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次数付きリー代数に対するトレース公式とモンストラス·ムーンシャイン

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1 序文

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有限単純群の分類において，リー型の単純群の１６無限系列と n 文字上の交代群 A_n $(n \geq 5)$ の無限系列以外に丁度 26 個の散在型単純群が存在する．これらの中の最大のものは位数

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71,$$

を持ち，その巨大さ故に，モンスターと呼ばれている．

モンスター単純群 G の自明な表現の次数は定義より 1 であり，自明でない最小の既約表現の次数は 196883 ([FLT]) である．マッカイは 1+196883=196884 であることに気付いた．この数は楕円モジュラー関数

$$j(q)-744=\sum_{n\geq -1} c(n)q^n = q^{-1}+196884q+21493760q^2+\cdots\cdots$$

の最初の自明でない係数である．

後に，トンプソンはモジュラー関数 $j(q)-744$ の最初の幾つかの係数が G の既約表現の次数の簡単な線形結合となっていることを見つけた ([T])．これらの観察に刺激されて，コンウエイとノートンはモンスター単純群 G の無限次元次数付き加群 $V=\bigoplus_{n\geq -1} V_n$ で $\dim V_n = c(n)$ を満足し，さらにそれのトンプソン級数

$$T_g(q)=\sum_{n\geq -1} Tr(g|V_n)q^n = \sum_{n\geq -1} c_g(n)q^n$$

が $PSL(2,\mathbf{R})$ のある離散部分群から出てくる種数 0 の関数体の正規化された生成元となっているものが存在するだろうと予想した ([CN]). この予想をムーンシャイン予想と呼ぶ.

ムーンシャイン予想におけるモンスター単純群 G の自然な次数付き表現 $V=\bigoplus_{n\geq -1}V_n$ は最終的にフレンケル, レポウスキ, ミュアマンによって構成された ([FLM]).

[$B5$] の中で, ボーチャードはモンスターリー代数と呼ばれる $II_{1,1}$-次数付きリー代数 $M=\bigoplus_{(m,n)\in II_{1,1}}M_{(m,n)}$ を構成することによってムーンシャイン予想の証明を完成させた. ここで, $II_{1,1}$ は行列$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$ を持つ 2 次元ローレンチアン格子である.

モンスターリー代数 M はモンスター単純群 G の $II_{1,1}$-次数付き表現であって, かつ $(m,n)\neq(0,0)$ に対して, G-加群として $M_{(m,n)}\cong V_{mn}$ となっているものである. それゆえ, 全ての $g\in G$, $(m,n)\neq(0,0)$ に対して,

$$Tr(g|M_{(m,n)})=Tr(g|V_{mn})=c_g(mn)$$

が成り立っている.

一方, モンスターリー代数 M は実単純ルート $(1,-1)$ と重複度 $c(i)$ を持つ虚単純ルート $(1,i)\quad(i\geq 1)$ を持つ $II_{1,1}$-次数付き一般カッツムーディ代数である. 一般カッツムーディ代数はボーチャードによって頂点代数とモンストラス· ムーンシャインの研究において導入された. ([$B1$]-[$B5$], [K]). 一般カッツムーディ代数の構造や表現論はカッツムーディ代数のものと非常に類似しており, カッツムーディ代数に関するほとんど全ての結果がほとんど同じ証明で一般カッツムーディ代数に拡張できる ([K]). 例えば, 対称化可能一般カッツムーディ代数上のユニタリ化可能既約最高次ウエイト加群に対してはワイル·カッツ·ボーチャード公式と呼ばれる指標公式を得ることができ, それを 1 次元の自明な表現に応用すると分母恒等式を得ることができる.

[$Ka2$] に於いて, ワイル·カッツ·ボーチャード公式と分母恒等式を使って, 著者の一人は全ての対称化可能一般カッツムーディ代数に対する closed form root multiplicity formula を得た. この root multiplicity formula をモンスターリー代数に応用すると, 楕円モジュラー関数 $j(q)-744$ ([$Ju2$] を参照) の係数 $c(n)$ に対する興味ある関係式を幾つか得ることができる. より正確に述べると, $k,l>0$ に対して,

$$T(k,l)=\{\underline{b}=(b_{ij})_{i,j\geq 1}|\ b_{ij}\in\mathbf{Z}_{\geq 0},\ \sum_{i,j\geq 1}b_{ij}(i,j)=(k,l)\},$$

を正の整数の順序付き組みの和への (k,l) の全ての分解の集合と定義する. この時, $m,n>0$ に対して, ([$Ju2$], [$Ka2$])

$$c(mn)=\sum_{d>0 d|(m,n)}\frac{1}{d}\mu(d)\sum_{\underline{b}\in T(m/d,n/d)}\frac{(\sum b_{ij}-1)!}{\Pi\,(b_{ij}!)}\prod c(i+j-1)^{b_{ij}}$$

を得る.

この論文に於いては, 少し一般化した" 同値" 条件で考察する. 即ち, Γ をある適切な有限条件を満足するアーベル半群とし, $L=\bigoplus_{\alpha\in\Gamma}L_\alpha$ を 有限次元ホモジニアス空間を持つΓ-次数付きリー代数とする. 群 G がΓ-次数を保つ自己同型としてリー代数 L に作用しているとする. この時, オイラー·ポアンカレ原理とメイビウス反転公式を使って, 全ての $g\in G$, $\alpha\in\Gamma$ に対してトレイス $Tr(g|L_\alpha)$ に対する良い closed form 公式を得ることが出来る.

我々のトレイス公式をモンスターリー代数に作用しているモンスター単純群に応用すると，上で述べた $c(n)$ に対する関係式の一般化であるトンプソン級数の係数 $c_g(n)$ に対する以下の興味ある関係式を得ることができる.

$$c_g(mn) = \sum_{d>0 d|(m,n)} \frac{1}{d}\mu(d) \sum_{\underline{b}\in T(m/d,n/d)} \frac{(\sum b_{ij}-1)!}{\Pi(b_{ij}!)} \prod c_{g^d}(i+j-1)^{b_{ij}}.$$

この種の関係式は S.J.Kang が 1994 年の春にオハイオ州立大学で一般カッツムーディ代数とモジュラ関数 j に関する講演をおこなったときに原田耕一郎教授に示唆されたものである．素晴らしい考察と多くの価値ある助言に対して原田教授に対して感謝をささげたい．我々の仕事の主要な部分は著者達が 1994 年 6 月にカナダの Banff におけるカナダ数学会年会セミナーに参加したときに完成させたものである．特に，その様な素晴らしい会議を開いていただいた Gerald Cliff 教授，Robert W. Moody 教授と，Arturo Pianzola 教授に感謝したい．また，トンプソン級数の係数　$c_g(n)$ に対する同様の関係式を独立に得た結果のプレプリントを送って頂いた Jurisich, James Lepowsky, と Robert L. Wilson の各教授に対して感謝する ($[JLW]$).

§1. Group characters and graded Lie algebras

Let Γ be an additive abelian semigroup and let $V = \bigoplus_{\alpha\in\Gamma} V_\alpha$ be a Γ-graded vector space such that $\dim V_\alpha < \infty$ for all $\alpha \in \Gamma$. Let G be a group, and suppose G acts on V in such a way that G preserves the Γ-gradation on V. That is, $g\cdot V_\alpha \subset V_\alpha$ for all $g \in G$, $\alpha \in \Gamma$. Then, for each $\alpha \in \Gamma$, every element $g \in G$ defines an invertible linear map $\varphi_g : V_\alpha \to V_\alpha$ given by $\varphi_g(v) = g \cdot v$ for all $g \in G$, $v \in V_\alpha$. Let us denote by $Tr(g|V_\alpha)$ the trace of φ_g on V_α. We define the *generalized character* $\mathrm{ch}_g(V)$ of g on V to be

$$\mathrm{ch}_g(V) = \sum_{\alpha\in\Gamma} Tr(g|V_\alpha)e^\alpha, \tag{1.1}$$

where e^α are the basis elements of the semigroup algebra $\mathbf{C}[\Gamma]$ with the multiplication $e^\alpha e^\beta = e^{\alpha+\beta}$ for $\alpha, \beta \in \Gamma$. In particular, when $g = 1$, the identity element of G, we obtain the usual character of V:

$$\mathrm{ch}(V) = \sum_{\alpha\in\Gamma} (\dim V_\alpha)e^\alpha. \tag{1.2}$$

In this paper, we assume that every element $\alpha \in \Gamma$ can be written as a sum of elements in Γ only in finitely many ways. For example, the semigroup $\mathbf{Z}_{>0}$ of positive integers satisfies our condition, whereas the monoid $\mathbf{Z}_{\geq 0}$ of nonnegative integers doesn't.

Now we consider a Γ-graded Lie algebra

$$L = \bigoplus_{\alpha\in\Gamma} L_\alpha, \tag{1.3}$$

and suppose that a group G acts on L by automorphisms preserving the Γ-gradation of L. We would like to derive a closed form formula for $Tr(g|L_\alpha)$ for all $g \in G, \alpha \in \Gamma$.

Recall that the homology modules $H_k(L) = H_k(L, \mathbf{C})$ are determined from the following complex

$$\begin{aligned} \cdots \longrightarrow \Lambda^k(L) \xrightarrow{d_k} \Lambda^{k-1}(L) \longrightarrow \cdots \\ \longrightarrow \Lambda^1(L) \xrightarrow{d_1} \Lambda^0(L) \xrightarrow{d_0} \mathbf{C} \longrightarrow 0, \end{aligned} \tag{1.4}$$

where the differentials $d_k : \Lambda^k(L) \to \Lambda^{k-1}(L)$ are defined by

$$d_k(x_1 \wedge \cdots \wedge x_k) = \sum_{s<t} (-1)^{s+t}[x_s, x_t] \wedge x_1 \wedge \cdots \wedge \widehat{x_s} \wedge \cdots \wedge \widehat{x_t} \wedge \cdots \wedge x_k \tag{1.5}$$

for $k \geq 2$, $x_i \in L$, and $d_1 = d_0 = 0$.

Each of the terms $\Lambda^k(L)$ has the Γ-gradation induced by that of L: for $\alpha \in \Gamma$, we define $\Lambda^k(L)_\alpha$ to be the subspace of $\Lambda^k(L)$ spanned by the vectors of the form $x_1 \wedge \cdots \wedge x_k$ ($x_i \in L$) such that $\deg(x_1) + \cdots + \deg(x_k) = \alpha$. We define the action of G on $\Lambda^k(L)$ by

$$g \cdot (x_1 \wedge \cdots \wedge x_k) = (g \cdot x_1) \wedge \cdots \wedge (g \cdot x_k) \tag{1.6}$$

for all $g \in G, x_i \in L$. Since the action of G on L preserves the Γ-gradation of L, the action of G on $\Lambda^k(L)$ also preserves the Γ-gradation of $\Lambda^k(L)$. Similarly, the homology modules $H_k(L)$ inherits the Γ-gradation from $\Lambda^k(L)$, and since G commutes with the d_k the group G acts on $H_k(L)$ preserving the Γ-gradation. Thus we can consider the generalized characters for $\Lambda^k(L)$ and $H_k(L)$:

$$\mathrm{ch}_g \Lambda^k(L) = \sum_{\alpha \in \Gamma} Tr(g|\Lambda^k(L)_\alpha) e^\alpha, \tag{1.7}$$

$$\mathrm{ch}_g H_k(L) = \sum_{\alpha \in \Gamma} Tr(g|H_k(L)_\alpha) e^\alpha \tag{1.8}$$

for $g \in G$, $\alpha \in \Gamma$.

By the Euler-Poincaré principle, we have

$$\sum_{k=0}^{\infty} (-1)^k \mathrm{ch}_g \Lambda^k(L) = \sum_{k=0}^{\infty} (-1)^k \mathrm{ch}_g H_k(L). \tag{1.9}$$

Recall that the alternating direct sum of the vector spaces $\sum_{k=0}^{\infty}(-1)^k \Lambda^k(L)$ is naturally isomorphic to $\exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} \Psi^k(L)\right)$, where Ψ^k is the k-th Adams operation ([A]). For $g \in G$ and $\alpha \in \Gamma$, the Adams operation Ψ^k on L is defined by $Tr(g|\Psi^k(L_\alpha)) = Tr(g^k|L_\alpha)$ and $\Psi^k(e^\alpha) = e^{k\alpha}$. It follows that

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k \mathrm{ch}_g \Lambda^k(L) &= \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} \mathrm{ch}_g \Psi^k(L)\right) \\ &= \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} \sum_{\alpha \in \Gamma} Tr(g|\Psi^k(L_\alpha)) e^{k\alpha}\right) \\ &= \exp\left(-\sum_{\alpha \in \Gamma} \sum_{k=1}^{\infty} \frac{1}{k} Tr(g^k|L_\alpha) e^{k\alpha}\right). \end{aligned} \tag{1.10}$$

Let

$$H = \sum_{k=1}^{\infty} (-1)^{k+1} H_k(L), \tag{1.11}$$

an alternating direct sum of G-modules. For $g \in G$ and $\alpha \in \Gamma$, we define

$$Tr(g|H_\alpha) = \sum_{k=1}^{\infty} (-1)^{k+1} Tr(g|H_k(L)_\alpha), \tag{1.12}$$

and

$$\mathrm{ch}_g(H) = \sum_{\alpha \in \Gamma} Tr(g|H_\alpha) e^\alpha = \sum_{k=1}^{\infty} (-1)^{k+1} \mathrm{ch}_g H_k(L). \tag{1.13}$$

Combining (1.10) and (1.13), (1.9) can be written as

$$\exp\left(-\sum_{\alpha \in \Gamma} \sum_{k=1}^{\infty} \frac{1}{k} Tr(g^k|L_\alpha) e^{k\alpha} \right) = 1 - \mathrm{ch}_g(H). \tag{1.14}$$

Let $P_g(H) = \{\alpha \in \Gamma |\ Tr(g|H_\alpha) \neq 0\}$ and $\{\tau_i |\ i \geq 1\}$ be an enumeration of $P_g(H)$. For $g \in G$, $\tau \in \Gamma$, let

$$T_g(\tau) = \{(n) = (n_i)_{i \geq 1} |\ n_i \in \mathbf{Z}_{\geq 0}, \sum n_i \tau_i = \tau\}. \tag{1.15}$$

Thus the set $T_g(\tau)$ is the set of all partitions of τ into a sum of τ_i's. We define a function

$$B_g(\tau) = \sum_{(n) \in T_g(\tau)} \frac{(\sum n_i - 1)!}{\Pi(n_i!)} \prod Tr(g|H_{\tau_i})^{n_i}. \tag{1.16}$$

We now obtain the following closed form formula for $Tr(g|L_\alpha)$ $(g \in G, \alpha \in \Gamma)$, which is a generalization of the closed form formula for $\dim L_\alpha$ obtained in [Ka3].

Theorem 1.1. *For $g \in G$, $\alpha \in \Gamma$, we have*

$$Tr(g|L_\alpha) = \sum_{\substack{d>0 \\ d|\alpha}} \frac{1}{d} \mu(d) B_{g^d}(\alpha/d), \tag{1.17}$$

where μ is the classical Möbius function.

Proof. By (1.14), we have

$$\exp\left(\sum_{\alpha \in \Gamma} \sum_{k=1}^{\infty} \frac{1}{k} Tr(g^k|L_\alpha) e^{k\alpha} \right) = \frac{1}{1 - \mathrm{ch}_g(H)} = \frac{1}{1 - \sum_{i=1}^{\infty} Tr(g|H_{\tau_i}) e^{\tau_i}}.$$

Using the formal power series $\log(1-t) = -\sum_{k=1}^{\infty} \frac{t^k}{k}$, we obtain from the right hand side

$$\log\left(\frac{1}{1-\sum_{i=1}^{\infty} Tr(g|H_{\tau_i})e^{\tau_i}}\right) = -\log\left(1-\sum_{i=1}^{\infty} Tr(g|H_{\tau_i})e^{\tau_i}\right)$$
$$= \sum_{m=1}^{\infty} \frac{1}{m}\left(\sum_{i=1}^{\infty} Tr(g|H_{\tau_i})e^{\tau_i}\right)^m$$
$$= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{(n)=(n_i) \\ \sum n_i = m}} \frac{(\sum n_i)!}{\Pi n_i!} \prod Tr(g|H_{\tau_i})^{n_i} e^{\sum n_i \tau_i}$$
$$= \sum_{\tau}\left(\sum_{(n)\in T_g(\tau)} \frac{(\sum n_i - 1)!}{\Pi n_i!} \prod Tr(g|H_{\tau_i})^{n_i}\right) e^{\tau}$$
$$= \sum_{\tau} B_g(\tau) e^{\tau}.$$

The left hand side yields

$$\log\exp\left(\sum_{\alpha\in\Gamma}\sum_{k=1}^{\infty} \frac{1}{k} Tr(g^k|L_\alpha)e^{k\alpha}\right) = \sum_{\alpha\in\Gamma}\sum_{k=1}^{\infty} \frac{1}{k} Tr(g^k|L_\alpha)e^{k\alpha}.$$

Hence we have

$$B_g(\tau) = \sum_{\substack{k>0 \\ \tau = k\alpha}} \frac{1}{k} Tr(g^k|L_\alpha).$$

Therefore, by Möbius inversion, we obtain

$$Tr(g|L_\alpha) = \sum_{\substack{d>0 \\ \alpha = d\tau}} \frac{1}{d}\mu(d) B_{g^d}(\tau). \qquad \square$$

Example. For $i \geq 1$, let V_i be a complex vector space of dimension d_i, and let $V = \bigoplus_{i\geq 1} V_i$. Consider the free Lie algebra L generated by V. For each $i \geq 1$, we let $\alpha_i = (0, \cdots, 0, 1, 0, \cdots)$, where 1 appears in the i-th place, and define an abelian semigroup $\Gamma = \left(\bigoplus_{i\geq 1} \mathbf{Z}_{\geq 0}\alpha_i\right) \setminus \{0\}$. Then the free Lie algebra L is a Γ-graded Lie algebra $L = \bigoplus_{\alpha\in\Gamma} L_\alpha$ by defining $\deg v = \alpha_i$ for $v \in V_i$.

Let $G = \prod_{i\geq 1} GL(d_i) = GL(d_1) \times GL(d_2) \times \cdots$, where $GL(d_i) = GL(V_i)$. Then G acts on L by automorphisms preserving the Γ-gradation. Thus we can apply our trace formula (1.17) to this setting.

Recall that, since L is the free Lie algebra generated by V, we have

$$H_1(L) = V = \bigoplus_{i\geq 1} V_i, \qquad (1.18)$$
$$H_k(L) = 0 \text{ for all } k \geq 2.$$

Therefore, for $g = (g_i)_{i\geq 1} \in G$ with $g_i \in GL(d_i)$, we have $H = H_1(L) = V$, $P_g(H) = \{\alpha_i |\ i \geq 1\}$, and $H_{\alpha_i} = V_i$, which implies

$$(1.19) \qquad Tr(g|H_{\alpha_i}) = Tr(g_i|V_i) \stackrel{\text{def}}{=} t_i(g_i).$$

Note that, for $\tau = \sum_{i\geq 1} s_i\alpha_i \in \Gamma$, we have

$$(1.20) \qquad T_g(\tau) = \{(s_1, s_2, s_3, \cdots)\},$$

since $\tau = s_1\alpha_1 + s_2\alpha_2 + \cdots$ is the only partition of τ into a sum of α_i's. It follows that

$$(1.21) \qquad B_g(\tau) = \frac{(\sum s_i - 1)!}{\Pi s_i!} \prod t_i(g_i)^{s_i},$$

and, for $\alpha = \sum_{i\geq 1} k_i\alpha_i$, our trace formula (1.17) yields

$$(1.22) \qquad Tr(g|L_\alpha) = \sum_{\substack{d>0 \\ d|k_i \text{ for all } i}} \frac{1}{d}\mu(d)\frac{(\sum k_i/d - 1)!}{\Pi(k_i/d)!} \prod t_i(g_i^d)^{k_i/d}. \qquad \square$$

§2. Generalized Kac-Moody algebras

The generalized Kac-Moody algebras were introduced by Borcherds in his study of vertex algebras and Monstrous Moonshine ([B1]–[B5], [K]). In this section, we recall the basic theory of generalized Kac-Moody algebras and discuss the application of our trace formula (1.17) to generalized Kac-Moody algebras.

Let I be a countable (possibly infinite) index set. A real matrix $A = (a_{ij})_{i,j\in I}$ is called a *Borcherds-Cartan matrix* if it satisfies: (i) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$, (ii) $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in \mathbf{Z}$ if $a_{ii} = 2$, (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. Let $I^{re} = \{i \in I|\ a_{ii} = 2\}$, $I^{im} = \{i \in I|\ a_{ii} \leq 0\}$, and let $\underline{m} = (m_i|\ i \in I)$ be a collection of positive integers such that $m_i = 1$ for all $i \in I^{re}$. We call $\underline{m}$ the *charge* of the matrix A. A Borcherds-Cartan matrix A is said to be *symmetrizable* if there is a diagonal matrix $D = diag(s_i|\ i \in I)$ with $s_i > 0$ $(i \in I)$ such that DA is symmetric. In this paper, we assume that A is symmetrizable.

Definition 2.1. *The generalized Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m})$ with a symmetrizable Borcherds-Cartan matrix A of charge $\underline{m} = (m_i|\ i \in I)$ is the Lie algebra over $\mathbf{C}$ generated by the elements h_i, d_i $(i \in I)$, e_{ik}, f_{ik} $(i \in I,\ k = 1, \cdots, m_i)$ with the defining relations:*

$$(2.1) \qquad \begin{aligned} &[h_i, h_j] = [h_i, d_j] = [d_i, d_j] = 0, \\ &[h_i, e_{jl}] = a_{ij}e_{jl}, \quad [h_i, f_{jl}] = -a_{ij}f_{jl}, \\ &[d_i, e_{jl}] = \delta_{ij}e_{jl}, \quad [d_i, f_{jl}] = -\delta_{ij}f_{jl}, \\ &[e_{ik}, f_{jl}] = \delta_{ij}\delta_{kl}h_i, \\ &(ade_{ik})^{1-a_{ij}}(e_{jl}) = (adf_{ik})^{1-a_{ij}}(f_{jl}) = 0 \quad \textit{if } a_{ii} = 2 \textit{ and } i \neq j, \\ &[e_{ik}, e_{jl}] = [f_{ik}, f_{jl}] = 0 \quad \textit{if } a_{ij} = 0 \end{aligned}$$

for $i, j \in I,\ k = 1, \cdots, m_i,\ l = 1, \cdots, m_j$.

The abelian subalgebra $\mathfrak{h} = (\bigoplus_{i\in I} \mathbf{C}h_i) \bigoplus (\bigoplus_{i\in I} \mathbf{C}d_i)$ is called the *Cartan subalgebra* of $\mathfrak{g}$. For each $j \in I$, we define a linear functional $\alpha_j \in \mathfrak{h}^*$ by

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d_i) = \delta_{ij} \quad \text{for } i, j \in I.$$

Let $\Pi = \{\alpha_i |\ i \in I\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{h_i |\ i \in I\} \subset \mathfrak{h}$. The elements of Π (resp. $\Pi^\vee$) are called the *simple roots* (resp. *simple coroots*) of $\mathfrak{g}$.

Let $Q = \bigoplus_{i\in I} \mathbf{Z}\alpha_i$ be the free abelian group generated by α_i's ($i \in I$). We call Q the *root lattice* of $\mathfrak{g}$. Set $Q_+ = \sum_{i\in I} \mathbf{Z}_{\geq 0}\alpha_i$, and $Q_- = -Q_+$. We define a partial ordering $\leq$ on $\mathfrak{h}^*$ by $\lambda \leq \mu$ if and only if $\lambda - \mu \in Q_-$. The generalized Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m})$ has the *root space decomposition* $\mathfrak{g} = \bigoplus_{\alpha\in Q} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} |\ [h, x] = \alpha(h)x \ \text{ for all } h \in \mathfrak{h}\}$ is the α*-root space*. Note that $\mathfrak{g}_{\alpha_i} = \mathbf{C}e_{i,1} \oplus \cdots \oplus \mathbf{C}e_{i,m_i}$, and $\mathfrak{g}_{-\alpha_i} = \mathbf{C}f_{i,1} \oplus \cdots \oplus \mathbf{C}f_{i,m_i}$. We say that $\alpha \in Q$ is a *root* if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. The number mult$\alpha$:=dim$\mathfrak{g}_\alpha$ is called the *multiplicity* of the root α. A root $\alpha > 0$ (resp. $\alpha < 0$) is called *positive* (resp. *negative*). We denote by Δ, Δ^+, and Δ^- the set of all roots, positive roots, and negative roots, respectively. Define the subspaces $\mathfrak{g}^\pm = \bigoplus_{\alpha\in\Delta^\pm} \mathfrak{g}_\alpha$. Then we have the *triangular decomposition* : $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$.

Since A is symmetrizable, there is a symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}^*$ satisfying $(\alpha_i|\alpha_j) = s_i a_{ij}$ for $i, j \in I$. We say that a root α is *real* if $(\alpha|\alpha) > 0$, and *imaginary* if $(\alpha|\alpha) \leq 0$. In particular, the simple root α_i is real if $a_{ii} = 2$, and imaginary if $a_{ii} \leq 0$. Note that the imaginary simple roots may have multiplicity > 1. For each $i \in I^{re}$, let $r_i \in \mathrm{GL}(\mathfrak{h}^*)$ be the *simple reflection* on $\mathfrak{h}^*$ defined by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ for $\lambda \in \mathfrak{h}^*$. The subgroup W of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the r_i's ($i \in I^{re}$) is called the *Weyl group* of $\mathfrak{g}$.

Let G be a group and suppose G acts on the generalized Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m})$ by automorphisms preserving the root space decomposition. We will apply our trace formula (1.17) to derive a closed form formula for $Tr(g|\mathfrak{g}_\alpha)$ ($g \in G, \alpha \in Q$).

Let S be a finite subset of I^{re}, let $\Delta_S = \Delta \cap (\sum_{i\in S} \mathbf{Z}\alpha_i)$, $\Delta_S^\pm = \Delta_S \cap \Delta^\pm$, $\Delta^\pm(S) = \Delta^\pm \setminus \Delta_S^\pm$, and let $W(S) = \{w \in W |\ w\Delta^- \cap \Delta^+ \subset \Delta^+(S)\}$. We also let $\mathfrak{g}_0^{(S)} = \mathfrak{h} \oplus (\sum_{\alpha\in\Delta_S} \mathfrak{g}_\alpha)$, and $\mathfrak{g}_\pm^{(S)} = \sum_{\alpha\in\Delta^\pm(S)} \mathfrak{g}_\alpha$. Then $\mathfrak{g}_0^{(S)}$ is the Kac-Moody algebra with Cartan matrix $A_S = (a_{ij})_{i,j\in S}$ (with an extended Cartan subalgebra $\mathfrak{h}$), and $\mathfrak{g}_-^{(S)}$ (resp. $\mathfrak{g}_+^{(S)}$) is a direct sum of irreducible highest (resp. lowest) weight modules over $\mathfrak{g}_0^{(S)}$. We denote by $P_S^+ = \{\lambda \in \mathfrak{h}^* |\ \lambda(h_i) \in \mathbf{Z}_{\geq 0} \text{ for all } i \in S\}$ the set of dominant integral weights for $\mathfrak{g}_0^{(S)}$ and $V_S(\lambda)$ the irreducible highest weight $\mathfrak{g}_0^{(S)}$-module with highest weight $\lambda \in P_S^+$. To apply our formula (1.17) to the Lie algebra $L = \mathfrak{g}_-^{(S)}$, we would like to compute the generalized characters of the homology modules $H_k(\mathfrak{g}_-^{(S)})$. The $\mathfrak{g}_0^{(S)}$-module structure of $H_k(\mathfrak{g}_-^{(S)})$ is determined by the following formula known as *Kostant's formula*.

Proposition 2.2 ([N]). *Let $\rho \in \mathfrak{h}^*$ be a linear functional satisfying $\rho(h_i) = \frac{1}{2}a_{ii}$ for all $i \in I$, and let T be the set of all imaginary simple roots counted with multiplicities. Then we have*

$$H_k(\mathfrak{g}_-^{(S)}) = \sum_{\substack{w \in W(S) \\ F \subset T \\ l(w)+|F|=k}} V_S(w(\rho - s(F)) - \rho), \tag{2.2}$$

where F runs over all the finite subsets of T such that any two elements of F are mutually perpendicular. We denote by $|F|$ the number of elements in F and $s(F)$ the sum of the elements in F. □

Therefore, the space H is the same as

$$\begin{aligned} H &= \sum_{k=1}^{\infty}(-1)^{k+1} H_k(\mathfrak{g}_-^{(S)}) \\ &= \sum_{\substack{w \in W(S) \\ F \subset T \\ l(w)+|F|\geq 1}} (-1)^{l(w)+|F|+1} V_S(w(\rho - s(F)) - \rho), \end{aligned} \tag{2.3}$$

and for all $g \in G$ and $\alpha \in Q_-$, we have

$$Tr(g|H_\alpha) = \sum_{\substack{w \in W(S) \\ F \subset T \\ l(w)+|F|\geq 1}} (-1)^{l(w)+|F|+1} Tr(g|V_S(w(\rho - s(F)) - \rho)_\alpha). \tag{2.4}$$

As in Section 1, let $P_g^{(S)}(H) = \{\alpha \in Q_- | \ Tr(g|H_\alpha) \neq 0\}$ and $\{\tau_i | \ i \geq 1\}$ be an enumeration of $P_g^{(S)}(H)$ compatible with the partial ordering $\leq$ of Q_-. For $\tau \in Q_-$ and $g \in G$, define the set $T_g^{(S)}(\tau)$ and the funciton $B_g^{(S)}(\tau)$ by (1.15) and (1.16). Then, by our trace formula (1.17), we obtain

Proposition 2.3. *For $g \in G$ and $\alpha \in \Delta^-(S)$, we have*

$$Tr(g|\mathfrak{g}_\alpha) = \sum_{\substack{d>0 \\ d|\alpha}} \frac{1}{d}\mu(d) B_{g^d}^{(S)}(\alpha/d). \tag{2.5}$$

In particular, when $g = 1$, we recover the closed form root multiplicity formula for symmetrizable generalized Kac-Moody algebras obtained in [Ka2]. □

§3. The Thompson series

In this section, we apply our trace formula (2.5) to the Monster Lie algebra $M = \bigoplus_{(m,n)\in II_{1,1}} M_{(m,n)}$ to derive some interesting relations among the coefficients $c_g(n)$ of the Thompson series

$$T_g(q) = \sum_{n \geq -1} Tr(g|V_n)q^n = \sum_{n\geq -1} c_g(n)q^n. \tag{3.1}$$

The main properties of the Monster Lie algebra M are summarized in the following proposition.

Proposition 3.1 ([B5]). *1) The Monster Lie algebra M is a $II_{1,1}$-graded generalized Kac-Moody algebra with the real simple root $(1,-1)$ and the imaginary simple roots $(1,i)$ $(i \geq 1)$ with multiplicity $c(i)$. Therefore, M is a generalized Kac-Moody algebra with Borcherds-Cartan matrix $A = (-(i+j))_{i,j\in I}$ of charge $\underline{m} = (c(i)|\ i \in I)$, where $I = \{-1\} \cup \{i|\ i \geq 1\}$ is the index set for the simple roots of M.*

2) M is a $II_{1,1}$-graded representation of the Monster simple group G acting by automorphisms of M such that $M_{(m,n)} \cong V_{mn}$ for $(m,n) \neq (0,0)$ as G-modules. In particular,

$$Tr(g|M_{(m,n)}) = Tr(g|V_{mn}) \quad \text{for } g \in G,\ (m,n) \neq (0,0). \qquad \square$$

Recall that $I = \{-1\} \cup \{i|\ i \geq 1\}$ is the index set for the simple roots of the Monster Lie algebra M, and M is a generalized Kac-Moody algebra with Borcherds-Cartan matrix $A = (-(i+j))_{i,j\in I}$ of charge $\underline{m} = (c(i)|\ i \in I)$. We denote by $e_{-1,1} = e_{-1}$, e_{ik} and $f_{-1,1} = f_{-1}$, f_{ik} $(i \in I, k = 1,2,\cdots,c(i))$ the positive and negative simple root vectors of M, respectively. Thus we have

$$\begin{aligned} M_{(1,-1)} &= \mathbf{C}e_{-1}, \\ M_{(-1,1)} &= \mathbf{C}f_{-1}, \\ M_{(1,i)} &= \mathbf{C}e_{i,1} \oplus \cdots \oplus \mathbf{C}e_{i,c(i)}, \\ M_{(-1,-i)} &= \mathbf{C}f_{i,1} \oplus \cdots \oplus \mathbf{C}f_{i,c(i)} \ (i \geq 1). \end{aligned} \tag{3.2}$$

Consider a basis of $M_{(-1,-i)}$ consisting of the eigenvectors $v_{i,k}$ of $g \in G$ with eigenvalues $\lambda_{i,k}$ $(k = 1,2,\cdots,c(i))$. Since $M_{(1,i)} \cong M_{(-1,-i)} \cong V_i$ $(i \geq 1)$ as representations of the Monster simple group G, we have

$$\sum_{k=1}^{c(i)} \lambda_{i,k} = Tr(g|M_{(-1,-i)}) = Tr(g|V_i) = c_g(i) \quad \text{for } g \in G,\ i \geq 1. \tag{3.3}$$

Moreover, since $M_{(1,-1)} \cong M_{(-1,1)} \cong V_{-1}$, the trivial G-module, we have

$$g \cdot e_{-1} = e_{-1}, \quad g \cdot f_{-1} = f_{-1} \quad \text{for all } g \in G. \tag{3.4}$$

To apply our trace formula (2.5), we take $S = \{-1\}$. Then $M_0^{(S)} \cong sl(2,\mathbf{C}) + \mathbf{C}^2$ and $W(S) = \{1\}$. Hence by Kostant's formula we obtain

$$\begin{aligned} H_1(M_-^{(S)}) &= \sum_{i\geq 1} c(i) V_S(-1,-i), \\ H_k(M_-^{(S)}) &= 0 \quad \text{for } k \geq 2, \end{aligned} \tag{3.5}$$

where $V_S(-1,-i)$ is an i-dimensional irreducible representation of the Lie algebra $sl(2,\mathbf{C})$ generated by e_{-1}, f_{-1}, h_{-1}. Since the weights of $V_S(-1,-i)$ are

$(-1,-i),(-2,-i+1),\cdots,(-i,-1)$, the space $H = H_1(M_-^{(S)})$ has the decomposition

$$H = \bigoplus_{i,j>0} H_{(-i,-j)}. \tag{3.6}$$

Note that each f_{ik} generates an i-dimensional irreducible represntation of the Lie algebra $sl(2,\mathbf{C})$ generated by e_{-1}, f_{-1}, h_{-1}, and hence so does each $v_{i,k}$ ($i \geq 1$, $k = 1,2,\cdots,c(i)$). Therefore we have

$$\begin{aligned} H_{(-i,-j)} &= \bigoplus_{k=1}^{c(i+j-1)} \mathbf{C}(\mathrm{ad} f_{-1})^{i-1}(f_{i+j-1,k}) \\ &= \bigoplus_{k=1}^{c(i+j-1)} \mathbf{C}(\mathrm{ad} f_{-1})^{i-1}(v_{i+j-1,k}). \end{aligned} \tag{3.7}$$

It follows from (3.4) that

$$\begin{aligned} g\cdot(\mathrm{ad} f_{-1})^{i-1}(v_{i+j-1,k}) &= (\mathrm{ad}(g\cdot f_{-1}))^{i-1}(g\cdot v_{i+j-1,k}) \\ &= \lambda_{i+j-1,k}(\mathrm{ad} f_{-1})^{i-1}(v_{i+j-1,k}) \end{aligned}$$

for all $g \in G$, $i,j \in \mathbf{Z}_{>0}$. Hence, by (3.3), we obtain

$$Tr(g|H_{(-i,-j)}) = \sum_{k=1}^{c(i+j-1)} \lambda_{i+j-1,k} = c_g(i+j-1) \tag{3.8}$$

for all $g \in G$, $i,j \in \mathbf{Z}_{>0}$. Therefore, for $g \in G$ and $k,l > 0$, we have

$$P_g^{(S)}(H) = \{(-i,-j) |\ i,j \in \mathbf{Z}_{>0}\}, \tag{3.9}$$

$$T_g^{(S)}(k,l) = \{\underline{b} = (b_{ij})_{i,j\geq 1} |\ b_{ij} \in \mathbf{Z}_{\geq 0},\ \sum_{i,j\geq 1} b_{ij}(i,j) = (k,l)\}, \tag{3.10}$$

the set of all partitions of (k,l) into a sum of ordered pairs of positive integers, and

$$B_g^{(S)}(k,l) = \sum_{\underline{b}\in T_g(k,l)} \frac{(\sum b_{ij}-1)!}{\Pi b_{ij}!} \prod c_g(i+j-1)^{b_{ij}}. \tag{3.11}$$

Now our trace formula (2.5) yields

$$Tr(g|M_{(m,n)}) = \sum_{\substack{d>0 \\ (m,n)=d(k,l)}} \frac{1}{d}\mu(d) B_{g^d}^{(S)}(k,l). \tag{3.12}$$

Since $Tr(g|M_{(m,n)}) = Tr(g|V_{mn}) = c_g(mn)$, we obtain the following interesting relations among the coefficients $c_g(n)$ of the Thompson series $T_g(q) = \sum_{n\geq -1} c_g(n)q^n$.

Theorem 3.2. *For $g \in G$ and $m, n \in \mathbf{Z}_{>0}$, we have*

$$(3.13) \quad c_g(mn) = \sum_{\substack{d>0 \\ d|(m,n)}} \frac{1}{d}\mu(d) \sum_{\underline{b} \in T_g(m/d,n/d)} \frac{(\sum b_{ij} - 1)!}{\Pi b_{ij}!} \prod c_{g^d}(i+j-1)^{b_{ij}}. \quad \square$$

Remark. When $g = 1$, we recover the relations for the coefficients $c(n)$ of the elliptic modular function $j(q) - 744$ obtained in [Ju2] and [Ka2]. Recently, we were informed that the relations (3.13) were obtained independently by Jurisich, Lepowsky, and Wilson ([JLW]). It is pointed out in [JLW] that these relations completely determine the coefficients $c_g(n)$ if the values of $c_h(1)$, $c_h(2)$, $c_h(3)$, and $c_h(5)$ are known for all $h \in G$. Hence, in this sense, our relations (3.13) are as good as Borcherds' relations (9.1) in [B5]. In particular, by taking $(m, n) = (2, 2k)$ and $(m, n) = (2, 2k+1)$, we recover the relations for $c_g(n)$ (n even) in [B5]:

$$(3.14) \quad \begin{aligned} c_g(4k) &= c_g(2k+1) + \sum_{j=1}^{k-1} c_g(j)c_g(2k-j) + \frac{1}{2}(c_g(k)^2 - c_{g^2}(k)), \\ c_g(4k+2) &= c_g(2k+2) + \sum_{j=1}^{k} c_g(j)c_g(2k+1-j). \end{aligned}$$

Moreover, by taking other factorizations of n (n even), we obtain more relations for $c_g(n)$ other than Borcherds' relations.

For n odd, we get different relations than Borcherds'. For example, for $n = 9 = 3^2$, our relation (3.13) implies

$$c_g(9) = c_g(5) + c_g(2)^2 + c_g(1)c_g(3) + \frac{1}{3}(c_g(1)^3 - c_{g^3}(1)),$$

whereas Borcherds' relation yields

$$\begin{aligned} c_g(9) = c_g(7) &+ \frac{1}{2}(c_g(4)^2 + c_{g^2}(4)) + \frac{1}{2}(c_g(3)^2 - c_{g^2}(3)) + c_g(1)c_g(5) \\ &+ c_{g^2}(1)c_g(4) - c_g(1)c_g(7) + c_g(2)c_g(6) - c_g(3)c_g(5). \end{aligned}$$

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