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MODULAR FORMS ASSOCIATED WITH THE MONSTER MODULE

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1. Introduction

In Harada-Lang [4], we associated to each irreducible character $\chi$ of the monster simple group $\mathbb{M}$ a modular function $t_\chi(z)$, called in [4], the McKay-Thompson series for $\chi$. $t_\chi(z)$ is a weighted average of all McKay-Thompson series $t_g(z)$ for the element $g$ of $\mathbb{M}$ as $g$ ranges over $\mathbb{M}$:

$$t_\chi(z) = \frac{1}{|\mathbb{M}|} \sum_{g \in \mathbb{M}} \chi(g) t_g(z).$$

If $\Gamma_\chi$ is the invariance subgroup of $t_\chi(z)$, then we showed

$$\Gamma_\chi = \Gamma_0(N_\chi) = \bigcap_{g \in \mathbb{M}} \Gamma_g,$$

where $g$ ranges over all the elements of $\mathbb{M}$ such that $\chi(g) \neq 0$ and

$$N_\chi = \text{lcm}\{n_g h_g : \text{for all } g \in \mathbb{M}\text{ with } \chi(g) \neq 0\}.$$

As shown in Conway-Norton [1], the invariance group $\Gamma_g$ of $t_g(z)$ is a certain subgroup of index $h$ of the conjugate by

$$\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$$

of

$$\Gamma_0(\frac{n}{h}) + e, f, \cdots$$

This is a preliminary version. A full version with a table will be published elsewhere.
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where $e$, $f$, etc. denote the Atkin-Lehner involutions. In [1], such a conjugate is denoted by

$$n|h + e, f, \cdots.$$ 

The numbers $n$, $h$ depend on $g$, hence our notation $n_g$, $h_g$. Obviously every $t_{\chi}(z)$ is invariant by

$$\bigcap_{g \in M} \Gamma_g = \Gamma_0(N_0)$$

where $N_0 = 2^6 3^3 5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \sim 10^{21}$. The level $N_\chi$ can be very large or relatively small. For example,

$$N_{\chi_1} = N_0, N_{\chi_{166}} = 2^6 3^3 7 = 4032$$

where $\chi_1 = 1$ is the trivial character and the character numbering such as $\chi_{166}$ is taken from the Atlas. In this paper, we will investigate the relation between $t_{\chi}(z)$ and the generating functions of the highest weight vectors (also called singular vectors, primary fields or lowest weight vectors.)

2. THE MONSTER MODULE AS A $Vir$ MODULE

The monster module $V$ is constructed in Frenkel-Lepowsky-Meurman [3] as a vertex operator algebra and is denoted by $V^\dagger$ there. Let $V$ be a vertex operator algebra. Then $V$ possesses two distinguished elements $1$ and $\omega$, called the vacuum and the conformal vector (or the Virasoro element) of $V$, respectively.

If $Y(\omega, z) = \sum \omega_n z^{-n-1}$ is the vertex operator corresponding to the conformal vector $\omega$ and if we set $L(n) = \omega_{n+1}$ for $n \in \mathbb{Z}$, then $L(n)$ satisfies the commutation relation:

$$[L(n), L(m)] = (n - m)L_{n+m} + \frac{1}{12}(n^3 - n)c\delta_{n+m,0}$$
where $c$ is a constant called the central charge of $V$. For the monster module $V$, $c = 24$. $c$ is also called the rank of the vertex operator algebra $V$.

Let $\mathcal{L}$ be the Lie algebra generated by $L(n)$, $n \in \mathbb{Z}$. $\mathcal{L}$ is denoted by $Vir$ else where. The subalgebras $\mathcal{L}^+$ and $\mathcal{L}^-$ are generated by $L(n), n \in \mathbb{Z}^+$ and $L(n), n \in \mathbb{Z}^-$, respectively. It is known that $V$ possesses a positive definite invariant bilinear form and so $V$ is completely reducible as an $\mathcal{L}$ module and is a sum of highest weight modules.

Let $M(h, c)$ be the Verma module of the Virasoro algebra of central charge $c$ generated by the highest weight vector $v$ of height $h$ : i.e.

$$M(h, c) = \mathcal{L}v, \mathcal{L}^+v = 0, L(0)v = hv.$$ 

The module structure of $M(h, c)$ has been determined by Feigin-Fuchs [2]. We will use their results to determine the module structure of $V$ as an $\mathcal{L}$ module. Feigin-Fuchs showed that every submodule of $M(h, c)$ is a sum of submodules that are also Verma modules. Therefore, the knowledge of all embeddings among Verma modules gives all submodules of a given Verma module. The main theorem of Feigin-Fuchs states that there are six types of embeddings of the Verma modules into other Verma modules. Let

$$\begin{align*}
p\alpha - q\beta &= m \\
c &= 24 = \frac{(3p-2q)(3q-2p)}{pq} \\
h &= \frac{m^2-(p-q)^2}{4pq}
\end{align*}$$

where $p$, $q$ and $m$ are complex variables. We next solve for integers $\alpha$ and $\beta$. Let

$$\epsilon = \frac{-11 \pm i\sqrt{23}}{2}, \quad \bar{\epsilon} = \frac{-11 \mp i\sqrt{23}}{2}$$
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We compute
\[ \epsilon \overline{\epsilon} = 1, \quad \epsilon + \overline{\epsilon} = \frac{-11}{6}, \quad \epsilon^2 + \overline{\epsilon}^2 = \frac{49}{36}. \]

Using the second equality of (1), we obtain
\[ (p \alpha - q \beta)^2 = m^2 = 4pq + (q - p)^2, \]
which may be rewritten as
\[ (\alpha - \epsilon \beta)^2 = 4\epsilon h + (\epsilon - 1)^2. \]

We therefore obtain two equations:
\[ \alpha^2 - 2\epsilon \alpha \beta + \epsilon^2 \beta^2 = 4\epsilon h + (\epsilon - 1)^2, \]
and
\[ \alpha^2 - 2\overline{\epsilon} \alpha \beta + \overline{\epsilon}^2 \beta^2 = 4\overline{\epsilon} h + (\overline{\epsilon} - 1)^2. \]

Taking the sum of them, we get
\[ 72\alpha^2 + 132\alpha \beta + 49\beta^2 = -264h + 253. \]

By subtracting one from the other, we get
\[ -12\alpha \beta - 11\beta^2 = 24h - 23. \]

Therefore
\[ \alpha^2 - \beta^2 = 0, \]
or \( \alpha = \pm \beta. \) Setting \( \alpha = \delta \beta \) with \( \delta = \pm 1, \) we have
\[ \beta^2 = \frac{24h - 1}{11 - 12\delta}. \]

If \( h = 0, \) then we must have \( \delta = 1 \) and so \( \alpha = \beta = \pm 1. \) In particular, \( \alpha \beta = 1 > 0. \) On the other hand, if \( h \in \mathbb{Z}^+, \) then \( \delta = -1 \) and so \( \alpha = -\beta = \pm 1, \)
and hence $\alpha \beta = -1 < 0$. Using the results of Feigin-Fuchs [2], we conclude (which must be well known to experts):

**Theorem.** $M(0,24)$ has a unique submodule, which is isomorphic to $M(1,24)$. For all positive integers $h$, $M(h,24)$ is irreducible.

Let $L(c,h)$ be the unique irreducible highest weight $\mathcal{L}$-module of central charge $c$ and height $h$. Then

**Corollary.** We have

1. $L(0,24) = M(0,24)/M(1,24)$, and,
2. $L(h,24) = M(h,24)$ if $h \in \mathbb{Z}^+$.

Let us now express the monster module $V$ as a sum of $L(h,24)$'s as follows

$$V = \sum_{h=0}^{\infty} s_h L(h,24).$$

Then $s_h$ is the number of linearly independent singular vectors $v_h$ of height $h$, hence $v_h \in V_h$. Since the Virasoro algebra $\mathcal{L}$ commutes with the action of the monster $\mathcal{M}$, we can actually split $s_h$ into the sum of $s_h^k$ where the index $k$ corresponds to the irreducible character $\chi_k$. More precisely, let

$$V_h^k = c_{hk} \chi_k$$

where $c_{hk}$ is the multiplicity of $\chi_k$ in $V_h$ and

$$V^k = \bigoplus_{h=0}^{\infty} V_h^k.$$  

Thus $V^k$ is an $\mathcal{M}$ submodule of $V$ consisting entirely of irreducible submodules isomorphic to $\chi_k$ and $V_h^k$ is an $\mathcal{M}$ submodule of $V^k$ of height $h$. We also define

$$W_h^k = \mathcal{L}(\bigcup_{0 \leq i < h} V_i^k) \cap V_h^k,$$
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which is an $\mathbb{M}$ submodule of $\mathcal{V}_h^k$ that is generated by elements of lower heights. Let

$$s_h^k = \dim \mathcal{V}_h^k / W_h^k.$$ 

Then $s_h^k$ is the number of linearly independent singular vectors in $\mathcal{V}_h^k$. Obviously

$$s_h = \sum_{k=1}^{194} s_h^k.$$ 

For a graded module $X = \sum_{h \in \mathbb{Z}} X_h$, the character of $X$ (or the partition function of $X$) is defined to be a formal sum

$$\text{char}(X) = \sum_{h \in \mathbb{Z}} \dim X_h x^h.$$ 

Using this notation, we have, as is well known,

$$\text{char} M(h, c) = x^h \sum_{n \geq 0} p(n) x^n$$

where $p(n)$ is the partition function of $n$. For convenience, set $p(0) = 1$, and $p(n) = 0$ if $n \in \mathbb{Z}^-$. Let us consider the $\mathcal{L}$ submodule generated by the vacuum 1. We have $V_1 = 0$ but the height 1 component of $M(0, 24)$ is nonzero, we conclude that

$$\mathcal{L} \cdot 1 \simeq M(0, 24) / M(1, 24).$$

Hence

$$\text{char}(\mathcal{L} \cdot 1) = \sum_{n \geq 0} p(n) x^n - x \sum_{n \geq 0} p(n) x^n = \sum_{n \geq 0} (p(n) - p(n-1)) x^n.$$ 

Writing

$$\text{char}(\mathcal{L} \cdot 1) = \sum_{h \geq 0} a_{h1} x^h,$$
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we get a partial list:

\[
\begin{array}{cccccccccccc}
 h & 0 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
 a_{h1} & 1 & 1 & 1 & 2 & 2 & 4 & 4 & 7 & 8 & 12 & 14 \\
\end{array}
\]

In [4], we had a partial list of \( c_{h1} \) where \( c_{h1} \) is the multiplicity of the trivial character \( \chi_1 \) occuring in \( \mathcal{V}_h \).

\[
\begin{array}{cccccccccccc}
 h & 0 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
 c_{h1} & 1 & 1 & 1 & 2 & 2 & 4 & 4 & 7 & 8 & 12 & 14 \\
\end{array}
\]

The coincidence \( c_{h1} = a_{h1} \) stops there and we have

\[
\begin{array}{cccc}
 h & 12 \\
 a_{h1} & 21 \\
 c_{h1} & 22 \\
\end{array}
\]

This means \( s_{12}^1 = 1 \), namely, \( \mathcal{V}_{12}^1 \) contains a singular vector, while \( \mathcal{V}_h^1, 0 < h \leq 11 \), do not. The number \( d \) of linearly independent singular vectors occuring in

\[
\mathcal{V}_h^1 (0 \leq h \leq 30)
\]

is as follows

\[
\begin{array}{cccccccccccc}
 h & 12 & 16 & 18 & 20 & 22 & 24 & 26 & 27 & 28 & 29 & 30 \\
 d & 1 & 1 & 1 & 1 & 1 & 3 & 2 & 1 & 4 & 2 & 6 \\
\end{array}
\]

We are now lead to consider its generating function for each \( k, 1 \leq k \leq 194 \).

Define

\[
G^k(x) = \sum_{h \geq 0} s_k^h x^h.
\]

The character of \( \mathcal{V}^k \) is

\[
\text{char}(\mathcal{V}^k) = \sum_{h \geq 0} c_h^k (\deg \chi_k) x^h = x \deg \chi_k t_\chi(x)
\]

where \( t_\chi(z) \) is the McKay-Thompson series for the irreducible character \( \chi \). On the other hand, using the expression

\[
\mathcal{V}^k = \sum_{h \geq 0} s_k^h L(h, 24),
\]
we obtain
\[
\text{char}(\mathcal{V}^k) = \sum_{h \geq 0} s_h^k \text{char} L(h, 24).
\]

Suppose \( k > 0 \). Then \( s_0^k = 0 \) and so
\[
\text{char}(\mathcal{V}^k) = \sum_{h \geq 1} s_h^k x^h \sum_{n \geq 0} p(n) x^n.
\]

On the other hand if \( k = 1 \), then \( L(0, 24) \) occurs only once as a constituent of \( \mathcal{V}^1 \). Therefore
\[
\text{char}(\mathcal{V}^1) = (1 - x + \sum_{h \geq 2} s_h^1 x^h) \sum_{n \geq 0} p(n) x^n.
\]

Using the Dedekind eta-function and replacing \( x \) by \( q = e^{2\pi i z} \), we obtain, by setting \( s_1^1 = -1 \) for convenience,
\[
\deg \chi_k t_{\chi_k}(q) = \frac{q^{-1} (\sum_{h \geq 0} s_h^k q^h) q^{\frac{1}{24}}}{\eta(q)}.
\]

Hence
\[
\deg \chi_k t_{\chi_k}(q) \eta(q) = q^{-\frac{23}{24}} \sum_{h \geq 0} s_h^k q^h,
\]

which implies
\[
q^{-\frac{23}{24}} G^k(q) = \deg \chi_k t_{\chi_k}(q) \eta(q)
\]

where as defined before \( G^k(q) \) is the generating function of the singular vectors in \( \mathcal{V}^k \). Writing \( G^k = G^x \) in general, we obtain:

**Theorem.** \( q^{-\frac{23}{24}} G^x(q) \) is a meromorphic modular form of weight \( \frac{1}{2} \) and level \( N_x \).

**Corollary.** \( q^{-\frac{23}{24}} G^x(q) \eta(q)^{23} \) is a holomorphic modular function of weight 12 and level \( N_x \).
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