A Collision Avoidance Control Problem for Moving Bodies in the Plane

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1 Introduction

There appeared several works related with the applicability of direct method of Liapunov to the find-path problems lies in the qualitative theory of differential games and the avoidance control strategies. We refer to, e.g., Vincent and Skowronski[4], Skowronski and Vincent[5], Stonier[7], and Skowronski and Stonier[6] for differential game aspects, and Leitman[3], Stonier[8] and JHN[9] for avoidance strategy aspects. In these works the generation of a suitable Liapunov function is the key in the view of the Liapunov stability theory. Recently, L.T. Grujić[1,2] has established that an asymptotically stable nonlinear system permits the construction of a Liapunov function to guarantee the asymptotic stability. That is, there is no Liapunov function which makes a given system be asymptotically stable if the given system is not asymptotically stable. But, it is very difficult or impossible to determine a suitable Liapunov function for the given complexed nonlinear system, because we have not to integrate the dini-derivative, see[1,2]. Therefore, in many cases we need to construct a Liapunov function, which implies that a system may be stable, and often we can obtain asymptotic stability under some restricted conditions.

Approaching the findpath problem to a collision avoidance strategy of robot arms, Stonier[8] adopts the Liapunov theory from the control and differential game literature for capture within targets, and for avoidance of antitargets. It may be the first good proposal in [8] to solve the collision avoidance control problem in the basis of Liapunov theory. The essential feature of his approach is to construct Liapunov functions for the approaching targets and collision avoiding of antitargets and to determine control
variables according to the time derivatives of Liapunov functions. However, in the
determination of feedback control variables, he used assumption called “right-of way”,
which is reasonable in numerical simulations but not meaningful in mathematical sense,
and unfortunately the generalized Liapunov functions do not satisfy the sufficient con-
dition of Liapunov stability theory; see[8]. In our previous work JHN[9], we can remove
assumption such as “right-of way”, and we introduce the elliptic Liapunov function to
obtain good paths of orbit of moving objects. But, in [9] we have failed to treat control
parameters which may make the path change freely, and futher the Liapunov function
does not satisfy the sufficient condition of the Liapunov stability.

In this paper, we introduce a new Liapunov function which satisfies the Liapunov
stability sufficient conditions, and by using the Liapunov function we may easily change
the paths freely via the control parameters. Finally, we note that almost all are “reg-
ular cases” in that we are getting in nice, smooth paths for the collision avoidance in
numerical simulations. These are illustrated by several examples, and the comparisions
of our numerical results with the cases of [8] and [9] are given.

2 Control plan for $m$ numbers of moving objects

Let us consider a system, containing $m$ numbers of moving objects and $m$ numbers
of fixed targets in a plane workspace, for the trajectories of the moving objects being
controlled to obtain collision avoidance and to reach the targets. The collision avoidance
control problem is to control the movement of the $i$-th moving object to reach the center
of the $i$-th target, while ensuring the $i$-th entire moving object to avoid the $j$-th target
and the $j$-th entire moving object which is regarded as an antitarget with respect to
the $i$-th one, where $i \neq j$, $1 \leq i, j \leq m$. We will use the Liapunov technique as known
as a powerful mathematical method to accomplish the plan for solving the collision avo-
dience control problem. Therefore, to utilize the Liapunov technique, it is necessary
to introduce the Liapunov function which can be applied to the given system, and we
give it below.
2.1 The Liapunov technique

Let $\mathbb{R}^+$ be the set of positive real numbers. We will denote by $A_i$ the $i$-th moving object and by $T_j$ the $j$-th target respectively, where $1 \leq i, j \leq m$. Let us regard the centers of moving objects $A_i$ as the points $(x_i, y_i)$ on the plane. When each moving object $A_i$ moves continuously depending on $t \in \mathbb{R}^+$, we can consider $(x_i, y_i)$ as a continuous function for $t \in \mathbb{R}^+$. In the paper, as studied in Stonier[8] and J-H-N[9], we suppose that the dynamics of $m$ point objects $(x_i, y_i), i = 1, 2, \cdots, m$, are described by the system of the controlled ordinary differential equations,

$$
\begin{aligned}
\dot{x}_i &= z_i \\
\dot{z}_i &= u_i \\
\dot{y}_i &= w_i \\
\dot{w}_i &= v_i,
\end{aligned}
$$

(2.1)

Here in (2.1), $(z_i, w_i) = (\dot{x}_i, \dot{y}_i)$ denotes the time derivatives of the $i$-th point object and $(u_i, v_i)$ denotes the $i$-th control variables pair. We remark that the special case where $m = 2$ is considered in Stonier[8] and J-H-N[9]. By the Liapunov technique, the controls $(u_i, v_i), 1 \leq i \leq m$ will be determined as feedback controls which are obtained by the result of differentiating the Liapunov function associated with the system equation (2.1). Let us define the target set $TS_i$ of the $i$-th target $T_i$ with center $(p_i c_1, p_i c_2)$ and radius $r p_i$ and the moving object set $AS_j$ of the $j$-th moving object $A_j$ with center $(x_j, y_j)$ and length $r a p_j$ of the $j$-th moving object $A_j$ as follows:

$$
TS_i = \{(x, y) : (x - p_i c_1)^2 + (y - p_i c_2)^2 \leq r p_i^2\}, \quad 1 \leq i \leq m,
$$

$$
AS_j = \{(x, y) : (x - x_j)^2 + (y - y_j)^2 \leq r a p_j^2\}, \quad 1 \leq j \leq m.
$$

In order to determine the controls which give the trajectories to avoid collision, we need to define the Liapunov functions such as approaching to the targets and avoiding the antitargets. Therefore, let us define such functions on the plane as follows. Let us define the following (sub)-Lapunov functions:

$V_i$ the Liapunov function to make the $i$-th moving object $A_i$ approach to the $i$-th target $T_i$:

$$
V_i = \frac{1}{2} \{(x_i - p_i c_1)^2 + (y_i - p_i c_2)^2 + z_i^2 + w_i^2\}, \quad 1 \leq i \leq m,
$$
\( W_{ij} \) the Liapunov function to make the \( i \)-th moving object \( A_i \) avoid the \( j \)-th target \( T_j, i \neq j; \)
\[
W_{ij} = \frac{1}{2} \{(x_i - p_j c_1)^2 + (y_i - p_j c_2)^2 - r_{p_j}^2\}, \quad 1 \leq i, j \leq m,
\]
\( V_{ij} \) the Liapunov function to avoid the \( i \)-th moving object \( A_i \) and the \( j \)-th moving object \( A_j, i \neq j \) each other;
\[
V_{ij} = \frac{1}{2} \{(x_i - x_j)^2 + (y_i - y_j)^2 - \max\{r_{a_i}^2, r_{a_j}^2\}\}, \quad 1 \leq i, j \leq m,
\]
\( G_i \) the function which denotes the distance between centers of the \( i \)-th moving object and the \( i \)-th target;
\[
G_i = \frac{1}{2} \{(x_i - p_{i_1} c_1)^2 + (y_i - p_{i_2} c_2)^2\}, \quad 1 \leq i \leq m.
\]
Using the above Liapunov functions \( V_i, W_{ij}, V_{ij} \) and \( G_i \), we can now define the total Liapunov function \( \mathcal{L} \) on \( D(\mathcal{L}) = \{(x, z) \in \mathbb{R}^{2m} \times \mathbb{R}^{2m} : V_{ij}(x_i, y_i, x_j, y_j) > 0, W_{ij}(x_i, y_i) > 0, 1 \leq i, j \leq m\} \) for the system (2.1) as follows,
\[
\mathcal{L}((x, z)) = \sum_{i=1}^{m} V_i(x_i, y_i, z_i, w_i) + \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\alpha_{ij} G_i(x_i, y_i)}{W_{ij}(x_i, y_i)} + \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\beta_{ij} G_i(x_i, y_i) G_j(x_j, y_j)}{V_{ij}(x_i, y_i, x_j, y_j)},
\]
where \( x = (x_1, y_1, \cdots, x_m, y_m) \in \mathbb{R}^{2m}, z = (z_1, w_1, \cdots, z_m, w_m) \in \mathbb{R}^{2m} \) and for all \( j, \alpha_{ii} = \beta_{ii} = 0, \alpha_{ij}, \beta_{ij} > 0 \) and \( \beta_{ij} = \beta_{ji} \) for \( 1 \leq i, j \leq m \). Then it is verified by the direct calculations of the time derivative \( \dot{\mathcal{L}}((x, z)) \) along the equation (2.1) is given by
\[
\dot{\mathcal{L}}((x, z)) = -\sum_{i=1}^{m} \left( \gamma_i z_i^2 + \mu_i w_i^2 \right)
\]
provided that the feedback control variables \((u_k, v_k)\) are given by
\[
u_k = -(x_k - p_k c_1) \left( 1 + \sum_{i=1}^{m} \frac{\alpha_{ki} G_i}{W_{ki}} + \sum_{i=1}^{m} \frac{\beta_{ki} G_i}{V_{ki}} \right) + \sum_{i=1}^{m} \frac{\alpha_{ki} G_k}{W_{ki}^2} (x_k - p_i c_1) - \sum_{i=1}^{m} \frac{\beta_{ik} G_k}{G_{ik}^2} (x_i - x_k) - \gamma_k z_k,\]
\[
v_k = -(y_k - p_k c_2) \left( 1 + \sum_{i=1}^{m} \frac{\alpha_{ki} G_i}{W_{ki}} + \sum_{i=1}^{m} \frac{\beta_{ki} G_i}{V_{ki}} \right) + \sum_{i=1}^{m} \frac{\alpha_{ki} G_k}{W_{ki}^2} (y_k - p_i c_2) - \sum_{i=1}^{m} \frac{\beta_{ik} G_k}{V_{ik}^2} (y_i - y_k) - \mu_k w_k,
\]
where \( k = 1, 2, \cdots, m \). From now on we will call \( \alpha_{ij}, \beta_{ij} \) the control parameters and \( \gamma_i > 0, \mu_i > 0 \) the convergence parameters. The role of the numerators \( G_i \) and \( G_i G_j \)
appeared in second and third terms of $\mathcal{L}$ is to wipe out the unnecessary effect of $W_{ij}$ and $V_{ij}$ when $A_i$ approach to $T_i$ or $A_j$ approach to $T_j$, where $1 \leq i, j \leq m$. Then we can easily know that $\mathcal{L}((x,z)) > 0$ and for $z \neq 0$, $\dot{\mathcal{L}}_{(2.1)}((x,z)) \leq 0$ for the solution $(x,z) \in D(\mathcal{L})$ associated with (2.1) and (2.2). Also, the Liapunov function $\mathcal{L}$ satisfies $\mathcal{L}(P) = 0$ which becomes a sufficient condition for the stability, and simultaneously, which guarantees that $\mathcal{L}((x,z)) \rightarrow 0$ as $t \rightarrow \infty$ implies $(x,z) \rightarrow P$, i.e., each moving object goes to each target, where $P \equiv ((p_1c_1,p_1c_2,p_2c_1,p_2c_2), 0)$ is an equilibrium point for the dynamics equation (2.1) with (2.2). But in Stonier\[8\] and JHN\[9\], for $m = 2$ they required some restricted conditions that the control parameters $\alpha_{ij}, i,j = 1,2$ and $\beta_{ij}, i,j = 1,2$ are sufficiently small in order that $V((x,z)) \rightarrow 0$ as $t \rightarrow \infty$, where $V$ is the Liapunov functions introduced by Stonier\[8\]

$$V_{\text{Stonier}} = \left( V_1 + \frac{\alpha_{12}}{V_{12}} + \frac{\beta_{12}}{W_{12}}, V_2 + \frac{\alpha_{21}}{V_{21}} + \frac{\beta_{21}}{W_{21}} \right),$$

and by JHN\[9\]

$$V_{\text{JHN}} = V_1 + V_2 + \frac{\alpha_{12}}{V_{12}} + \frac{\alpha_{21}}{V_{21}} + \frac{\beta_{12}}{W_{12}} + \frac{\beta_{21}}{W_{21}} + \text{Elliptic Liapunov Function.}$$

As the result, since they have to demand the control parameters $\alpha_{12}, \alpha_{21}, \beta_{12}$ and $\beta_{21}$, sufficiently small, it is difficult or impossible to control the trajectories for the system (2.1) with the controls which they determined under the Liapunov functions, $V_{\text{Stonier}}$ and $V_{\text{JHN}}$. That is to say, they failed to give their's control parameters intrinsic means owing to some constraints for all control parameters to be small. Beside, we can not expect the avoidance of collision between moving objects or moving objects and targets in the case where the control parameters are very small, because the effect of $V_{12}, V_{21}, W_{12}$ and $W_{21}$ disappeares for such the cases. For the new Liapunov function, it is easily verified that $\beta_{12}$ plays the role of adjusting the distance between moving objects, $A_1, A_2$ and the $\alpha_{12}$ (resp. $\alpha_{21}$) plays the part of modulating the distance between moving object $A_1$ (resp. $A_2$) and target $T_2$ (resp. $T_1$). Therefore, we have the advantage point of turning a trajectory for the system (2.1) with (2.3) into the best trajectory by artificial. We will survey such the points from some examples.
2.2 Analysis of trajectories for $m = 2$

For the case of $m = 2$, where it becomes an original problem introduced by Stonier[8], the forms of new Liapunov function and controls are given as follows:

**Forms of Liapunov function and controls for $m = 2$**

\[
\mathcal{L} = V_1 + V_2 + \frac{\alpha_{12} G_1}{W_{12}} + \frac{\alpha_{21} G_2}{W_{21}} + \frac{\beta_{12} G_1 G_2}{V_{12}}.
\]

\[
\begin{align*}
u_1 &= -A(x_1 - p_1 c_1) + \frac{\alpha_{12} G_1}{W_{12}^2}(x_1 - p_2 c_1) + \frac{\beta_{12}}{V_{12}}(x_1 - x_2)G_1 G_2 - \gamma_1 z_1, \\
u_2 &= -A(y_1 - p_1 c_2) + \frac{\alpha_{21} G_2}{W_{21}^2}(y_1 - p_2 c_2) + \frac{\beta_{12}}{V_{21}^2}(y_1 - y_2)G_1 G_2 - \gamma_2 z_2, \\
u_1 &= -B(x_2 - p_2 c_1) + \frac{\alpha_{12} G_1}{W_{12}^2}(x_2 - p_1 c_1) + \frac{\beta_{12}}{V_{12}^2}(x_2 - x_1)G_1 G_2 - \gamma_2 z_2, \\
u_2 &= -B(y_2 - p_2 c_2) + \frac{\alpha_{21} G_2}{W_{21}^2}(y_2 - p_1 c_2) + \frac{\beta_{12}}{V_{21}^2}(y_2 - y_1)G_1 G_2 - \mu_2 w_2,
\end{align*}
\]

where $A = 1 + \frac{\alpha_{12}}{W_{12}} + \frac{\beta_{12} G_2}{V_{12}}$ and $B = 1 + \frac{\alpha_{21}}{W_{21}} + \frac{\beta_{12} G_1}{V_{12}}$. Since the asymptotic stability of the system (2.1) with (2.2) was not expected in general, there exists a possibility such as $E = \{x \in \mathbb{R}^4 : \dot{\mathcal{L}}_{(2.1)}((x, z)) = 0, x \neq (p_1 c_1, p_1 c_2, p_2 c_1, p_2 c_2)\}$ is not empty. When the solution $x(t) \equiv (x_1(t), y_1(t), x_2(t), y_2(t))$ satisfies $x(t) \in E$ for all $t \geq 0$, we shall call such one the singular solution. It is difficult to find the conditions for $E = \emptyset$ because of the complexity of controls in (2.3), but we may search for the cases where the singular solutions exist under some initial conditions. In particular, one may guess that the trajectories caused by the symmetry of initial conditions belong to the set $E$. Indeed, firstly, let $x(t)$ satisfying

\[
x_1(t) - p_1 c_1 = -(x_2(t) - p_2 c_1), \quad y_1(t) = y_2(t), \quad p_2 c_1 = p_2 c_2, \quad \forall t \geq 0
\]

be the solution of (2.1) with (2.3) under initial conditions satisfying $z_1(0) + z_2(0) = 0$ and $w_1(0) = w_2(0)$. Then either $\gamma_1 = \gamma_2$ or $\mu_1 = \mu_2$ implies that $\alpha_{12} = \alpha_{21}, \gamma_1 = \gamma_2, \mu_1 = \mu_2$ and $r p_1 = r p_2$. Thus, either $\alpha_{12} \neq \alpha_{21}$ or $r p_1 \neq r p_2$ implies the fact that there is the time $t_1$ when trajectories satisfying above initial conditions don’t hold the equation (2.4), moreover the trajectories at $t_f$ when $x(t_f) \in E$ can’t satisfy the equation (2.4), where $t_f$ denotes the final time when all trajectories are stopped. Secondly, for given $m, n \in \mathbb{R}$, let $p_i c_2 = m p_i c_1 + n, i = 1, 2$ and let initial conditions satisfy $z_1(0) + z_2(0) = 0$ and $w_i(0) = mz_i(0)$. Then we can easily see that for each
$t \geq 0$, the solutions $y_i(t) = mx_i(t) + n, i = 1, 2$ satisfies (2.1) with (2.3) if $\gamma_1 = \mu_1$ and $\gamma_2 = \mu_2$. Therefore, the case where the $i$-th target or trajectory is between $j$-th target and trajectory indicates $x(t) \in E$. Similar to the first case, one can know when $\gamma_i \neq \mu_i, i = 1$ or $i = 2$, for the trajectories to escape from the line $y_i(t) = mx_i(t) + n$ and never to return to a parallel line with $y_i(t) = mx_i(t) + n$, because of considering the first case after rotating it proper.

**EXAMPLE 2.1** We start to compare with three results through this example. This example shows that an absolute value of controls is very small than two results, and the same time, reaching time to targets is to be shorten largely.

i) initial condition

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
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<tbody>
<tr>
<td></td>
<td>-20</td>
<td>1</td>
<td>5</td>
<td>5</td>
<td>20</td>
<td>-1</td>
<td>2</td>
</tr>
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</table>

ii) position of target and size of moving object, target and RK4 (Runge-Kutta 4th)

<table>
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<tr>
<th></th>
<th>$p_{1c_1}$</th>
<th>$p_{1c_2}$</th>
<th>$p_{2c_1}$</th>
<th>$p_{2c_2}$</th>
<th>$rp_1$</th>
<th>$rp_2$</th>
<th>$rap_1$</th>
<th>$rap_2$</th>
<th>RK4</th>
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<tr>
<td>12</td>
<td>0</td>
<td>-12</td>
<td>0</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>0.01</td>
<td></td>
</tr>
</tbody>
</table>

iii) control and convergence parameter

$\beta_{12}$ $\alpha_{12}$ $\alpha_{21}$ $\gamma_1$ $\gamma_2$ $\mu_1$ $\mu_2$

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
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<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

iv) maximum and minimum value of controls

<table>
<thead>
<tr>
<th></th>
<th>$u_1$</th>
<th>$v_1$</th>
<th>$u_2$</th>
<th>$v_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>max Stonier</td>
<td>21.99</td>
<td>38.62</td>
<td>1287.61</td>
<td>513.86</td>
</tr>
<tr>
<td>JHN</td>
<td>17.93</td>
<td>17.19</td>
<td>443.08</td>
<td>148.12</td>
</tr>
<tr>
<td>New</td>
<td>25.71</td>
<td>7.60</td>
<td>10.16</td>
<td>3.60</td>
</tr>
<tr>
<td>min Stonier</td>
<td>-69.14</td>
<td>-54.99</td>
<td>-86.48</td>
<td>-51.99</td>
</tr>
<tr>
<td>JHN</td>
<td>-46.02</td>
<td>-74.95</td>
<td>-37.27</td>
<td>-15.98</td>
</tr>
<tr>
<td>New</td>
<td>-4.90</td>
<td>-28.41</td>
<td>-16.84</td>
<td>-10.75</td>
</tr>
</tbody>
</table>

v) reaching time to targets
Stonier JHN New

\[ A_1 \rightarrow T_1 \quad 48.31 \quad 69.13 \quad 26.40 \]
\[ A_2 \rightarrow T_2 \quad 26.07 \quad 38.10 \quad 11.59 \]

Trajectories for three results in example 2.1 are illustrated in picture 2.1.

**EXAMPLE 2.2** In this example, we consider the case where initial condition and center of target are placed on the graph \( \{(x, y) : y = mx + n, \ m, n \in \mathbb{R}\} \). The case 1 where targets and initial points are put on two parallel lines is that the trajectories don’t go to the targets, but we can make the trajectories move to the targets by changing the value of \( \alpha_{12} \) different to \( \alpha_{21} \). The case 2 where all datum are put on the line \( y = 2x + 6 \) can become asymptotically stable by virtue of varying the values of convergent parameters, for example, \( \mu_1 = 4 \).

i) initial condition

\[
\begin{array}{cccccccc}
  x_1 & z_1 & y_1 & w_1 & x_2 & z_2 & y_2 & w_2 \\
  \text{Case 1} & -20 & 1 & 10 & 1 & 20 & -1 & 10 & 1 \\
  \text{Case 2} & -10 & 1 & -14 & 2 & 6 & -1 & 18 & -2 \\
\end{array}
\]

ii) position of target and size of moving object, target and RK4

\[
\begin{array}{ccccccccccc}
  p_1c_1 & p_1c_2 & p_2c_1 & p_2c_2 & r_{p_1} & r_{p_2} & r_{ap_1} & r_{ap_2} & \text{RK4} \\
  \text{Case 1} & 10 & 5 & -10 & 5 & 5 & 5 & 5 & 0.05 \\
  \text{Case 2} & 2 & 10 & -5 & -4 & 3 & 4 & 4 & 4 & 0.05 \\
\end{array}
\]

iii) control and convergence parameter

\[
\begin{array}{cccccccc}
  \beta_{12} & \alpha_{12} & \alpha_{21} & \gamma_1 & \mu_1 & \gamma_2 & \mu_2 \\
  \text{case 1} & 1 & 1(2) & 1 & 3 & 3 & 3 & 3 \\
  \text{case 2} & 1 & 2 & 3 & 3 & 3(4) & 3 & 3 \\
\end{array}
\]
iv) maximum and minimum value of controls

\[
\begin{array}{cccccc}
\text{max} & u_1 & v_1 & u_2 & v_2 \\
\text{case 1} & 28.32 & 17.95 & 22.88 & 28.23 \\
\text{case 2} & 9.61 & 17.22 & 7.63 & 15.49 \\
\text{min} & -14.83 & -9.29 & -29.57 & -11.02 \\
\text{case 2} & -5.78 & -13.79 & -7.95 & -15.91 \\
\end{array}
\]

v) reaching time to targets

<table>
<thead>
<tr>
<th>case 1</th>
<th>case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 \rightarrow T_1 )</td>
<td>23.6</td>
</tr>
<tr>
<td>( A_2 \rightarrow T_2 )</td>
<td>20.6</td>
</tr>
</tbody>
</table>

Trajectories for the case 1 and 2 in example 2.2 are illustrated in picture 2.2 and picture 2.3, respectively.

### 2.3 Analysis of the trajectories for \( m \geq 3 \)

In order to verify that even for \( m \geq 3 \) the new Liapunov function has no obstacle finding a path for the collision avoidance control problem, we will give some examples for the cases of \( m = 3, m = 4 \) and \( m = 5 \). It may occure the case, similar to the case \( m = 2 \), that the solution of the system (2.1) with (2.2) belongs to an invariant set or becomes asymptotically stable according to varying the values of parameters and initial conditions. Here we can get an information about the positions where the trajectories stop on the way, which is occurred when the trajectories fall into a dead alley. Therefore, it is necessary to block up the entrance of a dead alley for the trajectories not to enter into a dead alley, which can be obtained by taking the control parameter \( \alpha_{ij} \) around a target where a dead alley arises sufficiently large.

#### 2.3.1 An example for \( m = 3 \)

**Form of Liapunov function and controllers for \( m = 3 \)**

\[
\mathcal{L} = V_1 + V_2 + V_3 + G_1 \left( \frac{\beta_{12} G_2}{V_{12}} + \frac{\alpha_{12}}{W_{12}} + \frac{\alpha_{13}}{W_{13}} \right)
\]
\[ + \ G_2 \left( \frac{\beta_{23}G_3}{V_{23}} + \frac{\alpha_{21}}{W_{21}} + \frac{\alpha_{23}}{W_{23}} \right) + \ G_3 \left( \frac{\beta_{13}G_1}{V_{13}} + \frac{\alpha_{31}}{W_{31}} + \frac{\alpha_{32}}{W_{32}} \right) \]

\( u_1 = -(x_1 - p_1 c_1) \left( 1 + \frac{\alpha_{12}}{W_{12}} + \frac{\alpha_{13}}{W_{13}} + \frac{G_2 \beta_{12}}{V_{12}} + \frac{G_3 \beta_{13}}{V_{13}} \right) - \gamma_1 z_1 + \ G_1 \left[ \frac{\beta_{12}G_2}{V_{12}^2} (x_1 - x_2) - \frac{\beta_{13}G_3}{V_{13}^2} (x_3 - x_1) + \frac{\alpha_{12}}{W_{12}^2} (x_1 - p_2 c_2) + \frac{\alpha_{13}}{W_{13}^2} (x_1 - p_3 c_3) \right]\]

\( v_1 = -(y_1 - p_1 c_2) \left( 1 + \frac{\alpha_{12}}{W_{12}} + \frac{\alpha_{13}}{W_{13}} + \frac{G_2 \beta_{12}}{V_{12}} + \frac{G_3 \beta_{13}}{V_{13}} \right) - \mu_1 w_1 + \ G_1 \left[ \frac{\beta_{12}G_2}{V_{12}^2} (y_1 - y_2) - \frac{\beta_{13}G_3}{V_{13}^2} (y_3 - y_1) + \frac{\alpha_{12}}{W_{12}^2} (y_1 - p_2 c_2) + \frac{\alpha_{13}}{W_{13}^2} (y_1 - p_3 c_3) \right]\]

\( u_2 = -(x_2 - p_2 c_1) \left( 1 + \frac{\alpha_{21}}{W_{21}} + \frac{\alpha_{23}}{W_{23}} + \frac{G_1 \beta_{12}}{V_{12}} + \frac{G_3 \beta_{23}}{V_{23}} \right) - \gamma_2 z_2 + \ G_2 \left[ \frac{\beta_{23}G_3}{V_{23}^2} (x_2 - x_3) - \frac{\beta_{12}G_1}{V_{12}^2} (x_1 - x_2) + \frac{\beta_{21}}{W_{21}^2} (x_2 - p_1 c_2) + \frac{\beta_{23}}{W_{23}^2} (x_2 - p_3 c_3) \right]\]

\( v_2 = -(y_2 - p_2 c_2) \left( 1 + \frac{\alpha_{21}}{W_{21}} + \frac{\alpha_{23}}{W_{23}} + \frac{G_1 \beta_{12}}{V_{12}} + \frac{G_3 \beta_{23}}{V_{23}} \right) - \mu_2 w_2 + \ G_2 \left[ \frac{\beta_{23}G_3}{V_{23}^2} (y_2 - y_3) - \frac{\beta_{12}G_1}{V_{12}^2} (y_1 - y_2) + \frac{\beta_{21}}{W_{21}^2} (y_2 - p_1 c_2) + \frac{\beta_{23}}{W_{23}^2} (y_2 - p_3 c_3) \right]\]

\( u_3 = -(x_3 - p_3 c_1) \left( 1 + \frac{\alpha_{31}}{W_{31}} + \frac{\alpha_{32}}{W_{32}} + \frac{G_1 \beta_{13}}{V_{13}} + \frac{G_2 \beta_{23}}{V_{23}} \right) - \gamma_3 z_3 + \ G_3 \left[ \frac{\beta_{13}G_1}{V_{13}^2} (x_3 - x_1) - \frac{\beta_{23}G_2}{V_{23}^2} (x_2 - x_3) + \frac{\alpha_{31}}{W_{31}^2} (x_3 - p_1 c_1) + \frac{\alpha_{32}}{W_{32}^2} (x_3 - p_2 c_1) \right]\]

\( u_3 = -(y_3 - p_3 c_2) - \mu_3 w_3 + \ G_3 \left[ \frac{\beta_{13}G_1}{V_{13}^2} (y_3 - y_1) - \frac{\beta_{23}G_2}{V_{23}^2} (y_2 - y_3) + \frac{\alpha_{31}}{W_{31}^2} (y_3 - p_1 c_2) + \frac{\alpha_{32}}{W_{32}^2} (y_3 - p_2 c_2) \right]\]

**EXAMPLE 2.3** In this example, we consider the case where targets and initial points are concentrated around the origin, which are considered as a difficult situation to control the trajectory. In the case 1, the trajectories do not go to the targets in the desired time, and asymptotically stable under the case 2 where we change the control parameters \( \alpha_{12}, \alpha_{23} \) and \( \alpha_{31} \), which play a role of making \( A_i \) travel \( T_3 \) in the direction, \( A_2 \) to \( T_1 \) and \( A_3 \) to \( T_2 \) and removing of the entrance into three dead alleys simultaneously.
i) initial condition

\[
x_1 \quad z_1 \quad y_1 \quad w_1 \quad x_2 \quad z_2 \quad y_2 \quad w_2 \quad x_3 \quad z_3 \quad y_3 \quad w_3
\]
\[
-7 \quad 1 \quad 6 \quad 1 \quad 7 \quad 1 \quad 6 \quad 1 \quad 0 \quad 1 \quad -5 \quad 1
\]

ii) position of target and size of moving object, target and RK4

\[
p_{1c1} \quad p_{1c2} \quad p_{2c1} \quad p_{2c2} \quad p_{3c1} \quad p_{3c2}
\]
\[
3.5 \quad 0 \quad -3.5 \quad 0 \quad 0 \quad 6
\]

\[rpi = 3.5, \; rap_i = 3, \; i = 1, 2, 3, \; \text{and} \; RK4 = 0.05\]

iii) control and convergence parameter

\[
\beta_{12} \quad \beta_{13} \quad \beta_{23} \quad \alpha_{12} \quad \alpha_{13} \quad \alpha_{21} \quad \alpha_{23} \quad \alpha_{31} \quad \alpha_{32} \quad \gamma_1 \quad \mu_1 \quad \gamma_2 \quad \mu_2 \quad \gamma_3 \quad \mu_3
\]

\[
\begin{array}{cccccccccccccc}
\text{case 1} & 5 & 5 & 5 & 1 & 1 & 1 & 1 & 1 & 5 & 5 & 5 & 5 & 5 & 5 \\
\text{case 2} & 5 & 5 & 5 & 15 & 1 & 15 & 15 & 15 & 1 & 5 & 5 & 5 & 5 & 5
\end{array}
\]

iv) reaching time to targets

\[A_1 \rightarrow T_1 \quad A_2 \rightarrow T_2 \quad A_3 \rightarrow T_3\]

\[
\begin{array}{cccc}
\text{case 2} & 20.0 & 20.0 & 20.5
\end{array}
\]

Trajectories for the case 1 and 2 in example 2.3 are illustrated in picture 2.4.

2.3.2 An example for \(m = 4\)

**EXAMPLE 2.4** This example may not occur in a realistic system, but it is a very interesting case. Since the moving objects are closed up, they may get out of a workspace, otherwise they may collide each other or a moving object may collide with a target. Thus, it is necessary to adjust the strength between the moving objects to weak, which means making the control parameters \(\beta_{ij}\) be small enough.

i) initial condition

\[
x_1 \quad z_1 \quad y_1 \quad w_1 \quad x_2 \quad z_2 \quad y_2 \quad w_2 \quad x_3 \quad z_3 \quad y_3 \quad w_3
\]
\[
-16 \quad 1 \quad 0 \quad -1 \quad -13 \quad 1 \quad 0 \quad 1 \quad -10 \quad 1 \quad 0 \quad -1 \quad -7 \quad 1 \quad 0 \quad 1
\]

ii) position of target and size of moving object, target and RK4

\[
p_{1c1} \quad p_{1c2} \quad p_{2c1} \quad p_{2c2} \quad p_{3c1} \quad p_{3c2} \quad p_{4c1} \quad p_{4c2}
\]
\[
0 \quad 0 \quad 7 \quad 0 \quad 12 \quad 0 \quad 15 \quad 0
\]
\[ rp_1 = 4, \quad rp_2 = 3 \quad rp_3 = 2 \quad rp_4 = 1 \]
\[ rap_i = 2, \quad i = 1, 2, 3, 4. \]

iii) control and convergence parameter

1. \( \gamma_i = \mu_i = 5, \quad 1 \leq i \leq 4, \quad \beta_{ij} = 0.05, \quad 1 \leq i < j \leq 4 \) and \( \alpha_{ij} = 1, \quad 1 \leq i, j \leq 4, \quad i \neq j, \quad RK4 = 0.005. \)

2. \( \gamma_i = \mu_i = 5, \quad 1 \leq i \leq 4, \quad \beta_{ij} = 0.01, \quad 1 \leq i < j \leq 4 \) and \( \alpha_{ij} = 1, \quad 1 \leq i, j \leq 4, \quad i \neq j, \quad RK4 = 0.01. \)

Trajectories for the case 1 and 2 in example 2.4 are illustrated in picture 2.5(a) and picture 2.5(b).

2.3.3 Some examples for \( m = 5 \)

We present two interesting examples where the shape of located targets has four dead alleys and where all moving objects are concentrated on the very small workspace.

EXAMPLE 2.5 When we regard the \( T_1 \) as a big pillar placed on the plane and the other targets as some small bodies which are attached on the \( T_1 \), there exist four dead alleys which swallow all moving objects \( A_i, i = 2, 3, 4, 5. \)

i) initial condition

\[
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & -15 & 1 & 0 & -1 & 0 & 1 & -15 & 1 \\
0 & 0 & 0 & 0 & -15 & 1 & 0 & -1 & 0 & 1 & -15 & 1 \\
0 & 0 & 0 & 0 & -15 & 1 & 0 & -1 & 0 & 1 & -15 & 1 \\
15 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\
15 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\
\end{array}
\]

ii) position of target and size of moving object, target and RK4

\[
\begin{array}{cccccccccccc}
0 & 0 & 8 & 0 & 0 & 8 & -8 & 0 & 0 & -8 & -8 & -8 \\
0 & 0 & 8 & 0 & 0 & 8 & -8 & 0 & 0 & -8 & -8 & -8 \\
0 & 0 & 8 & 0 & 0 & 8 & -8 & 0 & 0 & -8 & -8 & -8 \\
0 & 0 & 8 & 0 & 0 & 8 & -8 & 0 & 0 & -8 & -8 & -8 \\
0 & 0 & 8 & 0 & 0 & 8 & -8 & 0 & 0 & -8 & -8 & -8 \\
\end{array}
\]

\[ rp_1 = 5, \quad rp_i = 3, \quad 2 \leq i \leq 5, \]
\[ rap_i = 2, \quad 1 \leq i \leq 5 \quad \text{and} \quad RK4 = 0.05. \]
ii) control and convergence parameter

\[
\gamma_i = \mu_i = 3, \quad 1 \leq i \leq 5, \quad \beta_{ij} = 1, \quad 1 \leq i < j \leq 4,
\]

\[
\alpha_{21} = \alpha_{31} = \alpha_{41} = \alpha_{51} = 20, \quad \alpha_{23} = \alpha_{32} = \alpha_{43} = \alpha_{54} = 0.5
\]
and other than then \( \alpha_{ij} = 1 \).

Trajectories for \( m=5 \) in example 2.5 are illustrated in picture 2.6.

EXAMPLE 2.6 Since all moving objects are closed up in the very small workspace, we have to make the control parameters \( \beta_{ii+1}, i = 1, 2, 3, 4 \) be small to prevent moving object and moving object or moving object and target from colliding each other, and then it is necessary to arrange the control parameters \( \alpha_{ij} \) to obtain the smooth of the trajectories. The results are below.

i) initial condition

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( z_1 )</th>
<th>( y_1 )</th>
<th>( w_1 )</th>
<th>( x_2 )</th>
<th>( z_2 )</th>
<th>( y_2 )</th>
<th>( w_2 )</th>
<th>( x_3 )</th>
<th>( z_3 )</th>
<th>( y_3 )</th>
<th>( w_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0</td>
<td>-5.0</td>
<td>1.0</td>
<td>4.75</td>
<td>-1.0</td>
<td>-1.55</td>
<td>1.0</td>
<td>2.95</td>
<td>1.0</td>
<td>3.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_4 )</th>
<th>( z_4 )</th>
<th>( y_4 )</th>
<th>( w_4 )</th>
<th>( x_5 )</th>
<th>( z_5 )</th>
<th>( y_5 )</th>
<th>( w_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.95</td>
<td>1.0</td>
<td>4.05</td>
<td>-1.0</td>
<td>-4.75</td>
<td>1.0</td>
<td>-1.55</td>
<td>1.0</td>
</tr>
</tbody>
</table>

ii) position of target and size of moving object, target and RK4

<table>
<thead>
<tr>
<th>( p_1c_1 )</th>
<th>( p_1c_2 )</th>
<th>( p_2c_1 )</th>
<th>( p_2c_2 )</th>
<th>( p_3c_1 )</th>
<th>( p_3c_2 )</th>
<th>( p_4c_1 )</th>
<th>( p_4c_2 )</th>
<th>( p_5c_1 )</th>
<th>( p_5c_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>-9.5</td>
<td>3.1</td>
<td>-5.9</td>
<td>5.9</td>
<td>-8.1</td>
<td>9.5</td>
<td>3.1</td>
<td></td>
</tr>
</tbody>
</table>

\( \text{r}_{pi} = 4, \quad 1 \leq i \leq 5 \), \( \text{rap}_i = 2, \quad 1 \leq i \leq 5 \) and RK4 = 0.05.

iii) control and convergence parameter

1. \( \gamma_i = \mu_i = 5, \quad 1 \leq i \leq 5, \quad \beta_{ij} = 0.05, \quad 1 \leq i < j \leq 5 \) and \( \alpha_{ij} = 1, \quad 1 \leq i, j \leq 4, \quad i \neq j \).

2. \( \gamma_i = \mu_i = 5, \quad 1 \leq i \leq 5, \quad \beta_{ij} = 0.05, \quad 1 \leq i < j \leq 5 \),

\( \alpha_{13} = \alpha_{24} = \alpha_{35} = \alpha_{41} = \alpha_{52} = 10 \) and other than then \( \alpha_{ij} = 1 \).

Trajectories for the case 1 and 2 in example 2.6 are illustrated in picture 2.7.
3 Conclusion

The most important feature of this paper is that the Liapunov function for the system (2.1) is setted up skilfully, so that the co-trolls and convergence parameters play their proper roles such as altering the trajectory of the system (2.1) into the desired one. It is obvious that the system (2.1) with (2.2) is stable. However, under the new Liapunov function, the asymptotic stability for the system (2.1) with (2.2) is not verified in general. In fact, for $m = 2$ there were many examples of the trajectories being stopped on the way, but we could avoid it by means of manipulating every condition to break out a symmetrical condition. We have hardly a stopping situation halfway, because the new Liapunov function have many parameters which are not necessary symmetry. When $m \geq 3$, we were confronted with many situations that the trajectories belong to the invariant set, but most situations were solved by adjusting the control parameters. If one want to make the state which is not asymptotically stable be asymptotically stable, we have to compose another Liapunov function with relation to a neural system, but it is a problem in the future.
2.1. Trojectories for three results.

Picture 2.1. Trajectories for the case 1.

Picture 2.2. Trajectories for the case 1.

Picture 2.3. Trajectories for the case 2.
2.4. Trajectories for the case 1 and 2.

Picture 2.4. Trajectories for the case 1 and 2.

2.5. (a) Trajectories for the case 1.

\[ \beta_{ij} = 0.05 \]
2.5. (b) Trajectories for the case 2.

Picture 2.5. (b) Trajectories for the case 2.

2.6. Trajectories for $m=5$.

Picture 2.6. Trajectories for $m=5$. 
Picture 2.7. Trajectories for the case 1 and 2.
References


