sp(n)/U(n)$. The operation of $\mathrm{Sq}^i$ is again obtained from the Wu formula since

$$H^*(Sp(n)/U(n)\,;\ \mathbf{Z}) = \mathbf{Z}[c_1, \cdots, c_n]\Big/\Big(\sum_{i+j=2k}(-1)^i c_i c_j\,;\ \ k \geq 1\Big),$$

where $c_i$ is the $i$-th Chern class ($i \geq 0$).

**1.3.** The space $SO(2n)/U(n)$. We extract from [6], Chap. 3, § 6. Using the fibration

$$SO(2n)/U(n) \xrightarrow{\ \iota\ } BU(n) \longrightarrow BSO(2n),$$

we have unique elements $e_{2i} = \dfrac{1}{2}\iota^* c_i \in H^{2i}(SO(2n)/U(n)\,;\ \mathbf{Z})$ $(1 \leq i \leq n-1)$. Let $p : SO(2n) \to SO(2n)/U(n)$ be the projection and $\sigma : H^*(BSO(2n)) \to H^*(SO(2n))$ the suspension. Then

(1.2) $\qquad p^*(e_{2i}) = \sigma(w_{2i+1}) \qquad$ ($w_j$ the Stiefel-Whitney classes);

(1.3) $\qquad H^*(SO(2n)/U(n)) = \Delta(e_2, e_4, \cdots, e_{2n-2}), \qquad e_{2i}^2 = e_{4i}$

(it should be understood that $e_{2j} = 0$ if $j \geq n$).

It follows that $p^*$ is injective. So we calculate in $SO(2n)$:

$$p^*\left(\mathrm{Sq}^{2i} e_{2k}\right) = \sigma\left(\mathrm{Sq}^{2i} w_{2k+1}\right) = \sigma\left(\sum_{0 \leq r \leq 2i} \binom{2k-2i+r}{r} w_{2k+1+r} w_{2i-r}\right)$$

by the Wu formula. Since $\sigma$ annihilates decomposables, we have

**Proposition 1.1.** $\mathrm{Sq}^{2i} e_{2k} = \dbinom{k}{i} e_{2k+2i} \qquad (i, k \geq 0).$

**1.4.** The complex quadric $Q_n = SO(n{+}2)/(SO(2) \times SO(n))$. We have the fibration

$$Q_n \xrightarrow{\ \iota\ } BSO(2) \times BSO(n) \longrightarrow BSO(n+2).$$

Let $t \in H^2(BSO(2)\,;\ \mathbf{Z})$ be the canonical generator, and put $t = \iota^*(t \times 1)$. Here we distinguish the two cases (a) $n$ is even, and (b) $n$ is odd.

(a) $n = 2m$. Let $\chi \in H^{2m}(BSO(2n);\ \mathbf{Z})$ be the Euler class and $p_i \in H^{2i}(BSO(n);\ \mathbf{Z})$ the $i$-th Pontrjagin class. Using the fibration above we see that

$$\iota^*(t \times \chi) = 0. \quad \iota^*(1 \times p_m) = (-1)^m t^{2m} \qquad \text{and}$$
$$\iota^*(1 \times \chi + t^m \times 1) \equiv 0 \mod (2).$$

Since $H^*(Q_{2m}\,;\ \mathbf{Z})$ has no torsion we have a unique element $s \in H^{2m}(Q_{2m}\,;\ \mathbf{Z})$ with $2s = \iota^*(1 \times \chi + t^m \times 1)$. Then the relations above yield

$$2st = t^{m+1} \qquad \text{and} \qquad 4s^2 = 2\,(1 + (-1)^m)\, st^m.$$

Considering the Serre spectral sequence for the fibration $SO(2m+2)/SO(2m) \to Q_{2m} \to BSO(2)$, we obtain

**Theorem 1.2.** $H^*(Q_{2m}\,;\ \mathbf{Z}) = \mathbf{Z}[t, s]/(t^{m+1} - 2st,\ s^2 - \delta_m st^m).$

Now consider the diagram:

$$
\begin{array}{ccccccc}
 & & SO(2m)/U(m) & & & & \\
 & & \downarrow & & & & \\
P_m(\mathbf{C}) = \dfrac{U(m+1)}{U(1)\times U(m)} & \xrightarrow{\ j\ } & \dfrac{SO(2m+2)}{U(1)\times U(m)} & \xrightarrow{\ q\ } & & SO(2m+2)/U(m+1) & \\
 & & \downarrow{\scriptstyle p} & \searrow{\scriptstyle \iota'} & & & \searrow \\
 & & & & BU(1)\times BU(m) & \xrightarrow{\ q'\ } & BU(m+1) \\
 & & & & \downarrow & & \\
 & & Q_{2m} = \dfrac{SO(2m+2)}{SO(2)\times SO(2m)} & \xrightarrow{\ \iota\ } & BSO(2)\times BSO(2m) & & 
\end{array}
\tag{1,4}
$$

Define $t,\ e_{2i}\ (i \geq 1)$ and $e \in H^*\left(\dfrac{SO(2m+2)}{U(1)\times U(m)}\,;\ \mathbf{Z}\right)$ by

$$t = p^*(t) = \iota'^*(c_1 \times 1), \qquad e_{2i} = q^*(e_{2i}) \quad \text{and} \quad e = \sum_{i=1}^{m}(-1)^i e_{2i} t^{m-i}.$$

Then

**Lemma 1.3.** (i) $q^*$ induces an isomorphism of algebras

$$H^*\left(\frac{SO(2m+2)}{U(1)\times U(m)}\,;\ \mathbf{Z}\right) \cong H^*\left(\frac{SO(2m+2)}{U(m+1)}\,;\ \mathbf{Z}\right)[t]\Big/\left(t^{m+1}+2te\right).$$

(ii) $$p^*(s) = \delta_m t^m + (-1)^m e.$$

*Proof.* By the definition of $t$ we see that $j^*t$ generates the ring $H^*(P_m(\mathbf{C})\,;\ \mathbf{Z})$. Since the spectral sequence for the row in (1.4) collapses, (i) holds as an isomorphism of modules. Let $c$, $c'$ and $c''$ be the total Chern classes of $BU(m)$, $BU(1)$ and $BU(m+1)$, respectively. Then $q'^*(c'') = c' \times c$. Applying $\iota'^*$ we have

$$1 + 2e_2 + 2e_4 + \cdots + 2e_{2m} = \left(\iota'^*(1\times c)\right)\cdot(1+t),$$

whence

$$\iota'^*(1\times c_m) = (-1)^m\,(t^m + 2e)\,.$$

Then $(t^m+2e)\,t = (-1)^m\,(\iota'^*(1\times c_m))\cdot t = 0$, which completes the proof of (i).

Next from the diagram (1.4) we have

$$2p^*(s) = \iota'^*(c_m\times 1) + t^m = 2\delta_m t^m + (-1)^m 2e,$$

which proves (ii) since $H^*\left(\dfrac{SO(2m+2)}{U(1)\times U(m)}\,;\ \mathbf{Z}\right)$ is torsion free. q.e.d.

**Theorem 1.4.** $\mathrm{Sq}^{2i}s = \dbinom{m+1}{i}\, st^i \qquad (i \geq 0).$

*Proof.* As is well known $\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots$ is an algebra homomorphism, and by 1.2 we can put $\mathrm{Sq}(s) = s\left(1 + \varepsilon_1 t + \varepsilon_2 t^2 + \cdots\right)$ $(\varepsilon_i \in \mathbf{F}_2)$. Applying $p^*$ we have

$$(1.5) \qquad \mathrm{Sq}\,(\delta_m t^m + e) = (\delta_m t^m + e)\,(1 + \varepsilon_1 t + \varepsilon_2 t^2 + \cdots).$$

Now put $A = H^*(P_m(\mathbf{C}) \times P_m(\mathbf{C})) = \mathbf{F}_2[a,b]/(a^{m+1}, b^{m+1})$. Then the correspondence $t \mapsto a,\ e_{2i} \mapsto b^i\ (i \geq 1)$ extends to an algebra homomorphism $\varphi : H^*\left(\dfrac{SO(2m+2)}{U(1) \times U(m)}\right) \to A$, which commutes with the squaring operations. Apply $\varphi$ to (1.5):

$$(1.6) \qquad \mathrm{Sq}\,((\delta_m + 1)a^m + c) = ((\delta_m + 1)a^m + c)\,(1 + \varepsilon_1 a + \varepsilon_2 a^2 + \cdots),$$

where $c = a^m + a^{m-1}b + \cdots + b^m$. First calculating in the quotient field of $\mathbf{F}_2[[a,b]]$, we obtain

$$\mathrm{Sq}(c) = \sum_{i=o}^{m} \binom{m+1}{i} \left(a^{m+i} + a^{m+i-1}b + \cdots + b^{m+i}\right) \cdot \sum_{j \geq 0} (a+b)^j,$$

and then in $A$ using the equalities $a^{m+i} + a^{m+i-1}b + \cdots + b^{m+i} = ca^i$ and $c(a+b) = 0$,

$$\mathrm{Sq}(c) = c(1+a)^{m+1}.$$

Comparing the coefficients of $a^m b^i$ in both sides of (1.6), we obtain $\varepsilon_i = \dbinom{m+1}{i}$, which proves the theorem. q.e.d.

(b) $n = 2m-1$. According to [6]

$$H^*(Q_{2m-1}\,;\ \mathbf{Z}) = \mathbf{Z}[t,e]/\left(t^m - 2e,\ e^2\right),$$

where $t$ is the same as ours. The inclusion $SO(2m+1) \subset SO(2m+2)$ $(X \mapsto X \oplus 1)$ yields a commutative diagram

$$\begin{array}{ccc} Q_{2m-1} & \xrightarrow{\quad f \quad} & Q_{2m} \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ BSO(2) \times BSO(2m-1) & \xrightarrow{\ f'\ } & BSO(2) \times BSO(2m). \end{array}$$

From $f'^*(1 \times \chi) = 0$ and $f'^*(t \times 1) = t \times 1$ it follows that $f^*(t) = t$ and $f^*(s) = e$, and we have

**Corollary 1.5.** $\mathrm{Sq}^{2i} e = \dbinom{m+1}{i} e t^i \qquad (i \geq 0).$

## § 2. Exceptional types

In this section $\ell = 6$ or $7$. Let $G$ be the simply connected exceptional Lie group of type $E_\ell$ and $T$ a maximal torus of $G$. Take the root system $\{\alpha_1, \cdots, \alpha_\ell\}$ as in [2] , and define $K$ to be the centralizer of the 1-dimensional torus defined by the equations $\alpha_i = 0$ $(i \neq \ell)$. Then the quotient space $G/K$ is the irreducible Hermitian symmetric space EIII $(\ell = 6)$ or EVII $(\ell = 7)$.

Consider the fibration $K/T \to G/T \xrightarrow{p} G/K$. By the classical theorem of Bott the odd dimensional parts of the cohomology of both the fibre and the base vanish. Hence the spectral sequence for the fibration collapses, and $p^* : H^*(G/K\,;\,A) \to H^*(G/T\,;\,A)$ is injective for any coefficient ring $A$. Therefore the action of $\mathrm{Sq}^i$ in $G/K$ is derived from that in $G/T$.

First we fix a system of generators of $H^*(BT;\,\mathbf{Z})$ after [7] and [9]. Let $\{w_1, \cdots, w_\ell\}$ be the fundamental weights of $G$. Being regarded as elements of $H^2(BT\,;\,\mathbf{Z})$, they form a basis of it. Let $R_j$ be the reflection in the plane $\alpha_j = 0$, and put

$$t_\ell = w_\ell, \quad t_i = R_{i+1}(t_{i+1})\,(\ell > i > 1), \quad t_1 = R_1(t_2) \quad \text{and} \quad c_i = \sigma_i(t_1, \cdots, t_\ell)\,(i \geq 0).$$

Then

$$H^*(BT\,;\,\mathbf{Z}) = \mathbf{Z}[t_1, \cdots, t_\ell, x]/(c_1 - 3x).$$

As the canonical mapping $i : G/T \to BT$ does not induce a surjection in $H^*(\ \ )$, we introduce $B\widetilde{G}$ the 4-connective fibre space over $BG$ to have commutative diagram with two fibrations

$$\begin{array}{ccccc} G/T & \xrightarrow{\ \widetilde{i}\ } & B\widetilde{T} & \longrightarrow & B\widetilde{G} \\ \Big\| & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow \\ G/T & \xrightarrow{\ i\ } & BT & \longrightarrow & BG \end{array}$$

In $H^*\left(B\widetilde{T}\,;\,\mathbf{Z}_{(2)}\right)$ we have new generators $g_i\,(i = 3, 5, 9)$ with

$$2g_3 = c_3, \quad 2g_5 = c_5' = c_5 + c_4c_1 \quad \text{and} \quad 2g_9 = c_9' = c_7c_1^2 + c_6c_1^3$$

(Note that the symbol $g^*$ is omitted here). We put

$$\gamma_3 = \rho(g_3), \quad \gamma_5 = \mathrm{Sq}^4\gamma_3 \quad \text{and} \quad \gamma_9 = \mathrm{Sq}^8\gamma_5 \qquad \in\ H^*\left(B\widetilde{T}\right).$$

Then

(2.1) $\quad \gamma_5 = \rho\left(g_5 + g_3c_1^2 + c_4c_1\right)$ and

$$\gamma_9 = \rho\left(g_9 + g_5\left(c_4 + c_1^4\right) + g_3\left(c_6 + c_4c_1^2 + c_1^6\right) + c_7c_1^2 + c_4^2c_1 + c_4c_1^5\right).$$

For details see [3]. Note that our $\gamma_i\,(i = 5, 9)$ are slightly different from those in [9].

Recall that the generator of maximum degree is $w$ in each case (see § 0). So it is sufficient for us to consider in the range of degree $\leq d_\ell$, where $d_6 = 14$ and $d_7 = 34$. Define polynomials

$$\begin{aligned} I_6 &= \gamma_3^2 + c_4c_1^2 + c_1^6, & I_8 &= c_6c_1^2 + c_4^2 + c_4c_1^4 + c_1^8, \\ I_{10} &= \gamma_5^2 + c_6c_1^4 + c_1^{10}, & I_{12} &= c_6^2 + c_6c_4c_1^2 + c_4^2c_1^4 + c_4c_1^8, \\ I_{14} &= c_7^2 + c_6c_4c_1^4 + c_6c_1^8 \end{aligned}$$

and sets

$$R_6 = \{c_2,\ c_3,\ c_5',\ I_6\}, \qquad R_7 = \{c_2,\ c_3,\ c_5',\ I_6,\ I_8,\ c_9',\ I_{10},\ I_{12},\ I_{14}\}$$

Then from § 3 in [4] we have

**Lemma** 2.1. (1) *Up to degree* $d_\ell$

$$H^*(G/T) = \begin{cases} \mathbf{F}_2[t_1, \cdots, t_6, \gamma_3]/(R_6) & (\ell = 6), \\ \mathbf{F}_2[t_1, \cdots, t_7, \gamma_3, \gamma_5, \gamma_9]/(R_7) & (\ell = 7). \end{cases}$$

(2) $\mathrm{Sq}^2\gamma_3 = c_4, \qquad \mathrm{Sq}^4\gamma_3 = \gamma_5 \ (= c_4c_1 + c_1^5 \ \ if \ \ \ell = 6)$ ;

$\mathrm{Sq}^2\gamma_5 = c_4c_1^2 + c_1^6, \quad \mathrm{Sq}^4\gamma_5 = c_7', \quad \mathrm{Sq}^8\gamma_5 = \gamma_9$ ;

$\mathrm{Sq}^2\gamma_9 = c_4c_1^6, \qquad \mathrm{Sq}^4\gamma_9 = 0, \quad \mathrm{Sq}^8\gamma_9 = c_7'c_6, \quad \mathrm{Sq}^{16}\gamma_9 = c_7'c_6c_4$

*where* $c_7' = c_7 + c_6c_1$.

Now we interpret the results in [7] and [9] to our situation.

(2.2) $$H^*(\mathrm{EIII}) = \mathbf{F}_2[t, w']/\left(w'^2t,\ w'^3 + t^{12}\right),$$

*where* $t,\ w' = w + t^4 \in H^*(E_6/T\,;\ \mathbf{Z})$ *satisfy*

$$t \equiv c_1 + t_1, \quad w' \equiv c_4 + \left(\gamma_3 + c_1^2t + c_1t^2\right)t \qquad \mathrm{mod}\ (2).$$

(2.3) $$H^*(\mathrm{EVII}) = \mathbf{F}_2[u, v, w]/\left(v^2,\ u^{14},\ w^2\right),$$

*where* $u = t_7,\ v,\ w \in H^*(E_7/T\,;\ \mathbf{Z})$ *satisfy*

$$2v \equiv \bar{c}_5 - \bar{c}_4\chi + \bar{c}_3\chi^2 - \bar{c}_2\chi^3 + 2\chi^5 + 2u^5 \qquad \mathrm{mod}\ (4),$$

$$\begin{aligned} 2w \equiv\ & \bar{c}_6\bar{c}_3 + \bar{c}_5\bar{c}_4 + 2\bar{c}_5\bar{c}_2^2 + \left(2\bar{c}_6\bar{c}_2 - \bar{c}_4^2 + 2\bar{c}_4\bar{c}_2^2\right)\chi \\ & - \left(\bar{c}_5\bar{c}_2 - \bar{c}_4\bar{c}_3 + 2\bar{c}_3\bar{c}_2^2\right)\chi^2 + \left(\bar{c}_6 + 2\bar{c}_2^3\right)\chi^3 - (\bar{c}_5 + \bar{c}_3\bar{c}_2)\chi^4 \\ & - \left(\bar{c}_4 - \bar{c}_2^2\right)\chi^5 - \bar{c}_3\chi^6 - \bar{c}_2\chi^7 + 2\chi^9 + 2vu^4 \qquad \mathrm{mod}\ (4) \end{aligned}$$

*with* $\chi = \dfrac{1}{3}c_1 - u$ *and* $\bar{c}_i = \sigma_i\left(t_1 - \dfrac{1}{3}u,\ \cdots,\ t_6 - \dfrac{1}{3}u\right)\ (i \geq 0)$.

We must describe $v$ and $w$ modulo 2 in terms of the $t_i$ and the $\gamma_j$. For EVII the results are a little complicated. So we calculate modulo $(c_1)$. Note that $c_2 \equiv 0 \bmod (4)$ (see [8]) and recall the relation (2.1). Then after some calculations modulo $(4,\ 2c_1)$ we obtain the following:

**Lemma 2.2.** *Modulo* $(c_1)$

$$t \equiv t_1, \qquad w' \equiv \gamma_3t_1 + c_4 \qquad (\ell = 6);$$

$$u \equiv t_7, \qquad v \equiv \gamma_5 + \gamma_3t_7^2 + c_4t_7, \qquad w \equiv \gamma_9 + \gamma_3t_7^6 + c_6t_7^3 \qquad (\ell = 7).$$

Fortunately we have

**Lemma 2.3.** (i) *In* $H^*(G/T)$ *the ideal* $(c_1)$ *is closed under the operation of the* $\mathrm{Sq}^i$.

(ii) *Up to degree* $d_\ell$ *the composition of* $p^* : H^*(G/K) \to H^*(G/T)$ *and the projection* $\pi : H^*(G/T) \to H^*(G/T)/(c_1)$ *is injective.*

*Proof.* (i) This follows from the Cartan formula.

(ii) For $\ell = 6$ put $b_i = \sigma_i(t_2, \cdots, t_6)\ (i \geq 0)$, and let

$$A = \mathbf{F}_2[t_1, \cdots, t_6, \gamma_3] \quad \text{and} \quad B = \mathbf{F}_2[t_1, b_1, \cdots, b_5, \gamma_3].$$

Then $A$ is free as a $B$-module, and hence for any subset $R \subset B$ the canonical map $B/BR \to A/AR$ is injective, where $BR$ denotes the ideal generated by $R$ in $B$, and similarly for $AR$.

Note that $B$ coincides with $\mathbf{F}_2[t_1, c_1, \cdots, c_5, \gamma_3]$. So it contains the set $R = \{c_1\} \cup R_6 = \{c_1, c_2, c_3, c_5', I_6\}$, and $B/BR$ can be regarded as a subset of $A/AR = H^*(G/T)/(c_1)$. Then by 2.2 the image of $\pi \circ p^*$ is contained in $B/BR = \mathbf{F}_2[t_1, c_4, \gamma_3]/(\gamma_3^2)$, and it is easily seen that up to degree $d_\ell$ the map $H^*(G/T) \to B/BR$ is injective.

Similarly for $\ell = 7$. q.e.d.

Now the operation of the $\mathrm{Sq}^i$ are obtained from 2.2 and 2.1,(2) by calculating modulo $(c_1)$. The results are as follows:

**Theorem 2.4.** (i) *In* $H^*(\mathrm{EIII})$ *we have*

$$\mathrm{Sq}^2 w' = w't, \qquad \mathrm{Sq}^4 w' = t^6, \qquad \mathrm{Sq}^8 w' = w'^2$$

(ii) *In* $H^*(\mathrm{EVII})$

$$\mathrm{Sq}^2 v = 0, \qquad \mathrm{Sq}^4 v = vu^2 + u^7, \qquad \mathrm{Sq}^8 v = w + vu^4 + u^9\,;$$

$$\mathrm{Sq}^2 w = u^{10}, \quad \mathrm{Sq}^4 w = vu^6 + u^{11}, \quad \mathrm{Sq}^8 w = vu^8 + u^{13}, \quad \mathrm{Sq}^{16} w = vu^{12}.$$

## § 3. Stiefel-Whitney classes

For the Hermitian symmetric spaces of classical type the Chern classes have been obtained in [1], whence so have the Stiefel -Whitney classes by mod 2 reduction. In this section as an application of the previous section we give the Stiefel-Whitney classes of EIII and EVII by using the Wu classes.

For a compact $n$-manifold $M$ the $i$-th Wu class $u_i \in H^i(M)$ is characterised by

$$u_i \cdot x = Sq^i x \qquad \text{for any } x \in H^{n-i}(M).$$

Using the Wu classes the Stiefel-Whitney classes $w_i$ are given by $w_i = \sum_{j \geq 0} \mathrm{Sq}^{i-j} u_j$, or equivalently

$$W = \mathrm{Sq} U, \tag{3.1}$$

where $W = \sum w_i$, and $U = \sum u_i$ (see [5], for example).

From the previous section it follows that

(3.2) $H^*(\mathrm{EIII})$ *has* $\{w'^n t^i,\ w'^2 \mid n = 0, 1\,;\ 0 \leq i \leq 12\}$ *as a basis, and*

$$\mathrm{Sq}^{2r} w' t^{12-r} = w' t^{12}\ (r = 0, 2, 6), \quad \mathrm{Sq}^{16} w^2 = w t^{12},$$

$$\mathrm{Sq}^{2r} b = 0 \quad \textit{for the other } b \textit{ of degree } 32 - 2r \textit{ in the basis};$$

(3.3) $H^*(\mathrm{EVII})$ *has* $\{w^n v^m u^i \mid n, m = 0, 1\,;\ 0 \leq i \leq 13\}$ *as a basis, and*

$$\mathrm{Sq}^{2r} wvu^{13-r} = wvu^{13}\ (r = 0, 4, 12),$$

$$\mathrm{Sq}^{2r} b = 0 \quad \textit{for the other } b \textit{ of degree } 54 - 2r \textit{ in the basis.}$$

Therefore

**Theorem 3.1.** *The non-zero Wu classes are given as follows*:

*for* EIII, $\quad u_0 = 1, \quad u_4 = t^2, \quad u_{12} = t^6, \quad u_{16} = w'^2\,;$

*for* EVII, $\quad u_0 = 1, \quad u_8 = u^4, \quad u_{24} = u^{12}.$

Then by use of the formula (3.1) we obtain

**Corollary 3.2.** *The total Stiefel-Whitney class $W$ is given as follows*:

*for* EIII, $W = 1 + t^2 + t^4 + t^6 + (w'^2 + t^8) + t^{10} + w't^{12}$;

*for* EVII, $W = 1 + u^4 + u^8 + u^{12}$.

Remark 3.3. For EIII the result coincides with the mod 2 reduction of that in [8].

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