数理解析研究所講究録
903 巻 1995 年 148-149

Two remarks on the subadditivity inequalities in von Neumann algebras [1]

by O. E. TIKHONOV

Taking into account a developed theory of operator monotone and operator convex functions (see e.g. [**1**–**3**]) it appears interesting to study operator subadditive functions. We do this within the context of von Neumann algebras though the main result is essentially a statement on 2×2-matrices.

In what follows we suppose that M is a von Neumann algebra and $\phi : [0,\infty) \to \mathbb{R}$ is a Borel measurable function bounded on bounded subsets of $[0,\infty)$. We say that ϕ is *operator subadditive with respect to M* or briefly *M-subadditive* if $\phi(a+b) \le \phi(a) + \phi(b)$ for every pair a, b of positive operators from M.

EXAMPLES. It is easy to see that the following functions on $[0,\infty)$ are M-subadditive for any M:

1) $\phi(t) = \alpha t + \beta$ ($\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^+$);

2) $\phi(t) = 1/(\alpha t + 1)$ ($\alpha \in \mathbb{R}^+$);

3) an arbitrary function ϕ satisfying $\alpha \le \phi(t) \le 2\alpha$ for some $\alpha \in \mathbb{R}^+$.

Subadditive real functions used in analysis often satisfy $\phi(0) = 0$. The following theorem shows that the class of operator subadditive functions satisfying this condition uses to be very small.

THEOREM 1. *Let M be a von Neumann algebra and let there exist a function $\phi : [0,\infty) \to \mathbb{R}$ such that $\phi(0) = 0$, ϕ is M-subadditive, and ϕ is not of the form $\phi(t) = \alpha t$ with $\alpha \in \mathbb{R}$. Then M is commutative.*

Proof. Let M be noncommutative. Suppose ϕ is M-subadditive and $\phi(0) = 0$. We will show that ϕ has to be of the form $\phi(t) = \alpha t$ for some $\alpha \in \mathbb{R}$.

Since M is noncommutative, it is easy to check that there exist two equivalent and mutually orthogonal nonzero projections in M, i.e., there exists a nonzero partial isometry $v \in M$ such that the projections $p = v^*v$ and $q = vv^*$ are mutually orthogonal (see e.g. [4]). Take positive reals ϵ, δ such that $\epsilon \le \delta$ and consider the pair of operators:

$$a = \epsilon p + \sqrt{\epsilon(\delta-\epsilon)}\, v + \sqrt{\epsilon(\delta-\epsilon)}\, v^* + (\delta-\epsilon)\, q,$$

$$b = \epsilon p - \sqrt{\epsilon(\delta-\epsilon)}\, v - \sqrt{\epsilon(\delta-\epsilon)}\, v^* + (\delta-\epsilon)\, q.$$

Observe that a and b are positive scalar multiples of projections and a straightforward computation shows that $\phi(a+b) = \phi(2\epsilon)p + \phi(2(\delta-\epsilon))q$, $\phi(a) = (\phi(\delta)/\delta)a$, $\phi(b) = (\phi(\delta)/\delta)b$. Whence, after multiplying the inequality $\phi(a+b) \le \phi(a) + \phi(b)$ by p from the left and the right we obtain $\phi(2\epsilon)p \le (2\epsilon\phi(\delta)/\delta)p$. Hence, $\phi(2\epsilon)/2\epsilon \le \phi(\delta)/\delta$. As the only restriction imposed on ϵ and δ is $0 < \epsilon \le \delta$, it follows that $\phi(t)/t$ is a constant, say α, on $(0,\infty)$. Thus, $\phi(t) = \alpha t$ on $[0,\infty)$.

[1] Supported by Russian Foundation for Basic Research, grant 93-011-16099.

By similar arguments, we can also prove the following.

THEOREM 2. *Let τ be a semifinite normal faithful trace on a noncommutative von Neumann algebra M. Let $\phi : [0,\infty) \to \mathbb{R}$ satisfy $\phi(0) = 0$ and*

$$\tau(\phi(a+b)) \leq \tau(\phi(a)) + \tau(\phi(b)) \qquad (*)$$

for every pair a, b of positive operators from M such that both sides of the inequality are well-defined. Then ϕ is concave on $[0,\infty)$.

This theorem may be viewed as a converse to the well-known assertion that, under certain restrictions, the inequality $(*)$ holds if ϕ is supposed to be a concave function on $[0,\infty)$ with $\phi(0) = 0$ (see [5], [6], [7]).

ACKNOWLEDGEMENT. The author would like to thank P. G. Ovchinnikov for a fruitful conversation.

REFERENCES

1. K. Löwner, Über monotone Matrixfunktionen, *Math. Z.* **38** (1934), 177–216.

2. J. Bendat and S. Sherman, Monotone and convex operator functions, *Trans. Amer. Math. Soc.* **79** (1955), 58–71.

3. F. Hansen and G. K. Pedersen, Jensen's inequality for operators and Löwner's theorem, *Math. Ann.* **258** (1982), 229–241.

4. M. Takesaki, *Theory of operator algebras* I (Springer-Verlag, 1979).

5. T. Fack and H. Kosaki, Generalized s-numbers of τ-measurable operators, *Pacific J. Math.* **123** (1982), 257–262.

6. L. G. Brown and H. Kosaki, Jensen's inequality in semi-finite von Neumann algebras, *J. Oper. Theory* **23** (1990), 3–19.

7. O. E. Tikhonov, Convex functions and inequalities for traces, in: *Konstr. Teor. Funktsiĭ i Funktsional. Anal.* **6** (Kazan Univ., 1987), 77–82 (in Russian).

Research Institute of Mathematics and Mechanics
Kazan University,
Universitetskaya 17,
Kazan, Tatarstan,
420008, Russia.
E-mail: tikhonov@niimm.kazan.su