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Metrizability of spaces having certain k-networks

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As is well-known, each of the following properties implies that X is metrizable.

(A) X is a paracompact developable space (R. H. Bing [2]).
(B) X is a paracompact space having a $\sigma$-locally countable base (V. V. Fedoruk [9]).
(C) X has a $\sigma$-hereditarily closure-preserving base (D. Burke, R. Engelking and D. Lutzer [5]).
(D) X is a paracompact $M$-space, and a $\sigma$-space (A. Okuyama [15]; the paracompactness can be omitted; see F. Siwiec and J. Nagata [16]).
(E) X is an $M$-space having a point-countable base (V. V. Filippov [10]).

In terms of these properties, we give some metrization theorems by means of certain k-networks, or generalizations of $M$-spaces, etc.

We assume that spaces are regular and $T_1$.

Definitions. (1) A cover $C$ of a space a $k$-network if, whenever $K \subset U$ with $K$ compact and $U$ open in $X$, then $K \subset \cup C' \subset U$ for some finite $C' \subset C$. If the $K$ is a single point, then such a cover is called network (or net). Recall that a space is an $H$-space (resp. $\sigma$-space) if it has a $\sigma$-locally finite $k$-network (resp. network).
(2) A space is countably bi-quasi-k [14] if, whenever \( \{F_n : n \in \mathbb{N} \} \) is a decreasing sequence with \( x \in \overline{F_n} \), there exists a decreasing sequence \( \{A_n : n \in \mathbb{N} \} \) such that \( x \in \overline{A_n \cap F_n} \) for each \( n \in \mathbb{N} \), and if \( x_n \in A_n \) for each \( n \in \mathbb{N} \), then the sequence \( \{x_n : n \in \mathbb{N} \} \) has a cluster point in \( \bigcap \{A_n : n \in \mathbb{N} \} \). Recall that a space \( X \) is a q-space if each point has a sequence \( \{V_n : n \in \mathbb{N} \} \) of nbds such that if \( x_n \in V_n \), the sequence \( \{x_n : n \in \mathbb{N} \} \) has a cluster point in \( X \). Every q-space is a countably bi-quasi-k-space. Every countably bi-quasi-k-space is precisely the countably bi-quotient image of an \( M \)-space; see [14].

(3) A space \( X \) is a monotonic \( w\Lambda \)-space (simply, \( mw\Lambda \)-space) [18] (resp. monotonic developable [7] (equivalently, space having a base of countable order in the sense of Arhangel'skii [1])), if there exists a sequence \( \{B_n \} \) of bases for \( X \) such that any decreasing sequence \( \{B_n : n \in \mathbb{N} \} \) with \( B_n \in \mathcal{B}_n \) satisfies (\(*\)):

\( (*) \) If \( x \in \bigcap B_n \), and \( x_n \in B_n \), then the sequence \( \{x_n : n \in \mathbb{N} \} \) has a cluster point in \( X \) (resp. the cluster point \( x \)).

If the sequence \( \{B_n : n \in \mathbb{N} \} \) with \( B_n \in \mathcal{B}_n \) is not necessarily decreasing, then such a space is called a \( w\Lambda \)-space; developable space respectively.

Every \( w\Lambda \)-space or every (monotonic) \( p \)-space [7] is an \( mw\Lambda \)-space. Every \( mw\Lambda \)-space is a q-space, hence countably bi-quasi-k. Among \( \theta \)-refinable (= submetacompact) spaces, \( mw\Lambda \)-spaces (resp. monotonic developable spaces) are \( w\Lambda \)-spaces (resp. developable spaces); see [18] (resp. [7]).

The lemma below holds by means of [13; Proposition 3.2], [17; Section 4], [4; Theorem 4.1], and [18], etc. Here, we note that, in a space having a \( \sigma \)-locally countable \( k \)-network, each point is a \( G_\delta \)-set. So, every countably bi-quasi-k-space with a \( \sigma \)-locally countable \( k \)-network is countably bi-k by [14; Theorem 7.3].
Lemma. (1) Suppose that $X$ is a k-space; a normal space in which every closed countably compact set is compact; or each point of $X$ is a G$_3$-set. Then (i) and (ii) below hold.

(i) $X$ has a point-countable base if and only if $X$ is a countably bi-quasi-k-space with a point-countable k-network.

(ii) $X$ is a nomotonically developable space with a point-countable base if and only if $X$ is an mw$\Delta$-space with a point-countable k-network (cf.[18]).

(2) (i) A space has a $\sigma$-locally countable base if and only if it is a countably bi-quasi-k-space with a $\sigma$-locally countable k-network.

(ii) A space is a nomotonically developable space with a $\sigma$-locally countable base if and only if it is an mw$\Delta$-space with a $\sigma$-locally countable k-network.

Metrization Theorem The following are equivalent.

(a) $X$ is metrizable,

(b) $X$ is a paracompact M-space having a point-countable k-network,

(c) $X$ is an M-space, and a k-space having a point-countable k-network,

(d) $X$ is an M-space having a point-countable k-network, and having a $\sigma$-locally countable network.

(e) $X$ is an M-space having a $\sigma$-locally countable k-network,

(f) $X$ is a paracompact, countably bi-quasi-k-space having a $\sigma$-locally countable k-network,

(g) $X$ is a countably bi-quasi-k-space having a point-countable k-network, and having a $\sigma$-closure preserving k-network; cf. [12].

(h) $X$ is an mw$\Delta$-space having a $\sigma$-closure preserving k-network; see [18].

(i) $X$ is a countably bi-quasi-k-space having a $\sigma$-hereditarily closure preserving k-network.
Remark. In the previous theorem, it is possible to replace "countably bi-quasi-k-space" by "countably bi-quasi-k-space, but the sequence $\{x_n: n \in \mathbb{N}\}$ has a cluster point in $X$ (instead of $\cap \{A_n: n \in \mathbb{N}\}$) in the definition of countably bi-quasi-k-spaces".

We see that each condition in (b) ~ (h) of Metrization Theorem is essential by means of the following examples.

Examples. (1) Not every countably compact (resp. countably compact, first countable) space having a point-countable k-network (resp. locally countable network) is a $\sigma$-space.

(2) Not every Čech-complete (resp. metacompact developable) space having a $\sigma$-locally countable base is a $\sigma$-space [8] (resp. metrizable; cf. [11]).

(3) Not every first countable, Lindelöf space having a $\sigma$-closure preserving base is developable [6].

But, the following holds by means of [3] and [18], etc.

Proposition. (1) Every $\theta$-refinable space $X$ is developable if (a) or (b) below holds.

(a) $X$ is an $\text{m} \Delta$-space having a point-countable k-network, or having a $\sigma$-locally countable network.

(b) $X$ is a countably bi-quasi-k-space having a $\sigma$-locally countable k-network.

(2) Every $\text{m} \Delta$-space which is the quotient compact image (resp. quotient s-image) of a metric space is developable (resp. monotonically developable); see [18].
We note that every \textit{wΔ}-space having a \(\sigma\)-locally finite \textit{network} (resp. \(\sigma\)-locally countable \(k\)-network) is developable (resp. monotonically developable; cf. [18]).

In view of the above results, we have the following questions.

\textit{Questions.} (1) Every \(\textit{wΔ}\)-space having a \(\sigma\)-locally countable \(k\)-network (resp. \(\sigma\)-locally countable \textit{network}) is developable (resp. monotonically developable)?

(2) Every \(\textit{wΔ}\)-space which is the quotient countable-to-one image of a metric space is developable?

\textbf{References}


