Two remarks on the subadditivity inequalities von Neumann algebras

Author(s)
Tikhonov, O.E.

Citation
数理解析研究所講究録 (1995), 903: 148-149

Issue Date
1995-03

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
Two remarks on the subadditivity inequalities in von Neumann algebras

by O. E. TIKHONOV

Taking into account a developed theory of operator monotone and operator convex functions (see e.g. [1–3]) it appears interesting to study operator subadditive functions. We do this within the context of von Neumann algebras though the main result is essentially a statement on 2 × 2-matrices.

In what follows we suppose that $M$ is a von Neumann algebra and $\phi : [0, \infty) \to \mathbb{R}$ is a Borel measurable function bounded on bounded subsets of $[0, \infty)$. We say that $\phi$ is operator subadditive with respect to $M$ or briefly $M$-subadditive if $\phi(a + b) \leq \phi(a) + \phi(b)$ for every pair $a, b$ of positive operators from $M$.

EXAMPLES. It is easy to see that the following functions on $[0, \infty)$ are $M$-subadditive for any $M$:

1) $\phi(t) = \alpha t + \beta \ (\alpha \in \mathbb{R}, \beta \in \mathbb{R}^+)$;
2) $\phi(t) = 1/((\alpha t + 1) (\alpha \in \mathbb{R}^+))$;
3) an arbitrary function $\phi$ satisfying $\alpha \leq \phi(t) \leq 2\alpha$ for some $\alpha \in \mathbb{R}^+$.

Subadditive real functions used in analysis often satisfy $\phi(0) = 0$. The following theorem shows that the class of operator subadditive functions satisfying this condition uses to be very small.

THEOREM 1. Let $M$ be a von Neumann algebra and let there exist a function $\phi : [0, \infty) \to \mathbb{R}$ such that $\phi(0) = 0$, $\phi$ is $M$-subadditive, and $\phi$ is not of the form $\phi(t) = \alpha t$ with $\alpha \in \mathbb{R}$. Then $M$ is commutative.

Proof. Let $M$ be noncommutative. Suppose $\phi$ is $M$-subadditive and $\phi(0) = 0$. We will show that $\phi$ has to be of the form $\phi(t) = \alpha t$ for some $\alpha \in \mathbb{R}$.

Since $M$ is noncommutative, it is easy to check that there exist two equivalent and mutually orthogonal nonzero projections in $M$, i.e., there exists a nonzero partial isometry $v \in M$ such that the projections $p = v^*v$ and $q = vv^*$ are mutually orthogonal (see e.g. [4]). Take positive reals $\epsilon, \delta$ such that $\epsilon \leq \delta$ and consider the pair of operators:

$$a = \epsilon p + \sqrt{\epsilon(\delta - \epsilon)} v + \sqrt{\epsilon(\delta - \epsilon)} v^* + (\delta - \epsilon) q,$$

$$b = \epsilon p - \sqrt{\epsilon(\delta - \epsilon)} v - \sqrt{\epsilon(\delta - \epsilon)} v^* + (\delta - \epsilon) q.$$

Observe that $a$ and $b$ are positive scalar multiples of projections and a straightforward computation shows that $\phi(a + b) = \phi(2\epsilon)p + \phi(2(\delta - \epsilon))q$, $\phi(a) = (\phi(\delta)/\delta)a$, $\phi(b) = (\phi(\delta)/\delta)b$. Whence, after multiplying the inequality $\phi(a+b) \leq \phi(a) + \phi(b)$ by $p$ from the left and the right we obtain $\phi(2\epsilon)p \leq (2\epsilon \phi(\delta)/\delta)p$. Hence, $\phi(2\epsilon)/2\epsilon \leq \phi(\delta)/\delta$. As the only restriction imposed on $\epsilon$ and $\delta$ is $0 < \epsilon \leq \delta$, it follows that $\phi(t)/t$ is a constant, say $\alpha$, on $(0, \infty)$. Thus, $\phi(t) = \alpha t$ on $[0, \infty)$.  

1Supported by Russian Foundation for Basic Research, grant 93-011-16099.
By similar arguments, we can also prove the following.

THEOREM 2. Let $\tau$ be a semifinite normal faithful trace on a noncommutative von Neumann algebra $M$. Let $\phi : [0, \infty) \to \mathbb{R}$ satisfy $\phi(0) = 0$ and

$$\tau(\phi(a + b)) \leq \tau(\phi(a)) + \tau(\phi(b))$$

(*)

for every pair $a, b$ of positive operators from $M$ such that both sides of the inequality are well-defined. Then $\phi$ is concave on $[0, \infty)$.

This theorem may be viewed as a converse to the well-known assertion that, under certain restrictions, the inequality (*) holds if $\phi$ is supposed to be a concave function on $[0, \infty)$ with $\phi(0) = 0$ (see [5], [6], [7]).

ACKNOWLEDGEMENT. The author would like to thank P. G. Ovchinnikov for a fruitful conversation.

REFERENCES


Research Institute of Mathematics and Mechanics
Kazan University,
Universitetskaya 17,
Kazan, Tatarstan,
420008, Russia.
E-mail: tikhonov@niimm.kazan.su