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<td>Author(s)</td>
<td>ISHITOYA, Kiminao</td>
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Squaring operations in the Hermitian symmetric spaces

By

Kiminao ISHTOYA

§ 0. Introduction

In this paper we calculate the squaring operations in the mod 2 cohomology of the irreducible Hermitian symmetric spaces of compact type. Each of them is obtained as a quotient of an appropriate compact simple Lie group by the centralizer of an appropriate 1-dimensional torus, and they are divided into six classes:

<table>
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<th>Class</th>
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<tr>
<td>AIII</td>
<td>$W(m, n) = U(m+n)/(U(m) \times U(n))$ (m, n ≥ 1)</td>
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<tr>
<td>BDI</td>
<td>$Q_n = SO(n+2)/(SO(2) \times SO(n))$ (n ≥ 3)</td>
</tr>
<tr>
<td>CI</td>
<td>$Sp(n)/U(n)$ (n ≥ 3)</td>
</tr>
<tr>
<td>DIII</td>
<td>$SO(2n)/U(n)$ (n ≥ 4)</td>
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<tr>
<td>EIII</td>
<td>$E_6/(Spin(10) \cdot T^1)$ (Spin(10) ∩ T^1 ≃ Z_4)</td>
</tr>
<tr>
<td>EVII</td>
<td>$E_7/(E_6 \cdot T^1)$ (E_6 ∩ T^1 ≃ Z_3)</td>
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Their cohomology rings have been obtained by several authors:

- $H^*(W(m, n); \mathbb{Z}) = \mathbb{Z}[a_1, \ldots, a_m, b_1, \ldots, b_n]/\left( \sum_{i+j=k} a_ib_j ; k ≥ 1 \right)$
- $H^*(Q_n; \mathbb{Z}) = \begin{cases} \mathbb{Z}[t, e]/(t^m - 2e, e^2) & (n = 2m - 1), \\ \mathbb{Z}[t, s]/(t^{m+1} - 2st, s^2 - \delta_mst^m) & (n = 2m) \end{cases}$
- $H^*(Sp(n)/U(n); \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n]/\left( \sum_{i+j=2k} (-1)^i c_ic_j ; k ≥ 1 \right)$
- $H^*(SO(2n)/U(n); \mathbb{Z}) = \mathbb{Z}[e_2, e_4, \ldots, e_{2n-2}]/\left( e_{4k} + \sum_{i=1}^{2k-1} (-1)^i e_2ie_{4k-2i} \right)$
  (it should be understood that $e_{2j} = 0$ if $j ≥ n$)
- $H^*(EIII; \mathbb{Z}) = \mathbb{Z}[t, w]/(t^9 - 3w^2t, w^3 + 15w^2t^4 - 9w^8)$
- $H^*(EVII; \mathbb{Z}) = \mathbb{Z}[u, v, w]/(v^2 - 2wu, u^{14} - 2A, w^2 - 2B)$

where $\delta_m = \frac{1 + (-1)^m}{2}$, $A$ and $B$ are appropriate integral cohomology classes, and

- $|a_i| = |b_i| = |c_i| = |e_{2i}| = 2i$, $|t| = 2$, $|s| = |e| = 2m$
- $|u| = 2$, $|v| = 10$ and $|w| = 8$ for EIII; $= 18$ for EVII.

For details see [1], [6], [7], [9] and § 1.4 in this paper.

For AIII and CI the cohomology rings are generated by Chern classes and for DIII by the suspension images of Stiefel-Whitney classes, whence the squaring operations are obtained by the Wu formula.
For BDI \((n = 2m)\) we calculate in \(H^* \left( \frac{SO(2m + 2)}{U(1) \times U(m)} \right)\) through the homomorphism induced by the projection \(\frac{SO(2m + 2)}{U(1) \times U(m)} \to Q_{2m}\), and obtain

**Theorem 1.4.** \(\text{Sq}^{2i} s = \binom{m+1}{i} s^i \) \((i \geq 0)\).

**Corollary 1.5.** \(\text{Sq}^{2i} c = \binom{m+1}{i} c^i \) \((i \geq 0)\).

For the exceptional types \(\text{EI} \text{II}\) and \(\text{EV} \text{II}\) we calculate in \(G/T \) \((T \text{ a maximal torus of } G)\) using the fibration \(G/T \xrightarrow{\iota} \tilde{B}T \to \tilde{B}G\), where \(\tilde{B}G\) is the 4-connective cover of \(B G\) and \(\tilde{B}T\) is defined in a similar way (for details see [3]). The results are

**Theorem 2.5.** (i) In \(\text{EI} \text{II}\) we have
\[\text{Sq}^2 w' = w't, \quad \text{Sq}^4 w' = t^6, \quad \text{Sq}^8 w' = w'^2 \quad \text{(where } w' = w + t^4).\]

(ii) In \(\text{EV} \text{II}\)
\[\text{Sq}^2 v = 0, \quad \text{Sq}^4 v = vu^2 + u^7, \quad \text{Sq}^8 v = w + vu^4 + u^9; \quad \text{Sq}^2 w = u^{10}, \quad \text{Sq}^4 w = vu^6 + u^{11}, \quad \text{Sq}^8 w = vu^8 + u^{13}, \quad \text{Sq}^{16} w = vu^{12}.\]

As an application we give in the final section the Stiefel-Whitney classes of \(\text{EI} \text{II}\) and \(\text{EV} \text{II}\) by use of the Wu classes.

Throughout the paper \(H^*(\quad )\) denotes exclusively the mod 2 cohomology (integral cohomology is always denoted by \(H^*(\quad ; \mathbb{Z})\)). \(\mathbb{F}_2\) denotes the prime field of characteristic 2. For an integral element \(x\) its mod 2 reduction is denoted by \(\rho(x)\), or simply by \(x\) unless there is danger of confusion. \(\sigma_i(x_1, \ldots, x_n)\) denotes the \(i\)-th elementary symmetric polynomial in \(x_1, \ldots, x_n\) \((i \geq 0)\). \(\Delta(a_1, \ldots, a_n)\) denotes an algebra with simple system of generators \(a_1, \ldots, a_n\).

**§ 1. Classical types**

First recall the Wu formula. Let \(x_1, \ldots, x_n\) be elements of degree \(d\) with \(\text{Sq}^i x_j = 0 \) \((0 < i < d)\), and put \(c_j = \sigma_j(x_1, \ldots, x_n) \) \((j \geq 0)\). Then we have
\begin{equation}
(1.1) \quad \text{Sq}^d c_j = \sum_{0 \leq r \leq i} \binom{j + r - i - 1}{r} c_{j+r} c_{i-r}.
\end{equation}

**1.1. The Grassmannian \(W(m, n)\).** We have the fibration
\[W(m, n) \xrightarrow{t} BU(m) \times BU(n) \longrightarrow BU(m+n).\]

Put \(a_i = t^*(c_i \times 1)\) and \(b_i = t^*(1 \times c_i)\), where \(c_i\) is the \(i\)-th universal Chern class in \(BU(m)\) or \(BU(n)\) \((i \geq 0)\). Then
\[H^*(W(m, n) ; \mathbb{Z}) = \mathbb{Z}[a_1, \ldots, a_m, b_1, \ldots, b_n] \left/ \left( \sum_{i+j=k} a_i b_j ; \ k \geq 1 \right) \right.,\]
whence by the naturality the operation of \(\text{Sq}^i\) is obtained from the Wu formula (1.1).
1.2. The space \( Sp(n)/U(n) \). The operation of \( Sq^i \) is again obtained from the Wu formula since

\[
H^*(Sp(n)/U(n); \mathbb{Z}) = \mathbb{Z}[c_1, \cdots, c_n]/\left( \sum_{i+j=2k} (-1)^i c_ic_j; \ k \geq 1 \right),
\]

where \( c_i \) is the \( i \)-th Chern class (\( i \geq 0 \)).

1.3. The space \( SO(2n)/U(n) \). We extract from \([6]\), Chap. 3, § 6. Using the fibration

\[
SO(2n)/U(n) \xrightarrow{\iota} BU(n) \xrightarrow{\pi} BSO(2n),
\]

we have unique elements \( e_{2i} = \frac{1}{2} \iota^* c_i \in H^{2i}(SO(2n)/U(n); \mathbb{Z}) \) (\( 1 \leq i \leq n - 1 \)). Let \( p: SO(2n) \to SO(2n)/U(n) \) be the projection and \( \sigma: H^*(BSO(2n)) \to H^*(SO(2n)) \) the suspension. Then

\[
\begin{align*}
(1.2) & \quad p^*(e_{2i}) = \sigma(w_{2i+1}) \quad (w_j \text{ the Stiefel-Whitney classes}); \\
(1.3) & \quad H^*(SO(2n)/U(n)) = \Delta(e_2, e_4, \cdots, e_{2n-2}), \quad e_{2i}^2 = e_{4i}
\end{align*}
\]

(it should be understood that \( e_{2j} = 0 \) if \( j \geq n \)).

It follows that \( p^* \) is injective. So we calculate in \( SO(2n) \):

\[
p^*\left(Sq^{2i}e_{2k}\right) = \sigma\left(Sq^{2i}w_{2k+1}\right) = \sigma\left( \sum_{0 \leq r \leq 2i} \binom{2k-2i+r}{r} w_{2k+1+r}w_{2i-r} \right)
\]

by the Wu formula. Since \( \sigma \) annihilates decomposables, we have

**Proposition 1.1.** \( Sq^{2i}e_{2k} = \binom{k}{i} e_{2k+2i} \) (\( i, k \geq 0 \)).

1.4. The complex quadric \( Q_n = SO(n+2)/(SO(2) \times SO(n)) \). We have the fibration

\[
Q_n \xrightarrow{\iota} BSO(2) \times BSO(n) \xrightarrow{\pi} BSO(n+2).
\]

Let \( t \in H^2(BSO(2); \mathbb{Z}) \) be the canonical generator, and put \( t = \iota^*(t \times 1) \). Here we distinguish the two cases (a) \( n \) is even, and (b) \( n \) is odd.

(a) \( n = 2m \). Let \( \chi \in H^{2m}(BSO(2n); \mathbb{Z}) \) be the Euler class and \( p_i \in H^{2i}(BSO(n); \mathbb{Z}) \) the \( i \)-th Pontrjagin class. Using the fibration above we see that

\[
\begin{align*}
t^*(t \times \chi) &= 0, \quad t^*(1 \times p_m) = (-1)^m t^{2m} \quad \text{and} \\
t^*(1 \times \chi + t^{m} \times 1) &\equiv 0 \mod (2).
\end{align*}
\]

Since \( H^*(Q_{2m}; \mathbb{Z}) \) has no torsion we have a unique element \( s \in H^{2m}(Q_{2m}; \mathbb{Z}) \) with \( 2s = t^*(1 \times \chi + t^{m} \times 1) \). Then the relations above yield

\[
2st = t^{m+1} \quad \text{and} \quad 4s^2 = 2(1 + (-1)^m) st^m.
\]

Considering the Serre spectral sequence for the fibration \( SO(2m+2)/SO(2m) \to Q_{2m} \to BSO(2) \), we obtain

**Theorem 1.2.** \( H^*(Q_{2m}; \mathbb{Z}) = \mathbb{Z}[t, s]/(t^{m+1} - 2st, s^2 - \delta_m st^m) \).
Now consider the diagram:

\[
P_m(C) = \frac{U(m+1)}{U(1) \times U(m)} \xrightarrow{j} \frac{SO(2m+2)}{U(1) \times U(m)} \xrightarrow{q} \frac{SO(2m+2)}{U(m+1)} \xrightarrow{\iota'} BU(1) \times BU(m) \xrightarrow{q'} BU(m+1)
\]

(1.4)

\[
Q_{2m} = \frac{SO(2m+2)}{SO(2) \times SO(2m)} \xrightarrow{\iota} BSO(2) \times BSO(2m)
\]

Define \( t, e_{2i} \) \((i \geq 1)\) and \( e \in H^* \left( \frac{SO(2m+2)}{U(1) \times U(m)} ; \mathbb{Z} \right) \) by

\[
t = p^* (t) = \iota^* (c_1 \times 1), \quad e_{2i} = q^* (e_{2i}) \quad \text{and} \quad e = \sum_{i=1}^{m} (-1)^i e_{2i} t^{m-i}.
\]

Then

\textbf{Lemma 1.3.} (i) \( q^* \) induces an isomorphism of algebras

\[
H^* \left( \frac{SO(2m+2)}{U(1) \times U(m)} ; \mathbb{Z} \right) \cong H^* \left( \frac{SO(2m+2)}{U(m+1)} ; \mathbb{Z} \right) \left[ t \right]/ \left( t^{m+1} + 2te \right).
\]

(ii) \( p^* (s) = \delta_m t^m + (-1)^m e. \)

\textit{Proof.} By the definition of \( t \) we see that \( j^* t \) generates the ring \( H^* (P_m(C) ; \mathbb{Z}) \). Since the spectral sequence for the row in (1.4) collapses, (i) holds as an isomorphism of modules. Let \( c, c' \) and \( c'' \) be the total Chern classes of \( BU(m) \), \( BU(1) \) and \( BU(m+1) \), respectively. Then \( q^*(c'') = c' \times c \). Applying \( \iota^* \) we have

\[
1 + 2e_2 + 2e_4 + \cdots + 2e_{2m} = (\iota^*(1 \times c)) \cdot (1 + t),
\]

whence

\[
\iota^*(1 \times c_m) = (-1)^m (t^{m} + 2e).
\]

Then \( (t^{m} + 2e) t = (-1)^m (\iota^*(1 \times c_m)) \cdot t = 0, \) which completes the proof of (i).

Next from the diagram (1.4) we have

\[
2p^*(s) = \iota^*(c_m \times 1) + t^m = 2\delta_m t^m + (-1)^m 2e,
\]

which proves (ii) since \( H^* \left( \frac{SO(2m+2)}{U(1) \times U(m)} ; \mathbb{Z} \right) \) is torsion free. q.e.d.

\textbf{Theorem 1.4.} \( Sq^{2i} = \binom{m+1}{i} st^i \) \((i \geq 0)\).

\textit{Proof.} As is well known \( Sq = 1 + Sq^1 + Sq^2 + \cdots \) is an algebra homomorphism, and by 1.2 we can put \( Sq(s) = s (1 + \varepsilon_1 t + \varepsilon_2 t^2 + \cdots) \) \((\varepsilon_i \in F_2)\). Applying \( p^* \) we have
(1.5) \[ \text{Sq}(\delta_m t^m + e) = (\delta_m t^m + e)(1 + \varepsilon_1 t + \varepsilon_2 t^2 + \cdots). \]

Now put \( A = H^*(P_m(C) \times P_m(C)) = \mathbb{F}_2[a, b]/(a^{m+1}, b^{m+1}) \). Then the correspondence \( t \mapsto a, \ v \mapsto b^i (i \geq 1) \) extends to an algebra homomorphism \( \varphi : H^*(SO(2m + 2) / U(1) \times U(m)) \rightarrow A \), which commutes with the squaring operations. Apply \( \varphi \) to (1.5):

(1.6) \[ \text{Sq}((\delta_m + 1)a^m + c) = ((\delta_m + 1)a^m + c)(1 + \varepsilon_1 a + \varepsilon_2 a^2 + \cdots), \]

where \( c = a^m + a^{m-1}b + \cdots + b^m \). First calculating in the quotient field of \( \mathbb{F}_2[a, b] \), we obtain

\[ \text{Sq}(c) = \sum_{i=0}^{m} \binom{m+1}{i} (a^{m+i} + a^{m+i-1}b + \cdots + b^{m+i}) \cdot \sum_{j=0}^{a+b} (a+b)^j, \]

and then in \( A \) using the equalities \( a^{m+i} + a^{m+i-1}b + \cdots + b^{m+i} = ca^i \) and \( c(a+b) = 0 \),

\[ \text{Sq}(c) = c(1+a)^{m+1}. \]

Comparing the coefficients of \( a^m b^i \) in both sides of (1.6), we obtain \( \varepsilon_i = \binom{m+1}{i} \), which proves the theorem.

(b) \( n = 2m - 1 \). According to [6]

\[ H^*(Q_{2m-1}; \mathbb{Z}) = \mathbb{Z}[t, e]/(t^m - 2e, e^2), \]

where \( t \) is the same as ours. The inclusion \( SO(2m+1) \subset SO(2m+2) (X \mapsto X \oplus 1) \) yields a commutative diagram

\[
\begin{array}{ccc}
Q_{2m-1} & \xrightarrow{f} & Q_{2m} \\
\downarrow t & & \downarrow t \\
BSO(2) \times BSO(2m-1) & \xrightarrow{f'} & BSO(2) \times BSO(2m).
\end{array}
\]

From \( f'^*(1 \times \chi) = 0 \) and \( f'^*(t \times 1) = t \times 1 \) it follows that \( f^*(t) = t \) and \( f^*(s) = e \), and we have

**Corollary 1.5.** \( \text{Sq}^i e = \binom{m+1}{i} e t^i \) \( (i \geq 0) \).

**§ 2. Exceptional types**

In this section \( \ell = 6 \) or 7. Let \( G \) be the simply connected exceptional Lie group of type \( E_\ell \) and \( T \) a maximal torus of \( G \). Take the root system \( \{\alpha_1, \cdots, \alpha_\ell\} \) as in [2], and define \( K \) to be the centralizer of the 1-dimensional torus defined by the equations \( \alpha_i = 0 \) \( (i \neq \ell) \). Then the quotient space \( G/K \) is the irreducible Hermitian symmetric space \( E_{III} (\ell = 6) \) or \( E_{VII} (\ell = 7) \).

Consider the fibration \( K/T \rightarrow G/T \xrightarrow{p} G/K \). By the classical theorem of Bott the odd dimensional parts of the cohomology of both the fibre and the base vanish. Hence
the spectral sequence for the fibration collapses, and \( p^* : H^*(G/K; A) \to H^*(G/T; A) \) is injective for any coefficient ring \( A \). Therefore the action of \( Sq^i \) in \( G/K \) is derived from that in \( G/T \).

First we fix a system of generators of \( H^*(BT; \mathbb{Z}) \) after \cite{7} and \cite{9}. Let \( \{w_1, \ldots, w_\ell\} \) be the fundamental weights of \( G \). Being regarded as elements of \( H^2(BT; \mathbb{Z}) \), they form a basis of it. Let \( R_j \) be the reflection in the plane \( \alpha_j = 0 \), and put

\[
t_\ell = w_\ell, \quad t_i = R_{i+1}(t_{i+1}) (\ell > i > 1), \quad t_1 = R_1(t_2) \quad \text{and} \quad c_i = \sigma_i(t_1, \ldots, t_\ell) (i \geq 0).
\]

Then

\[
H^*(BT; \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_\ell, x]/(c_1 - 3x).
\]

As the canonical mapping \( i : G/T \to BT \) does not induce a surjection in \( H^*(\quad) \), we introduce \( \tilde{B}G \) the 4-connective fibre space over \( BG \) to have commutative fibre diagram with two fibrations

\[
\begin{array}{ccc}
G/T & \xrightarrow{i} & \tilde{B}T \\
\downarrow & & \downarrow \gamma \\
G/T & \xrightarrow{i} & BT \\
\end{array}
\]

In \( H^*\left(\tilde{B}T; \mathbb{Z}\right) \) we have new generators \( g_i \) \((i = 3, 5, 9)\) with

\[
2g_3 = c_3, \quad 2g_5 = c_5 + c_4c_1 \quad \text{and} \quad 2g_9 = c_9 = c_7c_3^2 + c_6c_1^3
\]

(Note that the symbol \( g^* \) is omitted here). We put

\[
\gamma_3 = \rho(g_3), \quad \gamma_5 = Sq^4\gamma_3 \quad \text{and} \quad \gamma_9 = Sq^8\gamma_5 \in H^*\left(\tilde{B}T\right).
\]

Then

\[
(2.1) \quad \gamma_5 = \rho(g_5 + g_3c_4^2 + c_4c_1) \quad \text{and}
\]

\[
\gamma_9 = \rho(g_9 + g_5(c_4 + c_1) + g_3(c_6 + c_4c_1^2 + c_1^6) + c_7c_3^2 + c_3^3c_1 + c_4c_1^5).
\]

For details see \cite{3}. Note that our \( \gamma_i \) \((i = 5, 9)\) are slightly different from those in \cite{9}.

Recall that the generator of maximum degree is \( w \) in each case (see §.0). So it is sufficient for us to consider in the range of degree \( \leq d_\ell \), where \( d_6 = 14 \) and \( d_7 = 34 \).

Define polynomials

\[
I_6 = \gamma_3^2 + c_4c_1^2 + c_1^6, \quad I_8 = c_6c_3^2 + c_1^2 + c_4c_1^4 + c_1^8, \\
I_{10} = \gamma_3^2 + c_6c_1^2 + c_1^4, \quad I_{12} = c_3^2 + c_6c_4c_1^2 + c_4c_1^4 + c_4c_8, \\
I_{14} = c_3^2 + c_6c_4c_1^4 + c_6c_1^8
\]

and sets

\[
R_6 = \{c_2, c_3, c_5, I_6\}, \quad R_7 = \{c_2, c_3, c_5, I_6, I_8, c_9, I_{10}, I_{12}, I_{14}\}
\]

Then from §.3 in \cite{4} we have

\textbf{Lemma 2.1.} (1) \textit{Up to degree } \( d_\ell \)

\[
H^*(G/T) = \begin{cases} 
F_2[t_1, \ldots, t_6, \gamma_3]/(R_6) & (\ell = 6), \\
F_2[t_1, \ldots, t_7, \gamma_3, \gamma_5, \gamma_9]/(R_7) & (\ell = 7).
\end{cases}
\]
and \( u, v, w \).

Recall the relation (2.1). Then after some calculations modulo (4) to our situation.

(2.2) \[ H^*(\text{EIII}) = F_2[t, w']/ (w'^2 t, w'^3 + t^{12}), \]

where \( t, w' = w + t^4 \in H^*(E_6/T; \mathbb{Z}) \) satisfy

\[ t \equiv c_1 + t_1, \quad w' \equiv c_4 + (\gamma_3 + c_1^2 t_1 + c_4 t^2) t \pmod{(2)}. \]

(2.3) \[ H^*(\text{EVII}) = F_2[u, v, w]/ (v^2, v^{14}, w^2), \]

where \( u = t_7, \quad v, \quad w \in H^*(E_7/T; \mathbb{Z}) \) satisfy

\[ 2v \equiv \bar{c}_3 - \bar{c}_1 \chi + \bar{c}_3 \chi^2 - 2 \bar{c}_2 \chi^3 + 2 \chi^5 + 2u^5 \pmod{(4)}, \]
\[ 2w \equiv \bar{c}_6 \bar{c}_2 + \bar{c}_5 \bar{c}_4 + 2\bar{c}_5 \bar{c}_2^2 + (2\bar{c}_6 \bar{c}_2 - \bar{c}_1^2 + 2\bar{c}_4 \bar{c}_2^2) \chi \]
\[ - (\bar{c}_5 \bar{c}_2 - \bar{c}_4 \bar{c}_3 + 2\bar{c}_5 \bar{c}_2^2) \chi^2 + (\bar{c}_6 + 2\bar{c}_2) \chi^3 - (\bar{c}_5 + \bar{c}_3 \bar{c}_2) \chi^4 \]
\[ - (\bar{c}_4 - \bar{c}_2^2) \chi^5 - \bar{c}_3 \chi^6 - \bar{c}_2 \chi^7 + 2 \chi^9 + 2vu^4 \pmod{(4)} \]

with \( \chi = \frac{1}{3} c_1 - u \) and \( \bar{c}_i = \sigma_i \left( t_1 - \frac{1}{3} u, \cdots, t_6 - \frac{1}{3} u \right) \) \( (i \geq 0) \).

We must describe \( v \) and \( w \) modulo 2 in terms of the \( t_i \) and the \( \gamma_j \). For EVII the results are a little complicated. So we calculate modulo \((c_1)\). Note that \( c_2 \equiv 0 \pmod{(4)} \) (see [8]) and recall the relation (2.1). Then after some calculations modulo \((4, 2c_1)\) we obtain the following:

**Lemma 2.2.** Modulo \((c_1)\)

\[
\begin{align*}
t &\equiv t_1, & w' &\equiv \gamma_3 t_1 + c_4 & (\ell = 6); \\
u &\equiv t_7, & v &\equiv \gamma_5 + \gamma_3 t_2^2 + c_4 t_7, & w &\equiv \gamma_9 + \gamma_3 t_1^5 + c_6 t_3^2 & (\ell = 7).
\end{align*}
\]

Fortunately we have

**Lemma 2.3.** \((i)\) In \( H^*(G/T) \) the ideal \((c_1)\) is closed under the operation of the \( \text{Sq}^i \).

(ii) Up to degree \( d_\ell \) the composition of \( p^* : H^*(G/K) \to H^*(G/T) \) and the projection \( \pi : H^*(G/T) \to H^*(G/T)/(c_1) \) is injective.

**Proof.** \((i)\) This follows from the Cartan formula.

(ii) For \( \ell = 6 \) put \( b_i = \sigma_i(t_2, \cdots, t_6) \) \((i \geq 0)\), and let

\[ A = F_2[t_1, \cdots, t_6, \gamma_3] \quad \text{and} \quad B = F_2[t_1, b_1, \cdots, b_5, \gamma_3]. \]

Then \( A \) is free as a \( B \)-module, and hence for any subset \( R \subseteq B \) the canonical map \( B/BR \to A/AR \) is injective, where \( BR \) denotes the ideal generated by \( R \) in \( B \), and similarly for \( AR \).
Note that $B$ coincides with $F_2[t_1, c_1, \ldots, c_5, \gamma_3]$. So it contains the set $R = \{c_1\} \cup R_6 = \{c_1, c_2, c_3, c_5, I_6\}$, and $B/BR$ can be regarded as a subset of $A/AR = H^*(G/T)/(c_1)$. Then by 2.2 the image of $\pi \circ p^*$ is contained in $B/BR = F_2[t_1, c_4, \gamma_3]/(\gamma_3^2)$, and it is easily seen that up to degree $d_\ell$ the map $H^*(G/T) \to B/BR$ is injective.

Similarly for $\ell = 7$.

q.e.d.

Now the operation of the Sq$^i$ are obtained from 2.2 and 2.1,(2) by calculating modulo $(c_1)$. The results are as follows:

**Theorem 2.4.** (i) In $H^*(EIII)$ we have
\[ Sq^2w' = w't, \quad Sq^4w' = t^6, \quad Sq^8w' = w^2 \]
(ii) In $H^*(EVII)$
\[ Sq^2v = 0, \quad Sq^4v = vu^2 + u^7, \quad Sq^8v = w + vu^4 + u^9; \]
\[ Sq^2w = u^{10}, \quad Sq^4w = vu^6 + u^{11}, \quad Sq^8w = vu^8 + u^{13}, \quad Sq^{16}w = vu^{12}. \]

§ 3. Stiefel-Whitney classes

For the Hermitian symmetric spaces of classical type the Chern classes have been obtained in [1], whence so have the Stiefel-Whitney classes by mod 2 reduction. In this section as an application of the previous section we give the Stiefel-Whitney classes of EIII and EVII by using the Wu classes.

For a compact $n$-manifold $M$ the $i$-th Wu class $u_i \in H^i(M)$ is characterised by
\[ u_i \cdot x = Sq^ix \quad \text{for any } x \in H^{n-i}(M). \]

Using the Wu classes the Stiefel-Whitney classes $w_i$ are given by $w_i = \sum_{j \geq 0} Sq^{i-j}u_j$, or equivalently
\[ W = SqU, \quad (3.1) \]
where $W = \sum w_i$, and $U = \sum u_i$ (see [5], for example).

From the previous section it follows that
\[ H^*(EIII) \text{ has } \{w^{ni}, w^2 | n = 0, 1; 0 \leq i \leq 12\} \text{ as a basis, and} \]
\[ Sq^{2r}w't^{12-r} = w't^{12} (r = 0, 2, 6), \quad Sq^{16}w^2 = wt^{12}; \]
\[ Sq^{2r}b = 0 \text{ for the other } b \text{ of degree } 32 - 2r \text{ in the basis}; \]
\[ H^*(EVII) \text{ has } \{w^n v^m u^i | n, m = 0, 1; 0 \leq i \leq 13\} \text{ as a basis, and} \]
\[ Sq^{2r}wvu^{13-r} = wvu^{13} (r = 0, 4, 12), \]
\[ Sq^{2r}b = 0 \text{ for the other } b \text{ of degree } 54 - 2r \text{ in the basis}. \]

Therefore

**Theorem 3.1.** The non-zero Wu classes are given as follows:

for EIII, $u_0 = 1, \quad u_4 = t^2, \quad u_{12} = t^6, \quad u_{16} = w^2$;

for EVII, $u_0 = 1, \quad u_8 = u^4, \quad u_{24} = u^{12}$.
Then by use of the formula (3.1) we obtain

**Corollary 3.2.** The total Stiefel-Whitney class $W$ is given as follows:

for $E_{III}$, \[ W = 1 + t^2 + t^4 + t^6 + (w'^2 + t^8) + t^{10} + w't^{12}; \]

for $E_{VII}$, \[ W = 1 + u^4 + u^8 + u^{12}. \]

Remark 3.3. For $E_{III}$ the result coincides with the mod 2 reduction of that in [8].

**References**


