the generalized Numerical Range and its Boundary

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Abstract. We consider the boundary of $C$-numerical range of a matrix and its related topics in the theory of invariant polynomials

1. the generalized numerical range of a matrix as a semi-algebraic set.

We recall the definition of the $C$-numerical range. Denote by $M_n(C)$ the set of all $n \times n$ complex matrices.

Def. Let $A, C$ be $n \times n$ complex matrices. Set

$$ W_C(A) = \{tr(CUAU^*) : U \in U(n)\} \tag{1.1} $$

where $U(n)$ is the compact group of all unitary matrices of order $n$. $W_C(A)$ is called the $C$-numerical range of $A$.

It is clear that $C$-numerical range is a unitary invariant of the square matrix $A$: If $B$ is unitarily similar to $A$, i.e., $B = U A U^*$, then $W_C(B) = W_C(A)$. If $C$ is a rank one orthogonal projection, then the $C$-numerical range $W_C(A)$ coincides with the classical numerical range $W(A)$,

$$ W(A) = \{(A\xi, \xi) : \xi \in C^n, ||\xi|| = 1\}.$$ 

The generalized numerical range $W_C(A)$ is the range of the real algebraic variety

$$ U(n) = \{U = \{u_{i,j}\}_{1 \leq i,j \leq n} \in C^{n^2} \cong \mathbb{R}^{2n^2} : \sum_{k=1}^{n} u_{i,k} u_{j,k} = \delta_{i,j} (1 \leq i,j \leq n)\} \tag{1.2} $$

under the polynomial mapping

$$ X (\in M_n(C) = C^{n^2} \cong \mathbb{R}^{2n^2}) \mapsto tr(CXAX^*) (\in C \cong \mathbb{R}^2) \tag{1.3} $$

We recall Tarski=Seidenberg’s theorem.

Def. A subset $\Lambda$ of the $n$-dimensional affine space $\mathbb{R}^n$ is said to be semi-algebraic if $\Lambda$ is an element of the Boolean algebra generated by the family of sets

$$ Z_{0}(f) = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : f(x_1, x_2, \ldots, x_n) = 0\}, $$

and

$$ Z_{1}(f) = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : f(x_1, x_2, \ldots, x_n) > 0\} $$

where $f$ varies over the real polynomial ring $\mathbb{R}[X_1, X_2, \ldots, X_n]$.

In other words, $\Lambda$ is a semi-algebraic set ($\subset \mathbb{R}^n$) \begin{definition} $\Lambda = \bigcup_{j=1}^{m} \bigcap_{k=1}^{l_{j,k}} Z_{\epsilon_{j,k}}(f_{j,k})$ \end{definition}

where $\epsilon_{j,k} \in \{0,1\}, f_{j,k} \in \mathbb{R}[X_1, X_2, \ldots, X_n]$.

Of course, an algebraic set ($\subset \mathbb{R}^n$) is semi-algebraic.

a Corollary of Tarski=Seidenberg’s theorem.

The range $\phi(\Lambda)$ of a semi-algebraic set $\Lambda \subset \mathbb{R}^n$ under a polynomial mapping $\phi = (\phi_1, \phi_2, \ldots, \phi_n) : \mathbb{R}^m \to \mathbb{R}^n$, is semi-algebraic, where $\phi_j \in \mathbb{R}[X_1, X_2, \ldots, X_m]$ (1 \leq j \leq n).

A Simple Example

$$ S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3 $$

projection

$$ \downarrow \text{projection} $$

A closed disc

$$ \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}. $$

We remark that the closed disc is semi-algebraic and not algebraic.
Another Result about Semi-Algebraic Sets. The boundary $\partial \Lambda$ of a semi-algebraic set $\Lambda(\subset \mathbb{R}^n)$ is also semi-algebraic.

Using the result above, Takaguchi, Nishikawa and I proved the following.

Theorem. [Takaguchi=Nishikawa=Nakazato[1]] The boundary $\partial W_C(A)$ of $C$-numerical range $W_C(A) \subset C \cong \mathbb{R}^2$ is the union of a finite number of real algebraic arcs.

Furthermore, using the theorem above and Bezout's theorem, we proved the following.

Theorem. The number of non $\omega$-regular points of $\partial W_C(A)$ is finite.

2. Concrete $C$-numerical ranges

Second I consider the $q$-numerical ranges of matrices.

We set

$$C = C_q = \left( \begin{array}{cccc} q & \sqrt{1-|q|^2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{array} \right).$$

We denote $W_{C_q}(A)$ by $W(A : q)$. We recall N.K. Tsing's formula.

A formula due to Tsing.

$$W(A : q) = \cup \{q(A\xi,\xi) + \sqrt{1-|q|^2} r \exp(i\theta) \sqrt{||A\xi||^2 - |(A\xi,\xi)|^2} : \xi \in \mathbb{C}^n, ||\xi|| = 1, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi \}$$

(2.2)

This formula is rewritten as

$$W(A : q) = \cup \{B(qz : \sqrt{\phi_A(z)} : z \in W(A : 1)) \}$$

(2.3),

where $\phi_A(z) = \max\{||A\xi||^2 - |z|^2 : \xi \in \mathbb{C}^n, ||\xi|| = 1, (A\xi,\xi) = z\}$

(2.5),

for $z \in W(A) = W(A : 1)$.

We set

$$\psi_A(z) = \max\{||A\xi||^2 : \xi \in \mathbb{C}^n, ||\xi|| = 1, (A\xi,\xi) = z\}.$$

We review N.K. Tsing's results.

Theorem. [N.K. Tsing [6]] $\psi_A$ is a concave function on $W(A)$ and hence $\phi_A$ and $\sqrt{\phi_A}$ are also concave on $W(A)$.

Using the theorem above we can prove the following two claims.

Proposition. $A, B_1, \ldots, B_m :$ matrices. $W(A) = \cup_{j=1}^m W(B_j), \phi_A(z) = \max\{\phi_{B_j}(z) : 1 \leq j \leq m\},$

$$\Rightarrow W(A : q) = \cup_{j=1}^m W(B_j : q)$$

for every $q$ with $|q| \leq 1$.

Theorem. [2] (1) Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are arbitrary complex numbers ($n \geq 3$). Then the following equation holds for every $z \in \text{Conv}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$:

$$\phi_{\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)}(z) = \max\{\phi_{\text{diag}(\lambda_{j_1}, \lambda_{j_2}, \lambda_{j_3})}(z) : 1 \leq j_1 < j_2 < j_3 \leq n\}.$$

Hence

$$W(\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) : q) = \cup \{W(\text{diag}(\lambda_{j_1}, \lambda_{j_2}, \lambda_{j_3}) : q) : 1 \leq j_1 < j_2 < j_3 \leq n\}.$$

(2) If $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ and $|\lambda_1| = |\lambda_2| = |\lambda_3| = 1$, then

$$\partial W(\text{diag}(\lambda_1, \lambda_2, \lambda_3) : q) \subseteq \{z \in \mathbb{C} : |z| = 1\} \cup \{\partial W(\text{diag}(\lambda_j, \lambda_k) : q) : 1 \leq j < k \leq 3\}.$$
3. C-numerical ranges of 2 by 2 matrices
The study of C-numerical ranges of 2 × 2 matrices is reduces to that of

$$W\left( \begin{array}{cc} 0 & \alpha \\ \beta & 0 \end{array} \right) : q$$

for $0 \leq \beta \leq \alpha, 0 \leq q \leq 1$.

We have the following formula(cf.[3]):

$$W\left( \begin{array}{cc} 0 & \alpha \\ \beta & 0 \end{array} \right) : q$$

$$=\{r[((\alpha + \beta)/2 + \sqrt{1-q^2}(\alpha - \beta)/2)\cos(\theta) + \sqrt{1-\text{tr}((\alpha - \beta)/2 + \sqrt{1-q^2}(\alpha + \beta)/2)\sin(\theta)} : 0 \leq r \leq 1,$$

$$0 \leq \theta \leq 2\pi\}, \text{where } 0 \leq \beta \leq \alpha, 0 \leq q \leq 1.$$

4. Relation between generalized numerical ranges and invariant polynomials
It is well known that the following three conditions for 2×2 matrices $x, y$ are mutually equivalent: (i) $W(x) = W(y); (ii) \text{tr}(x) = \text{tr}(y), \text{tr}(x^2) = \text{tr}(y^2), \text{tr}(x^*x) = \text{tr}(y^*y); (iii) y = UxU^*$ for some unitary matrix.

For $n \times n$ matrices $x, y$ the following three conditions are mutually equivalent: (iv) $W_R(x) = W_R(y)$ for every orthogonal projection $P,(v) W_H(x) = W_H(y)$ for every Hermitian matrix $H$; (vi) $\det(\lambda I_n - (\alpha x + \beta x^*)) = \det(\lambda I_n - (\alpha y + \beta y^*))$ for every $\lambda, \alpha, \beta \in \mathbb{C}$.

In the case $n = 3$, the condition (iv) is equivalent to the condition: $W(x) = W(y), \text{tr}(x) = \text{tr}(y)$.

Question. $x, y: 3 \times 3$ matrices such that

$$\text{tr}(x) = \text{tr}(y), W(x : q) = W(y : q)$$

for every $0 \leq q \leq 1$.

$$\Rightarrow 3U: \text{a unitary matrix such that } y = UxU^* \text{or } y = U(x)U^*.$$

I had the question above under the influence of the following theorem.

Theorem. (S. Teranishi, Nagoya Math. J. 1986[8]). Let $x, y$ be arbitrary $3 \times 3$ complex matrices. (1) If

$$\text{tr}(x^k) = \text{tr}(y^k) (k = 1, 2, 3) \quad (4.1),$$

$$\text{tr}(x^*x) = \text{tr}(y^*y) \quad (4.2),$$

$$\text{tr}(x^2x) = \text{tr}(y^2y) \quad (4.3),$$

$$\text{tr}(x^*x^2) = \text{tr}(y^*y^2) \quad (4.4),$$

then there exits a unitary matrix $U$ such that

$$y = UxU^* \text{ or } y = U(x)U^*.$$

(2) If the conditions (4.1),(4.2),(4.3),(4.4) holds and if

$$\text{tr}(x^2x^*x^2) = \text{tr}(y^2y^2y^*y) \quad (4.5),$$

then there exits a unitary matrix $U$ such that $y = UxU^*$.

We denote by $M_n(\mathbb{C})$ the $C^*$-algebra of all $n \times n$ complex matrices. I want to find a finite subset $F$ of $M_n(\mathbb{C})$ such that $\{W_C(\cdot) : C \in F\}$ is a complete unitary invariant system(cf.[7],Theorem 10). This desire is influenced by the following theorem.
Theorem (cf. [9]) Let $x, y$ be $n \times n$ complex matrices. Set $x_1 = x, x_2 = x^*, y_1 = y, y_2 = y^*$. If
\[
\text{tr}(x_1 x_2 \cdots x_p) = \text{tr}(y_1 y_2 \cdots y_p).
\]
for every $1 \leq p \leq 2^n - 1, 1 \leq i_1, i_2, \ldots, i_p \leq 2$, then there exists a unitary matrix $U$ such that $y = U x U^*$.

We can obtain another criterion by using the following lemma.

Lemma. Let $A, B$ be $C^*$-subalgebras of $M_n(C)$ such that $I_n \in A, I_n \in B$. If $\phi$ is a $*$-isomorphism of $A$ onto $B$ satisfying $\text{tr}(\phi(x)) = \text{tr}(x)$ for every $x \in A$, then there exists a unitary matrix $U$ such that $\phi(x) = U x U^*$ for every $x \in A$.

Proposition. If
\[
\text{tr}(x_1 x_2 \cdots x_p) = \text{tr}(y_1 y_2 \cdots y_p)
\]
for every $1 \leq p \leq 4(n^2 - 3), 1 \leq i_1, i_2, \ldots, i_p \leq 2$, then there exists a unitary matrix $U$, such that $y = U x U^*$.

This proposition is stronger than the theorem above under the condition
\[
n \geq 8.
\]

We have an analogous result.

Proposition. If
\[
\text{tr}(x_1 x_2 \cdots x_q) = \text{tr}(y_1 y_2 \cdots y_q)
\]
for every $1 \leq q \leq 2(n^2 - 3), 1 \leq i_1, i_2, \ldots, i_q \leq 2$, then there exists a $*$-preserving, Hilbert-Schmidt norm preserving linear bijection of the $C^*$-algebra $C^*(x)$ onto the $C^*$-algebra $C^*(y)$.

5. 3 by 3 orthostochastic matrices

We consider a $3 \times 3$ real matrix
\[
A(x, y, z : u, v, w) = \begin{pmatrix} x + w & y + u & z + v \\ x + u & x + v & y + w \\ y + v & z + w & x + u \end{pmatrix} = xe_1 + ye_2 + ze_3 + u e_4 + v e_5 + w e_6,
\]
for $x \geq 0, y \geq 0, z \geq 0, u \geq 0, v \geq 0, w \geq 0, x + y + z + u + v + w = 1$. Here
\[
e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
\[
e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
\]
\[
e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]
\[
e_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
\[
e_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]
\[
e_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Such a matrix $A(x, y, z : u, v, w)$ is called a $3 \times 3$ doubly stochastic matrix. A ($3 \times 3$) doublyochastic matrix $D$ is said to be orthostochastic if it is the Hadamard product (entrywise product) of some ($3 \times 3$) unitary matrix $U$ and its conjugate matrix $\overline{U}$. The necessary and sufficient condition of Au-Yeng and Poon for a doubly stochastic matrix
\[
A(x, y, z : u, v, w)
\]
to be orthostochastic (cf. [5]) is stated in another fashion (cf. [4]):
\[
x^2 y^2 + x^2 z^2 + y^2 z^2 - 2xyz(x + y + z) + u^2 v^2 + u^2 w^2 + v^2 w^2 - 2uvw(u + v + w)
\]
\[-4uvw(x + y + z) - 4xyz(u + v + w) - 2(xy + xz + yz)(uv + uw + vw) \leq 0. \quad (5.1)
\]

Using this characterization we can obtain the following proposition.
Proposition ([4]) Suppose that $A_3$ is the affine hull of the compact convex set of all $3 \times 3$ doubly stochastic matrices and $O_3$ is the compact set of all $3 \times 3$ orthostochastic matrices. Denote by $\partial O_3$ the boundary of $O_3$ in the affine space $A_3$. Then the following equation holds:

$$\partial O_3 = \{g \circ g : g \in SO(3)\},$$

where $SO(3)$ is the group of rotations in the 3-dimensional Euclidean space.

Consider a linear functional $\Psi$ of the complex vector space $M_3(\mathbb{C})$,

$$\Psi: X = \{x_{i,j} : 1 \leq i,j \leq 3\} \in M_3(\mathbb{C}) \mapsto (x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33}).$$

Suppose that $A$ and $C$ are complex $3 \times 3$ diagonal matrices with diagonal entries

$$a = \{a_1, a_2, a_3\}, \quad c = \{c_1, c_2, c_3\}$$

respectively. Consider the $3 \times 3$ matrix $[A : C]$ given as the product $'(a)c$, that is,

$$[A : C] = \begin{pmatrix} a_1 c_1 & a_1 c_2 & a_1 c_3 \\ a_2 c_1 & a_2 c_2 & a_2 c_3 \\ a_3 c_1 & a_3 c_2 & a_3 c_3 \end{pmatrix}.$$

Then the $C$-numerical range of $A$ coincides with the set

$$\{\Psi([A : C] \circ G) : G \in O_3\}.$$

The following proposition is effective to determine the $C$-numerical range of a matrix $A$ under the condition $A = C$ and $A$ is a $3 \times 3$ complex diagonal matrix.

**Proposition ([4])** Suppose that $A$ is a complex $3 \times 3$ diagonal matrix. Then the $A$-numerical range $W_A(A)$ coincides with the set

$$\{\Psi([A : A] \circ G) : G \in O_3 \text{ and } (\mathbf{t} G) = G\}.$$

Under the condition

$$A = \text{diag}\{1, 0, \alpha\},$$

the description of the boundary of the range $W_A(A)$ concerns deeply the curve " (Steiner's) Deltoid". We define the Deltoid $D[x_0 : K : \exp[i \eta]]$ ( $x_0$ is a complex number, $K$ is a positive number, $\eta$ is a real number ) in the complex plane $C$ by the following: Choose $\exp[i \theta]$ ( $\theta$ is a real number ) in an arbitrary cubic root of $\exp[i \eta]$.

$$D[x_0 : K : \exp[i \eta]] = \{x_0 + (K/3) \exp[i \theta](2 \exp[2i \phi] + \exp[-i \phi]) : 0 \leq \phi \leq 2 \pi\}.$$

We obtain the following theorem.

**Theorem ([4])** Suppose that $A = \text{diag}\{1, 0, \alpha\}$ and $\alpha = a + i b$ where $a, b$ are real numbers with $b \neq 0$. Then the boundary of the range $W_A(A)$ is a subset of the following set

$$\{(1-t)(1+\alpha^2) + t : 0 \leq t \leq 1\} \cup \{(1-t)(1+\alpha^2) + 2t\alpha : 0 \leq t \leq 1\} \cup \{(1-t)(1+\alpha^2) + t(\alpha)^2 : 0 \leq t \leq 1\}$$

$$\cup D[z_0 : K : \exp[i \eta]]$$

where

$$z_0 = (1/(4b^2))\alpha(\overline{\alpha} - 1)(3a^2 - 3a - b^2 + i [b + 4ab]),$$

$$K = (3/(4b^2))|\alpha - 1|^2 |\alpha|^2.$$
\[ |\alpha|^2 |\alpha-1|^2 \exp[i \eta] = -(a-a^2 + b - 2a b + b^2)(a-a^2 - b + 2a b + b^2) + i(-2 b)(2a-1)(a^2 - a - b^2). \]

We give you some graphics about $C$-numerical ranges after this document.

References


\[ W_C(C) \]

(11-3) \( C = \text{diag}\{ 1, 0, 2 + 3 i \} \).
Graphics of C-numerical Ranges of Matrices and the set of
3 x 3 symmetric orthostochastic matrices

(I) The q-numerical range of a 3 x 3 complex diagonal matrix A.
(i) The diagonal entries of A are mutually distinct cubic
   roots of 1.
(ii) q = 4/5.

(II) The C-numerical range W c(C) of the 3 x 3 complex diagonal
    matrix.
(II-1) The diagonal entries of C are mutually disdistinct cubic
    roots of 1. This generalized numerical range coincides with the
    closed domain surrounded by the Deltoid D(0,3,1):
    x = 2 cos(2t) + cos(2t), y = 2 sin(2t) - sin(2t)
(II-2) \( C = \text{diag}\{1, 0, (1/4) + i (1/2)\} \).

One graphic describes the range \( W \circ (C) \) based on the command "ParametricPlot3D" (2-parameter plot) in "Mathematica" (ver2.0, hardware : Apple, Quadra 900). Other three graphics describe its boundary based on the command "ParametricPlot" (1-parameter plot on the plane).
(III) The set of $3 \times 3$ symmetric orthostochastic matrices in the $3$-dimensional affine space $H$ of all $3 \times 3$ symmetric real matrices $\{(a_{11},a_{12},a_{13}),(a_{12},a_{22},a_{23}),(a_{13},a_{23},a_{33})\}$ such that $a_{11}+a_{12}+a_{13}=1$.

We use a non-orthogonal coordinates system $\langle , , \rangle$ in $H$ for which:

$\{(1/3,1/3,1/3),(1/3,1/3,1/3),(1/3,1/3,1/3)\}=<0,0,0>$,
$\{(0,1,0),(1,0,0),(0,0,1)\}=<0,0,1>$,
$\{(0,0,1),(0,1,0),(1,0,0)\}=-1/2,1/2,0>$,
$\{(1,0,0),(0,0,1),(0,1,0)\}=-1/2,-1/2,0>$,
$\{(1,0,0),(0,1,0),(0,0,1)\}=0,0,b$ for some $b>0$. 

![Diagram](image-url)