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Kyoto University
COVERING PROPERTIES CHARACTERIZED BY ORTHOCOMPACTNESS AND SUBNORMALITY OF PRODUCTS

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All spaces are assumed to be $T_1$, but compact spaces and paracompact spaces are assumed to be Hausdorff.

A space $X$ is assumed to be Tychonoff when we consider the product $X \times \gamma X$, where $\gamma X$ denotes a compactification of $X$. An infinite cardinal $\kappa$ is assumed to be no less than $L(X)$ when we consider the product $X \times 2^\kappa$ or the product $X \times (\kappa + 1)$, where $L(X)$ denotes the Lindelöf number of the space $X$.

The main purpose of this note is to give some partial answers to Problems A and C stated in Section 1.

1. CHARACTERIZATIONS OF COVERING PROPERTIES BY PRODUCTS

Let us begin with a classical result of Dowker [D].

**Theorem 1.1 [D].** For a normal space $X$, the following are equivalent.

(a) $X$ is countably paracompact.
(b) $X \times (\omega + 1)$ is normal.
(c) $X \times [0, 1]$ is normal.

Theorem 1.1 is the first result which indicated an important implication between covering properties and products. Moreover, this led up to a beautiful characterization of paracompactness in terms of products.

**Theorem 1.2 [T,M].** For a Hausdorff space $X$, the following are equivalent.

(a) $X$ is paracompact.
(b) $X \times \gamma X$ is normal.
(c) $X \times 2^\kappa$ is normal.
(d) $X \times (\kappa + 1)$ is normal.

**Remark.** The equivalence (a) and (d) in Theorem 1.2 was proved by Kunen. It is found in [P, Corollary 3.7].
An open cover $\mathcal{V}$ of a space $X$ is **interior-preserving** if $\bigcap \mathcal{V}'$ is open in $X$ for each $\mathcal{V}' \subset \mathcal{V}$. A space $X$ is **orthocompact** if every open cover of $X$ has an interior-preserving open refinement.

Subsequently, as a nice analogue of Theorem 1.2, a characterization of metacompactness was obtained as follows.

**Theorem 1.3** [Ju1,S]. For a space $X$, the following are equivalent.

(a) $X$ is metacompact.
(b) $X \times \gamma X$ is orthocompact.
(c) $X \times 2^\kappa$ is orthocompact.

This means that there are some closed relations between normality and orthocompactness of products (see [S,KY]). Moreover, as an analogue of Theorem 1.3, we proved a characterization of submetacompactness as follows.

**Theorem 1.4** [Y1]. For a space $X$, the following are equivalent.

(a) $X$ is submetacompact.
(b) $X \times \gamma X$ is suborthocompact.
(c) $X \times 2^\kappa$ is suborthocompact.

Seeing Theorems 1.2 and 1.3, it is natural to raise the following problem.

**Problem A** [Y2]. If $X \times (\kappa + 1)$ is orthocompact, is $X$ metacompact?

Moreover, it is natural to ask whether there is an analogous characterization of subparacompactness in terms of products.

Recall that a space $X$ is **subnormal** [C, Kr] (normal) if for any disjoint closed sets $A$ and $B$ in $X$, there are disjoint $G_\delta$-sets (open sets) $G$ and $H$ such that $A \subset G$ and $B \subset H$. Note that a space $X$ is subnormal (normal) if and only if every binary open cover of $X$ has a countable (finite) closed refinement.

**Problem B** [Ju3]. If $X \times \gamma X$ is subnormal, is $X$ subparacompact?

**Problem C** [Y2]. If $X \times 2^\kappa$ is subnormal, is $X$ subparacompact?

**Remark.** As is shown later, it suffices for these three problems to prove that $X$ is submetacompact. In fact, this follows from Lemma 2.9 and Theorem 3.3 (or Corollary 3.5) below.

2. **METACOMPACTNESS AND SUBMETACOMPACTNESS OF $\beta$-SPACES**

In this section, we give an affirmative answer to our Problem A under the assumption of $X$ being a $\beta$-space.
A space $X$ is called a $\beta$-space if there is a function $g : X \times \omega \to \text{Top}(X)$, satisfying

(i) $x \in \bigcap_{n \in \omega} g(x, n)$,
(ii) if $x \in g(x_n, n)$ for each $n \in \omega$, then $\{x_n\}$ has a cluster point in $X$.

Since the class of $\beta$-spaces contains the classes of $\Sigma$-spaces and semi-stratifiable spaces, it is very broad as a class of generalized metric spaces.

A well-ordered sequence $\{y_\alpha : \alpha \in \kappa\}$ of length $\kappa$ in a space $Y$ is a free sequence if $\text{Cl}\{y_\beta : \beta < \alpha\} \cap \text{Cl}\{y_\gamma : \alpha \leq \gamma < \kappa\} = \emptyset$ for each $\alpha \in \kappa$.

**Theorem 2.1.** Let $X$ be a $\beta$-space and $C$ a compact space with a free sequence of length $\geq L(X)$. Then $X$ is metacompact if and only if $X \times C$ is orthocompact.

Since $\kappa + 1$ has a free sequence of length $\kappa$, Theorem 2.1 yields a partial answer to Problem A.

**Corollary 2.2.** A $\beta$-space $X$ is metacompact if and only if $X \times (\kappa + 1)$ is orthocompact.

Moreover, Arhangel'skii's theorem in [A] and Theorem 2.1 yield

**Corollary 2.3.** Let $X$ be a $\beta$-space and $C$ a compact space with tightness $> L(X)$. Then $X$ is metacompact if and only if $X \times C$ is orthocompact.

Now, we will give only a course of the proof of Theorem 2.1. On the way, we will obtain a characterization of submetacompactness of $\beta$-spaces.

A well-ordered open cover $\{U_\alpha : \alpha \in \kappa\}$ of a space $X$ is well-monotone if $\beta < \alpha$ implies $U_\beta \subset U_\alpha$.

**Lemma 2.4.** Let $X$ be a space and $C$ a compact space with a free sequence of length $\geq L(X)$. If $X \times C$ is orthocompact, then every well-monotone open cover of $X$ has a closure-preserving closed refinement.

By this, it seems to be effective to consider well-monotone open covers and their closure-preserving closed refinements. So we think of the following Junnila's theorem.

**Theorem 2.5 [Ju1, Ju2].** The following are equivalent for a space $X$.

(a) $X$ is metacompact (submetacompact).
(b) Every well-monotone open cover of $X$ has a point-finite open refinement ($\theta$-sequence of open refinements).
(c) Every interior-preserving directed open cover of $X$ has a ($\sigma$-)closure-preserving closed refinement.

Seeing Lemma 2.4 and Theorem 2.5, we raise the following problem.

**Problem D.** If every well-monotone open cover of a space $X$ has a $\sigma$-closure-preserving closed refinement, when is $X$ submetacompact?
Lemma 2.6 [Ji]. Let $X$ be a $\beta$-space and $\mathcal{U}$ a well-monotone open cover of $X$. If $\mathcal{H}$ is an open refinement of $\mathcal{U}$, then there is a sequence $\{\mathcal{G}_{\mathcal{H}, s^j}: s \in \omega^{<\omega}\}$ of partial refinements by open sets in $X$, satisfying

1. $\mathcal{G}_{\mathcal{H}, s^j} \subset \mathcal{G}_{\mathcal{H}, s'^j}$ for $s \subset s'$,
2. if $x \in X$ with ord$(x, \mathcal{H}) \leq n$, then $x \in \bigcup \mathcal{G}_{\mathcal{H}, s}$ for each $s \in \omega^{n+1}$,
3. for each $x \in X$, there is some $\sigma \in \omega^\omega$ such that ord$(x, \mathcal{G}_{\mathcal{H}, (\sigma | n)}) < \omega$ for each $n \in \omega$.

Making use of this, we prove the following lemma. A basic idea for the proof is also due to Jiang [Ji].

Lemma 2.7 (main). Let $X$ be a $\beta$-space and $\mathcal{U}$ a well-monotone open cover of $X$. If $\mathcal{U}$ has a closure-preserving closed refinement, then it has a $\theta$-sequence of open refinements.

By Lemma 2.7, we can easily obtain an answer to our Problem D.

Theorem 2.8. A $\beta$-space $X$ is submetacompact if and only if every well-monotone open cover of $X$ has a $\sigma$-closure-preserving closed refinement.

Now, let us return the proof of Theorem 2.1.

Let $X$ be a space and $\mathcal{F}$ a collection of subsets of $X$. A collection $\{G(F): F \in \mathcal{F}\}$ of subsets in $X$ is an open expansion (a $G_\delta$-expansion) if $G(F)$ is an open set (a $G_\delta$-set) in $X$ such that $F \subset G(F)$ for each $F \in \mathcal{F}$.

A space $X$ is almost expandable [SK] if every locally finite collection of closed sets in $X$ has a point-finite open expansion.

A well-ordered sequence $\{y_\alpha: \alpha \in \kappa\}$ of length $\kappa$ in a space $Y$ is right separated if $y_\alpha \not\in \mathrm{Cl}\{y_\delta: \delta > \alpha\}$ for each $\alpha \in \kappa$. Note that each free sequence is right separated.

Lemma 2.9. Let $X$ be a space and $C$ a compact space with a right separated sequence of length $\geq L(X)$. If $X \times C$ is orthocompact, then $X$ is almost expandable.

Since submetacompact, almost expandable spaces are metacompact (see [SK]), Theorem 2.1 follows from Lemmas 2.4 and 2.9, and Theorem 2.8. \Box

As a similar problem to Problem D, we raise

Problem D'. If every well-monotone open cover of an orthocompact space $X$ has a closure-preserving closed refinement, is $X$ metacompact?

If problem D' would be affirmatively solved, it follows from Lemma 2.4 that Problem A would be affirmative.

Concerning Problem D', we get an additional result.
Lemma 2.10 [HV, Theorem 3.1]. For a (an orthocompact) space $X$, the following are equivalent.

(a) For every well-monotone open cover $\{U_\alpha : \alpha \in \kappa\}$ of $X$, there is a well-monotone closed cover $\{F_\alpha : \alpha \in \kappa\}$ of $X$ such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \kappa$.
(b) Every well-monotone open cover of $X$ has a cushioned (closure-preserving) closed refinement.
(c) Every infinite open cover $\mathcal{U}$ of $X$ has an open refinement $\mathcal{V}$ with $\text{ord}(x, \mathcal{V}) < |\mathcal{U}|$ for each $x \in X$.

Let $(\lambda + 1)_\lambda$ denote the space $\lambda + 1$ with the topology such that the point $\lambda$ has a neighborhood base in the usual order topology and that all other points are isolated.

Using Lemma 2.10, we obtain

Theorem 2.11. For an orthocompact space $X$, every well-monotone open cover of $X$ has a closure-preserving closed refinement if and only if $X \times (\lambda + 1)_\lambda$ is orthocompact for each $\lambda (\leq L(X))$.

We close this section with the following two unsolved problems, which seem to be related to Problems D and D'.

Problem E [Ka,Y1]. If every directed open cover of a (suborthocompact) space $X$ has a $\sigma$-cushioned closed refinement, is $X$ submetacompact?

Problem E' [Ka,Ju3]. If every directed open cover of a space $X$ has a cushioned closed refinement, is $X$ metacompact?

Problem E' was affirmatively solved under the assumption of $X$ being suborthocompact (see [Y1]).

3. Countable Subparacompactness

In this section, we give some partial answers to our Problem C.

A space $X$ is countably subparacompact [Kr] if every countable open cover of $X$ has a countable closed refinement. Note that countably subparacompact spaces are, equivalently, countably metacompact and subnormal (see [Kr, Theorem 2.5]).

Recently, a list of analogues of Theorem 1.1 was given in [GT, p.118]. Here we can add another analogue, answering to Problem C in the case of $\kappa = \omega$.

Theorem 3.1. For a space $X$, the following are equivalent.

(a) $X$ is countably subparacompact.
(b) $X \times 2^\omega$ is subnormal.
(c) $X \times [0,1]$ is subnormal.
Remark 1. The equivalence of (a) and (c) in Theorem 3.1 was stated in [GT, p.127] without proof. However, at the 10th Summer Conference on General Topology and Application (Amsterdam, August 1994), Good and Tree kindly informed the author that this equivalence had not been proved yet, because they misunderstood the proof.

Theorem 3.1 immediately yields a generalization of Theorem 1.1.

Corollary 3.2. For a normal space $X$, the following are equivalent.

(a) $X$ is countably paracompact.
(b) $X \times (\omega + 1)$ is normal.
(c) $X \times [0,1]$ is subnormal.

Remark 2. It should be noticed that Theorem 3.1 and Corollary 3.2 are essentially different from all the analogues in the list of [GT, p.118]. Because we can replace $[0,1]$ with $\omega + 1$ in all of them, but we cannot do in Theorem 3.1 and Corollary 3.2. In fact, consider a Dowker space $Y$, whose existence is assured by Rudin [R1]. Since the product of a subnormal space and a countable space is subnormal, $Y \times (\omega + 1)$ is subnormal. On the other hand, $Y$ is normal, but not countably metacompact.

A space $X$ is collectionwise $\delta$-normal [Ju3] if every discrete collection of closed sets in $X$ has a disjoint $G_\delta$-expansion.

Theorem 3.3 [R2]. Let $X$ be a space and $C$ a compact space with weight $\geq L(X)$. If $X \times C$ is subnormal, then $X$ is collectionwise $\delta$-normal.

A space $X$ is collectionwise subnormal [C, Kr] if for each discrete collection $\mathcal{F}$ of closed sets in $X$, there is a sequence $\{\mathcal{U}_n\}$ of open expansions of $\mathcal{F}$ such that for each $x \in X$, there is some $n \in \omega$ such that at most one member of $\mathcal{U}_n$ contains $x$. Note that

"subparacompact $\Rightarrow$ collectionwise subnormal $\Rightarrow$ collectionwise $\delta$-normal".

Now, we get another partial answer to Problem C.

Theorem 3.4. If $X \times 2^\kappa$ is subnormal, then $X$ is collectionwise subnormal.

Since collectionwise $\delta$-normal and submetacompact spaces are subparacompact [Ju3], Theorems 1.4 and 3.3 yields a partial answer to Problems B and C.

Corollary 3.5. For a space $X$, the following are equivalent.

(a) $X$ is subparacompact.
(b) $X \times \gamma X$ is subnormal and suborthocompact.
(c) $X \times 2^\kappa$ is subnormal and suborthocompact.

4. LINDELÖF SPACES

Recall that a space $X$ is $\omega_1$-compact if every closed discrete subset in $X$ is at most countable. Note that Lindelöf spaces are $\omega_1$-compact.
Lemma 4.1. Let $C$ be a countably compact space and $X$ a subspace of $C$. If the subspace $(X \times C) \cup (C \times X)$ of the square $C^2$ is subnormal, then $X$ is $\omega_1$-compact.

Using this, we can obtain an analoguous characterization of Lindelöf spaces to Tamano's theorem for paracompactness (see Theorem 1.2).

Theorem 4.2. For a Tychonoff space $X$, the following are equivalent.

(a) $X$ is Lindelöf.
(b) The subspace $(X \times \gamma X) \cup (\gamma X \times X)$ of the square $(\gamma X)^2$ is normal.
(c) $X$ is submetacompact and the subspace $(X \times \gamma X) \cup (\gamma X \times X)$ of the square $(\gamma X)^2$ is subnormal.

In Theorem 4.2, we can find a kind of similarity to the form of Corollary 3.2.

REFERENCES