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INTRODUCTION TO VERTEX OPERATOR ALGEBRAS, III

YI-ZHI HUANG*

CONTENTS

In this exposition, we continue the discussions of Dong [D2] and Li [L]. We shall prove an S_3-symmetry of the Jacobi identity, construct the contragredient module for a module for a vertex operator algebra and apply these to the construction of the vertex operator map for the moonshine module. We shall introduce the notions of intertwining operator, fusion rule and Verlinde algebra. We shall also describe briefly the geometric interpretation of vertex operator algebras. We end the exposition with an explanation of the role of vertex operator algebras in conformal field theories.

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Notations:

$\mathbb{C}$: the (structured set of) complex numbers.
$\mathbb{C}^{\times}$: the nonzero complex numbers.
$\mathbb{R}$: the real numbers.
$\mathbb{Z}$: the integers.
$\mathbb{Z}_{+}$: the positive integers.
$\mathbb{N}$: the nonnegative integers.

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1. S_3-SYMMETRY OF THE JACOBI IDENTITY AND CONTRAGREDIENT MODULES

The results and constructions discussed in this section are all natural from the axiomatic viewpoint. But they also have practical uses in some very concrete problems. Before going into the detailed discussions, let us first recall one of those problems.

One of the most important examples of vertex operator algebra is the moonshine module constructed by Frenkel, Lepowsky and Meurman [FLM1] [FLM2]. (See the introduction of [FLM2] for a historical discussion, including the important role of Borcherds' announcement [B].) The construction can be briefly described as follows: From the Leech lattice Λ, one can construct an untwisted vertex operator algebra V_Λ. The automorphism $\theta : \Lambda \to \Lambda$ defined by $\theta(x) = -x$ for any $x \in \Lambda$ induces an automorphism of V_Λ which is still denoted θ. One can construct a unique irreducible θ-twisted module V_Λ^T for V_Λ. The automorphism $\theta : \Lambda \to \Lambda$ also induces an automorphism of V_Λ and is also denoted θ. Let V_Λ^+ and $(V_\Lambda^T)^+$ be spaces of fixed points of θ in V_Λ and V_Λ^T, respectively. Then the moonshine module is $V^\natural = V_\Lambda^+ \oplus (V_\Lambda^T)^+$ as a $\mathbb{Z}$-graded vector space. In [FLM2], the vertex operator map for the moonshine module is defined and it is shown that $V^\natural$ is indeed a vertex operator algebra.

The definition of vertex operator map for $V^\natural$ in [FLM2] uses some special features in the construction of the moonshine module. In fact, there is a conceptual way to define the vertex operator map which is motivated by the S_3-symmetry of the Jacobi identity and contragredient modules and which works also in much more general cases (see [FHL] and also [DGM] in physicists' language). The hard part is to prove that the moonshine module together with this abstractly defined vertex operator map is a vertex operator algebra. This was first proved in [DGM] using techniques developed in string theory, and recently, this has also been proved conceptually by the author [Hu7] using the tensor product theory for modules for a vertex operator algebra developed by Lepowsky and the author [HL1] [HL4]–[HL6] [Hu6] and some results of Dong [D1] on modules for the vertex operator algebra V_Λ^+. (Note that in more general cases in which we can still define the vertex operator maps abstractly, it is not always true that we will obtain a vertex operator algebra.)

We now turn to the main subjects of this section. We follow the discussions in [FHL]. At the end of this section, we shall apply these results to the problem above.

We first list some properties of the formal δ-function and some easy consequences of the definition of vertex operator algebra. First there is the fundamental property of the δ-function:

$$f(x)\delta(x) = f(1)\delta(x) \quad \text{for} \quad f(x) \in \mathbb{C}[x, x^{-1}]. \tag{1.1}$$

This property has many variants; in general, whenever an expression is multiplied by the δ-function, we may formally set the argument appearing in the δ-function equal to 1, provided the relevant algebraic expressions make sense. There are two basic identities for the δ-function:

$$x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right), \tag{1.2}$$

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right). \tag{1.3}$$

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. We have the following immediate consequences of the definition of vertex operator algebra:

$$\begin{aligned}
[L(-1), Y(v,x)] &= Y(L(-1)v, x), && (1.4)\\
[L(0), Y(v,x)] &= Y(L(0)v,x)+xY(L(-1)v,x), && (1.5)\\
[L(1), Y(v,x)] &= Y(L(1)v,x)+2xY(L(0)v,x)+x^2Y(L(-1)v,x) && (1.6)
\end{aligned}$$

for any $v \in V$. From the $L(-1)$-derivative property and bracket formulas (1.4), we obtain

$$e^{x_0L(-1)}Y(v,x)e^{-x_0L(-1)}=Y(e^{x_0L(-1)}v,x)=Y(v,x+x_0) \tag{1.7}$$

Applying (1.7) to $\mathbf{1}$ and then taking the constant term in x_0, we have

$$Y(v,x)\mathbf{1}=e^{xL(-1)}v. \tag{1.8}$$

Finally, one very important consequence is the skew-symmetry, that is, for any $u, v \in V$,

$$Y(u,x)v=e^{xL(-1)}Y(v,-x)u. \tag{1.9}$$

We derive (1.9) as follows: We have

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2)\\
&\qquad -x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_1)Y(u,x_2)\\
&= (-x_0)^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)\\
&\qquad -(-x_0)^{-1}\delta\left(\frac{x_1-x_2}{-(-x_0)}\right)Y(u,x_1)Y(v,x_2).
\end{aligned} \tag{1.10}$$

By the Jacobi identity and (1.10),

$$x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)=x_1^{-1}\delta\left(\frac{x_2-(-x_0)}{x_1}\right)Y(Y(v,-x_0)u,x_1). \tag{1.11}$$

Using the fundamental property of the δ-function and the identity (1.2), we obtain

$$x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(v,-x_0)u,x_2+x_0). \tag{1.12}$$

In particular (taking the coefficient of x_1^{-1} in (1.12)),

$$Y(Y(u,x_0)v,x_2)=Y(Y(v,-x_0)u,x_2+x_0). \tag{1.13}$$

But by the second equality in (1.7),

$$Y(Y(v,-x_0)u,x_2+x_0)=Y(e^{x_0L(-1)}Y(v,-x_0)u,x_2). \tag{1.14}$$

By the creation property, (1.13) and (1.14),

$$\begin{aligned} Y(u,x_0)v &= \lim_{x_2\to 0} Y(Y(u,x_0)v,x_2)\mathbf{1} \\ &= \lim_{x_2\to 0} Y(e^{x_0L(-1)}Y(v,-x_0)u,x_2)\mathbf{1} \\ &= e^{x_0L(-1)}Y(v,-x_0)u. \end{aligned} \tag{1.15}$$

Now we discuss the S_3-symmetry of the Jacobi identity. For the Jacobi identity for Lie algebras, if we call

$$[u,[v,w]]-[v,[u,w]]=[[u,v],w] \tag{1.16}$$

"the Jacobi identity for the ordered triple (u,v,w)," then the Jacobi identity for (u,v,w) implies the Jacobi identity for any permutation of the ordered triple (u,v,w). The S_3-symmetry for the Jacobi identity for vertex operator algebra is an analogous statement. (The analogy between Lie algebras and vertex operator algebras is the reason why Frenkel, Lepowsky and Meurman called the main axiom for vertex operator algebras the "Jacobi identity." It would be more accurate and less confusing to call this identity the Frenkel-Lepowsky-Meurman identity or simply the FLM identity.) Let us retain the axioms for a vertex operator algebra except for the Jacobi identity, and let us call

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2)w - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)w \\ = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)w \end{aligned} \tag{1.17}$$

"the Jacobi identity for the ordered triple (u,v,w)." We also assume that the consequences (1.7) and (1.9) hold. By skew-symmetry (1.9) for the pair (u,v) and the second equality in (1.7) for the vector $Y(v,-x_0)u$ we have

$$Y(Y(u,x_0)v,x_2)=Y(e^{x_0L(-1)}Y(v,-x_0)u,x_2)=Y(Y(v,-x_0)u,x_2+x_0). \tag{1.18}$$

Thus from (1.18) and the identity (1.2), the Jacobi identity (1.17) for (u, v, w) gives

$$(-x_0)^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1)w - (-x_0)^{-1}\delta\left(\frac{x_1 - x_2}{-(-x_0)}\right) Y(u, x_1)Y(v, x_2)w$$
$$= x_1^{-1}\delta\left(\frac{x_2 - (-x_0)}{x_1}\right) Y(Y(v, -x_0)u, x_1)w, \tag{1.19}$$

which is the Jacobi identity for (v, u, w) (with (x_1, x_2, x_0) replaced by $(x_2, x_1, -x_0)$).

On the other hand, multiplying both sides of the Jacobi identity (1.17) for (u, v, w) by $e^{-x_2 L(-1)}$ and using (1.9) for the pairs (v, w), $(v, Y(u, x_1)w)$ and $(Y(u, x_0)v, w)$ and the outer equality in (1.7) for the vector u, we obtain

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1 - x_2)Y(w, -x_2)v - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(Y(u, x_1)w, -x_2)v$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(w, -x_2)Y(u, x_0)v. \tag{1.20}$$

Using the fundamental property of the δ-function and (1.2), we can write (1.20) as

$$x_1^{-1}\delta\left(\frac{x_0 + x_2}{x_1}\right) Y(u, x_0)Y(w, -x_2)v + x_2^{-1}\delta\left(\frac{x_0 - x_1}{-x_2}\right) Y(Y(u, x_1)w, -x_2)v$$
$$= x_1^{-1}\delta\left(\frac{-x_2 - x_0}{-x_1}\right) Y(w, -x_2)Y(u, x_0)v, \tag{1.21}$$

that is,

$$x_1^{-1}\delta\left(\frac{x_0 - (-x_2)}{x_1}\right) Y(u, x_0)Y(w, -x_2)v$$
$$-x_1^{-1}\delta\left(\frac{(-x_2) - x_0}{-x_1}\right) Y(w, -x_2)Y(u, x_0)v$$
$$= (-x_2)^{-1}\delta\left(\frac{x_0 - x_1}{-x_2}\right) Y(Y(u, x_1)w, -x_2)v, \tag{1.22}$$

the Jacobi identity for (u, w, v) (and $(x_0, -x_2, x_1)$). Since the two permutation above of (u, v, w) generate S_3, the permutation group of (u, v, w), we conclude:

Proposition 1.1. *Under the assumptions indicated in the argument above, the Jacobi identity for an ordered triple implies the Jacobi identity for any permutation of this triple.*

We turn next to the contragredient module for a module for a vertex operator algebra. Let (W, Y), with

$$W = \coprod_{n \in \mathbb{C}} W_{(n)}, \tag{1.23}$$

be a module for a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$,

$$W' = \coprod_{n\in\mathbb{C}} W_{(n)}^{*} \tag{1.24}$$

the graded dual space of W and $\langle \cdot, \cdot \rangle$ the pairing between W' and W. We define the *contragredient vertex operators* $Y'(v, x)$ $(v \in V)$ by means of the linear map

$$\begin{aligned} V &\to (\text{End } W')[[x, x^{-1}]] \\ v &\mapsto Y'(v, x) = \sum_{n\in\mathbb{Z}} v_n' x^{-n-1} \quad (\text{where } v_n' \in \text{End } W'), \end{aligned} \tag{1.25}$$

determined by the condition

$$\langle Y'(v, x)w', w \rangle = \langle w', Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w \rangle \tag{1.26}$$

for $v \in V$, $w' \in W'$, $w \in W$. The operator $(-x^{-2})^{L(0)}$ has the obvious meaning; it acts on a vector of weight $n \in \mathbb{Z}$ as multiplication by $(-x^{-2})^n$. Also note that $e^{xL(1)}(-x^{-2})^{L(0)}v$ involves only finitely many (integral) powers of z, that the right-hand side of (1.26) is a Laurent polynomial in x, and that the components v_n' of the formal Laurent series $Y'(v, x)$ defined by (1.26) indeed preserve W'.

We give the space W' a $\mathbb{C}$-grading by setting

$$W_{(n)}' = W_{(n)}^{*} \quad \text{for } n \in \mathbb{C}. \tag{1.27}$$

The following proposition defines the *V-module contragredient to W*:

Theorem 1.2. *The pair (W', Y') carries the structure of a V-module.*

Proof. The axioms on the grading are clear. For the Virasoro algebra properties, we note that

$$\langle Y'(\omega, x)w', w \rangle = \langle w', Y(x^{-4}\omega, x^{-1})w \rangle \tag{1.28}$$

since

$$L(1)\omega = L(-1)L(-2)\mathbf{1} = L(-2)L(-1)\mathbf{1} = 0. \tag{1.29}$$

Thus, defining component operators $L'(n)$ by

$$Y'(\omega, x) = \sum_{n\in\mathbb{Z}} L'(n)x^{-n-2}, \tag{1.30}$$

we have

$$\begin{aligned}
&\langle \sum_{n\in\mathbb{Z}} L'(n)x^{-n}w', w\rangle = \\
&\quad = \langle x^2 Y'(\omega, x)w', w\rangle \\
&\quad = \langle w', x^{-2}Y(\omega, x^{-1})w\rangle \\
&\quad = \langle w', \sum_{n\in\mathbb{Z}} L(-n)x^{-n}w\rangle,
\end{aligned} \tag{1.31}$$

and so

$$\langle L'(n)w', w\rangle = \langle w', L(-n)w\rangle \quad \text{for} \quad n \in \mathbb{Z}. \tag{1.32}$$

This immediately gives us the Virasoro commutator relation for $L'(n)$, $n \in \mathbb{Z}$.

We shall give proofs of the Jacobi identity and the $L(-1)$-derivative property. For these two axioms, we shall use some commutator formulas motivated by the Lie group $\mathrm{SL}(2,\mathbb{C})$, but formulated and proved in terms of formal series. We shall omit the proofs of these formulas; they can be found in [FHL] and are all direct calculations.

Lemma 1.3. *Let*

$$f(x) \in x\mathbb{C}[[x]]. \tag{1.33}$$

We have the following identities, valid on any module for the Lie algebra $\mathfrak{sl}(2)$ spanned by $L(-1)$, $L(0)$, $L(1)$:

$$L(-1)e^{f(x)L(0)} = e^{f(x)L(0)}L(-1)e^{-f(x)}, \tag{1.34}$$

$$L(1)e^{f(x)L(0)} = e^{f(x)L(0)}L(1)e^{f(x)}, \tag{1.35}$$

$$\begin{aligned}
&L(-1)e^{f(x)L(1)} = \\
&\quad = e^{f(x)L(1)}L(-1) - 2f(x)L(0)e^{f(x)L(1)} - f(x)^2L(1)e^{f(x)L(1)} \\
&\quad = e^{f(x)L(1)}L(-1) - 2f(x)e^{f(x)L(1)}L(0) + f(x)^2e^{f(x)L(1)}L(1).
\end{aligned} \tag{1.36}$$

These identities also hold for more general f for which the series are well defined, such as

$$f(x, x_0) \in x\mathbb{C}[[x, x_0]]. \tag{1.37}$$

Now we establish the $L(-1)$-derivative property. For convenience, we assume that $v \in V$ is homogeneous of weight $n \in \mathbb{Z} : L(0)v = nv$. Using the definition $Y'(\cdot, x)$ and the chain rule we get

$$\begin{aligned}
&\langle \frac{d}{dx} Y'(v, x) w', w\rangle = \\
&\quad = \frac{d}{dx} \langle w', Y(e^{xL(1)}(-x^{-2})^{L(0)} v, x^{-1}) w\rangle \\
&\quad = \langle w', \frac{d}{dx} Y(e^{xL(1)}(-x^{-2})^{L(0)} v, x^{-1}) w\rangle \\
&\quad = \langle w', Y(\frac{d}{dx}(e^{xL(1)}(-x^{-2})^{L(0)}) v, x^{-1}) w\rangle \\
&\quad\quad + \langle w', \frac{d}{dx} Y(v_1, x^{-1}) \Big|_{v_1 = e^{xL(1)}(-x^{-2})^{L(0)} v} w\rangle,
\end{aligned} \tag{1.38}$$

where w' and w are arbitrary elements of W' and W, respectively. We perform the indicated calculations:

$$\begin{aligned}
&\frac{d}{dx}(e^{xL(1)}(-x^{-2})^{L(0)}) = \\
&\quad = L(1)e^{xL(1)}(-x^{-2})^{L(0)} - 2x^{-1}e^{xL(1)}L(0)(-x^{-2})^{L(0)},
\end{aligned} \tag{1.39}$$

$$\begin{aligned}
&\frac{d}{dx} Y(v_1, x^{-1}) \Big|_{v_1 = e^{xL(1)}(-x^{-2})^{L(0)} v} = \\
&\quad = -x^{-2} \frac{d}{dx^{-1}} Y(v_1, x^{-1}) \Big|_{v_1 = e^{xL(1)}(-x^{-2})^{L(0)} v} \\
&\quad = -x^{-2} Y(L(-1)v_1, x^{-1}) \Big|_{v_1 = e^{xL(1)}(-x^{-2})^{L(0)} v} \\
&\quad = -x^{-2} Y(L(-1)e^{xL(1)}(-x^{-2})^{L(0)} v, x^{-1}) \\
&\quad = -x^{-2} Y((e^{xL(1)}L(-1) - 2xe^{xL(1)}L(0) \\
&\quad\quad + x^2 L(1)e^{xL(1)})(-x^{-2})^n v, x^{-1}) \\
&\quad = Y(e^{xL(1)}(-x^{-2})^{n+1} L(-1)v, x^{-1}) \\
&\quad\quad + Y(2x^{-1}e^{xL(1)}L(0)(-x^{-2})^n v, x^{-1}) \\
&\quad\quad - Y(L(1)e^{xL(1)}(-x^{-2})^n v, x^{-1}) \\
&\quad = Y(e^{xL(1)}(-x^{-2})^{L(0)} L(-1)v, x^{-1}) \\
&\quad\quad + Y(2x^{-1}e^{xL(1)}L(0)(-x^{-2})^{L(0)} v, x^{-1}) \\
&\quad\quad - Y(L(1)e^{xL(1)}(-x^{-2})^{L(0)} v, x^{-1}).
\end{aligned} \tag{1.40}$$

Here we have used the outer equality in (1.36) and the fact that

$$L(0)L(-1)v = L(-1)(L(0) + 1)v = (n + 1)L(-1)v. \tag{1.41}$$

Substituting (1.39) and (1.40) into (1.38) we get

$$\begin{aligned}
&\langle \frac{d}{dx} Y'(v,x)w', w\rangle = \\
&\quad = \langle w', Y(L(1)e^{xL(1)}(-x^{-2})^{L(0)}v \\
&\qquad -2x^{-1}e^{xL(1)}L(0)(-x^{-2})^{L(0)}v, x^{-1})w\rangle \\
&\qquad +\langle w', Y(e^{xL(1)}(-x^{-2})^{L(0)}L(-1)v, x^{-1})w\rangle \\
&\qquad +\langle w', Y(2x^{-1}e^{xL(1)}L(0)(-x^{-2})^{L(0)}v, x^{-1})w\rangle \\
&\qquad -\langle w', Y(L(1)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w\rangle \\
&\quad = \langle w', Y(e^{xL(1)}(-x^{-2})^{L(0)}L(-1)v, x^{-1})w\rangle \\
&\quad = \langle Y'(L(-1)v, x)w', w\rangle,
\end{aligned} \tag{1.42}$$

proving the $L(-1)$-derivative property.

Finally, we shall prove the Jacobi identity. Let $v_1, v_2 \in V$, $w \in W$ and $w' \in W'$. What we want to prove can be written as follows:

$$\begin{aligned}
&\langle x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) Y'(v_1,x_1)Y'(v_2,x_2)w', w\rangle \\
&\qquad -\langle x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right) Y'(v_2,x_2)Y'(v_1,x_1)w', w\rangle \\
&\quad = \langle x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) Y'(Y(v_1,x_0)v_2,x_2)w', w\rangle.
\end{aligned} \tag{1.43}$$

But by the definition (1.26) of contragredient vertex operator, we have

$$\begin{aligned}
&\langle Y'(v_1,x_1)Y'(v_2,x_2)w', w\rangle \\
&\quad = \langle w', Y(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v_2, x_2^{-1})Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v_1, x_1^{-1})w\rangle
\end{aligned} \tag{1.44}$$

$$\begin{aligned}
&\langle Y'(v_2,x_2)Y'(v_1,x_1)w', w\rangle \\
&\quad = \langle w', Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v_1, x_1^{-1})Y(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v_2, x_2^{-1})w\rangle
\end{aligned} \tag{1.45}$$

$$\begin{aligned}
&\langle Y'(Y(v_1,x_0)v_2,x_2)w', w\rangle \\
&\quad = \langle w', Y(e^{x_2L(1)}(-x_2^{-2})^{L(0)}Y(v_1,x_0)v_2, x_2^{-1})w\rangle,
\end{aligned} \tag{1.46}$$

and from the Jacobi identity for W we have

$$\begin{aligned}
&\langle w', \left(\frac{-x_0}{x_1x_2}\right)^{-1}\delta\left(\frac{x_1^{-1}-x_2^{-1}}{-x_0/x_1x_2}\right)Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v_1, x_1^{-1})\cdot\\
&\quad\cdot Y(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v_2, x_2^{-1})w\rangle\\
&\quad-\langle w', \left(\frac{-x_0}{x_1x_2}\right)^{-1}\delta\left(\frac{x_2^{-1}-x_1^{-1}}{x_0/x_1x_2}\right)Y(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v_2, x_2^{-1})\cdot\\
&\quad\cdot Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v_1, x_1^{-1})w\rangle\\
&=\langle w', (x_2^{-1})^{-1}\delta\left(\frac{x_1^{-1}+x_0/x_1x_2}{x_2^{-1}}\right)\cdot\\
&\quad\cdot Y(Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v_1, -x_0/x_1x_2)e^{x_2L(1)}(-x_2^{-2})^{L(0)}v_2, x_2^{-1})w\rangle,
\end{aligned}\tag{1.47}$$

or equivalently,

$$\begin{aligned}
&-\langle w', x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v_1, x_1^{-1})\cdot\\
&\quad\cdot Y(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v_2, x_2^{-1})w\rangle\\
&\quad+\langle w', x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(e^{x_2L(1)}(-x_2^{-2})^{L(0)}v_2, x_2^{-1})\cdot\\
&\quad\cdot Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v_1, x_1^{-1})w\rangle\\
&=\langle w', x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\cdot\\
&\quad\cdot Y(Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v_1, -x_0/x_1x_2)e^{x_2L(1)}(-x_2^{-2})^{L(0)}v_2, x_2^{-1})w\rangle.
\end{aligned}\tag{1.48}$$

(As usual, the reader should be observing that the formal Laurent series which arise are well defined.) Thus (by (1.2)) the desired result (1.43) is equivalent to

$$\begin{aligned}
&x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y(e^{x_2L(1)}(-x_2^{-2})^{L(0)}Y(v_1, x_0)v_2, x_2^{-1})\\
&\quad=x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y(Y(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v_1, -x_0/x_1x_2)\cdot\\
&\quad\quad\cdot e^{x_2L(1)}(-x_2^{-2})^{L(0)}v_2, x_2^{-1}),
\end{aligned}\tag{1.49}$$

or to

$$\begin{aligned}
&Y(e^{x_2L(1)}(-x_2^{-2})^{L(0)}Y(v_1, x_0)v_2, x_2^{-1})\\
&\quad=Y(Y(e^{(x_2+x_0)L(1)}(-(x_2+x_0)^{-2})^{L(0)}v_1, -x_0/(x_2+x_0)x_2)\cdot\\
&\quad\quad\cdot e^{x_2L(1)}(-x_2^{-2})^{L(0)}v_2, x_2^{-1}).
\end{aligned}\tag{1.50}$$

If we can prove

$$\begin{aligned} & e^{x_2L(1)}(-x_2^{-2})^{L(0)}Y(v_1,x_0) \\ & \quad = Y(e^{(x_2+x_0)L(1)}(-(x_2+x_0)^{-2})^{L(0)}v_1, -x_0/(x_2+x_0)x_2)\cdot \\ & \qquad \cdot e^{x_2L(1)}(-x_2^{-2})^{L(0)} \end{aligned} \tag{1.51}$$

or equivalently, the conjugation formula

$$\begin{aligned} & e^{xL(1)}(-x^{-2})^{L(0)}Y(v,x_0)(-x^{-2})^{-L(0)}e^{-xL(1)} \\ & \quad = Y(e^{(x+x_0)L(1)}(-(x+x_0)^{-2})^{L(0)}v, -x_0/(x+x_0)x) \end{aligned} \tag{1.52}$$

for any element v of a vertex operator algebra, where the operators act on the algebra itself, then we will be done. But for this, it is sufficient to prove the following lemma:

Lemma 1.4. *Let V be a vertex operator algebra. The following conjugation formulas hold on V :*

$$x^{L(0)}Y(v,x_0)x^{-L(0)} = Y(x^{L(0)}v, xx_0) \tag{1.53}$$

$$e^{xL(1)}Y(v,x_0)e^{-xL(1)} = Y(e^{x(1-xx_0)L(1)}(1-xx_0)^{-2L(0)}v, x_0/(1-xx_0)). \tag{1.54}$$

The proof of this lemma, which we omit here, can be found in [FHL]. This finishes the proof of the theorem. □

The functor taking a V-module to its contragredient module has some important properties which we state without proof (see [FHL]):

Proposition 1.5. *There is a natural isomorphism between the double contragredient module (W'', Y'') and (W, Y).*

Proposition 1.6. *The module (W, Y) is irreducible if and only if (W', Y') is irreducible.*

Proposition 1.7. *The module (W, Y) is isomorphic to its contragredient module (W', Y') if and only if there exists a nondegenerate bilinear form $(\cdot,\cdot)_W$ on W such that*

$$(W_{(m)}, W_{(n)})_W = 0, \quad m \neq n \tag{1.55}$$

and

$$(Y'(v,x)w_1, w_2)_W = (w_1, Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w_2)_W. \tag{1.56}$$

If V as a V-module is isomorphic to V', the bilinear form $(\cdot,\cdot)_V$ is symmetric.

We return to our problem of defining the vertex operator map for $V^\natural$. Let V be a vertex operator algebra and W a V-module. Assume that both V and W as V-modules are isomorphic to themselves. By Proposition 1.7, there are nondegenerate bilinear forms $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$ satisfying the two conditions in Proposition 1.7. In addition, $(\cdot,\cdot)_V$ is symmetric. Assume that there is a vertex operator map

$$Y_{V\oplus W} : (V\oplus W)\otimes (V\oplus W) \to (\mathrm{End}\,(V\oplus W))[[x,x^{-1}]]$$

such that $(V\oplus W, Y_{V\oplus W}, \mathbf{1}, \omega)$ ($\mathbf{1}$ and ω are the vacuum and the Virasoro element of V, respectively) is a vertex operator algebra satisfying the following:

(1) The vertex operator algebra structure on V and the module structure on W are substructures of it.
(2) As a module for itself, it is isomorphic its contragredient module and the corresponding symmetric nondegenerate bilinear form $(\cdot,\cdot)_{V\oplus W}$ is defined by

$$((v_1,w_2),(v_2,w_2))_{V\oplus W} = (v_1,v_2)_V + (w_1,w_2)_W \tag{1.57}$$

for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$.
(3) The involution which is the identity on V and is -1 on W is an automorphism of $(V\oplus W, Y_{V\oplus W}, \mathbf{1}, \omega)$.

Then we must have the following:

(1) The module W is $\mathbb{Z}$-graded.
(2) The bilinear form $(\cdot,\cdot)_W$ is symmetric.
(3) We have the following formulas: For any $v\in V$ and $w\in W$,

$$Y_{V\oplus W}(w,x)v = e^{xL(-1)}Y_W(v,-x)w \tag{1.58}$$

and for any $v\in V$, $w_1, w_2, w_3 \in W$,

$$\begin{aligned}(w_3, Y_{V\oplus W}(w_1,x)w_2)_W &= 0, & (1.59)\\ (v, Y_{V\oplus W}(w_1,x)w_2)_V &= (Y_W(v,-x^{-1})e^{xL(1)}(-x^2)^{-L(0)}w_1, e^{x^{-1}L(1)}w_2)_W, & (1.60)\end{aligned}$$

where Y_V and Y_W are the vertex operator maps for V and W, respectively.

We see that the vertex operator map $Y_{V\oplus W}$ is determined completely by the vertex operator maps Y_V, Y_W, the bilinear forms $(\cdot,\cdot)_V$, $(\cdot,\cdot)_W$ and (1.58)–(1.60). Thus even if we do not know whether $V\oplus W$ is such a vertex operator algebra, we can still define a vertex operator map $Y_{V\oplus W}$ using Y_V, Y_W, the bilinear forms $(\cdot,\cdot)_V$, $(\cdot,\cdot)_W$ and (1.58)–(1.60). In particular, since V_Λ^+ and $(V_\Lambda^T)^+$ as V_Λ^+-modules are both isomorphic to their contragredient modules, we can define a vertex operator map $Y_{V^\natural}$ for $V^\natural = V_\Lambda^+ \oplus (V_\Lambda^T)^+$.

2. Intertwining operators, fusion rules and Verlinde algebras

We first define intertwining operators and fusion rules for a vertex operator following [FHL].

Let

$$V\{x\} = \left\{ \sum_{n\in\mathbb{C}} v_n x^n | v_n \in V \right\} \tag{2.1}$$

be the vector space of V-valued formal series involving the complex powers of x with coefficients in a vector space V.

Definition 2.1. Let V be a vertex operator algebra and let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be three V-modules (not necessarily distinct, and possibly equal to V). An *intertwining operator of type* $\binom{3}{12}$ (or *of type* $\binom{W_3}{W_1\ W_2}$) is a linear map $W_1 \otimes W_2 \to W_3\{x\}$, or equivalently,

$$\begin{aligned} W_1 &\to (\mathrm{Hom}(W_2, W_3))\{x\} \\ w &\mapsto \mathcal{Y}(w, x) = \sum_{n\in\mathbb{Q}} w_n x^{-n-1} \quad (\text{where } w_n \in \mathrm{Hom}(W_2, W_3)) \end{aligned} \tag{2.2}$$

such that "all the defining properties of a module action that make sense hold" (cf. the definition of V-module). That is, for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$w_{(1)n} w_{(2)} = 0 \text{ for } n \text{ whose real part is sufficiently large;} \tag{2.3}$$

the following Jacobi identity holds for the operators $Y_i(v, \cdot)$, $i = 1, 2, 3$, $\mathcal{Y}(w_{(1)}, \cdot)$ acting on the element $w_{(2)}$:

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_3(v, x_1)\mathcal{Y}(w_{(1)}, x_2)w_{(2)} \\ &\quad -x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}(w_{(1)}, x_2)Y_2(v, x_1)w_{(2)} \\ &= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y_1(v, x_0)w_{(1)}, x_2)w_{(2)} \end{aligned} \tag{2.4}$$

(note that the first term on the left-hand side is algebraically meaningful because of condition (2.3), and the other terms are meaningful by the usual properties of modules; also note that this Jacobi identity involves integral powers of x_0 and x_1 and complex powers of x_2);

$$\frac{d}{dx}\mathcal{Y}(w_{(1)}, x) = \mathcal{Y}(L(-1)w_{(1)}, x), \tag{2.5}$$

where $L(-1)$ is the operator acting on W_1.

We may denote the intertwining operator just defined by

$$\mathcal{Y}_{12}^{3} \quad \text{or} \quad \mathcal{Y}_{W_1 W_2}^{W_3}, \tag{2.6}$$

if necessary, to indicate its type.

Note that $Y(\cdot, x)$ acting on V is an example of an intertwining operator of type $\binom{V}{V\,V}$, and $Y(\cdot, x)$ acting on a V-module W is an example of an intertwining operator of type $\binom{W}{V\,W}$. These intertwining operators satisfy the normalization condition $Y(\mathbf{1}, x) = 1$.

The intertwining operators of type $\binom{3}{1\,2}$ clearly form a vector space, which we denote by $\mathcal{V}_{12}^{3}$ or $\mathcal{V}_{W_1 W_2}^{W_3}$. We set

$$N_{12}^{3} = N_{W_1 W_2}^{W_3} = \dim\ \mathcal{V}_{12}^{3} \ \ (\leq \infty). \tag{2.7}$$

These numbers are called the *fusion rules* associated with the algebra and modules. Then for example, assuming that V and the V-module W are nonzero, the corresponding fusion rules are positive:

$$N_{VV}^{V} \geq 1, \tag{2.8}$$
$$N_{VW}^{W} \geq 1, \tag{2.9}$$
$$N_{WV}^{W} \geq 1. \tag{2.10}$$

In [FHL] and [HL5], it is shown that the fusion rules have the following symmetry property: Define

$$N_{ijk} = N_{W_i W_j W_k} = N_{W_i W_j}^{W_k'} \tag{2.11}$$

for $i, j, k = 1, 2, 3$. Then for any element $\sigma \in S_3$, we have

$$N_{\sigma(1)\sigma(2)\sigma(3)} = N_{123}. \tag{2.12}$$

If the vertex operator algebra V is rational, that is, V satisfies the conditions: (i) there are only finitely many irreducible V-modules (up to equivalence), (ii) every V-module is completely reducible, (iii) all the fusion rules are finite, then we can define an algebra called the *fusion algebra* or the *Verlinde algebra* using fusion rules for the irreducible modules as follows: Assume that there are m inequivalent irreducible V-modules. Let A be the abelian group tensor product of the K-group of the V-modules with $\mathbb{C}$. Then A has a natural structure of a vector space. Since V is rational, we have

$$A = \sum_{i=1}^{m} \mathbb{C}\phi_i \tag{2.13}$$

where ϕ_i, $i = 1, \dots, m$, are all the equivalence classes containing irreducible modules. We define a product on A by

$$\phi_i \cdot \phi_j = \sum_{k=1}^{m} N_{ij}^k \phi_k \tag{2.14}$$

for all $i, j = 1, \dots, m$, where N_{ij}^k, $1 \leq i, j, k \leq m$, are the fusion rules $N_{W_i W_j}^{W_k}$ for any $W_i \in \phi_i$, $W_j \in \phi_j$ and $W_k \in \phi_k$. By the symmetry (2.12), it is clear that this product is commutative. When the intertwining operators for the vertex operator algebra satisfy certain additional conditions, it can be proved that this product is also associative. One condition that we need is that all irreducible V-modules are $\mathbb{R}$-graded. If V is rational, then this condition implies that every V-module is $\mathbb{R}$-graded, that is, the weight of an element of a V-module is always a real number. We also need an additional condition. Given any V-modules W_1, W_2, W_3, W_4 and W_5, let $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$ and $\mathcal{Y}_4$ be intertwining operators of type $\binom{W_4}{W_1 W_5}$, $\binom{W_5}{W_2 W_3}$, $\binom{W_5}{W_1 W_2}$ and $\binom{W_4}{W_5 W_3}$, respectively. Consider the following conditions for the product of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ and for the iterate of $\mathcal{Y}_3$ and $\mathcal{Y}_4$, respectively:

Convergence and extension property for products: There exists an integer N (depending only on $\mathcal{Y}_1$ and $\mathcal{Y}_2$), and for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w'_{(4)} \in W'_4$, there exist $j \in \mathbb{N}$, $r_i, s_i \in \mathbb{R}$, $i = 1, \dots, j$, and analytic functions $f_i(z)$ on $|z| < 1$, $i = 1, \dots, j$, satisfying

$$\mathrm{wt}\ w_{(1)} + \mathrm{wt}\ w_{(2)} + s_i > N, \quad i = 1, \dots, j, \tag{2.15}$$

such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_2)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_1^n = e^{n \log z_1},\ x_2^n = e^{n \log z_2},\ n \in \mathbb{C}} \tag{2.16}$$

is convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to the multi-valued analytic function

$$\sum_{i=1}^{j} z_2^{r_i}(z_1 - z_2)^{s_i} f_i\left(\frac{z_1 - z_2}{z_2}\right) \tag{2.17}$$

when $|z_2| > |z_1 - z_2| > 0$.

Convergence and extension property for iterates: There exists an integer $\tilde{N}$ (depending only on $\mathcal{Y}_3$ and $\mathcal{Y}_4$), and for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w'_{(4)} \in W'_4$, there exist $k \in \mathbb{N}$, $\tilde{r}_i, \tilde{s}_i \in \mathbb{R}$, $i = 1, \dots, k$, and analytic functions $\tilde{f}_i(z)$ on $|z| < 1$, $i = 1, \dots, k$, satisfying

$$\mathrm{wt}\ w_{(2)} + \mathrm{wt}\ w_{(3)} + \tilde{s}_i > \tilde{N}, \quad i = 1, \dots, k, \tag{2.18}$$

such that

$$\langle w'_{(4)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_0^n=e^{n\log(z_1-z_2)},\ x_2^n=e^{n\log z_2},\ n\in\mathbb{C}} \tag{2.19}$$

is convergent when $|z_2| > |z_1 - z_2| > 0$ and can be analytically extended to the multi-valued analytic function

$$\sum_{i=1}^{k} z_1^{\tilde{r}_i} z_2^{\tilde{s}_i} \tilde{f}_i\left(\frac{z_2}{z_1}\right) \tag{2.20}$$

when $|z_1| > |z_2| > 0$.

If for any V-modules W_1, W_2, W_3, W_4 and W_5 and any intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of the types as above, the convergence and extension property of products holds, we say that *the products of the intertwining operators for V have the convergence and extension property.* Similarly we can define the meaning of the phrase *the iterates of the intertwining operators for V have the convergence and extension property.*

We also need the notion of generalized module: A *generalized V module* is a pair (W, Y) satisfying all the axioms for a V-module except the two grading axioms: $\dim W_{(n)} < \infty$ for all $n \in \mathbb{C}$ and $W_{(n)} = 0$ for $n \in \mathbb{C}$ whose real part is sufficiently small. If a generalized V-module $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ satisfies the second grading axiom above, we say that W is *lower-truncated.* We have the following result:

Theorem 2.1. *Let V be a rational vertex operator algebra for which all irreducible modules are $\mathbb{R}$-graded. Assume that V satisfies the following conditions:*

1. *Every finitely-generated lower-truncated generalized V-module is a V-module.*
2. *The products or the iterates of the intertwining operators for V have the convergence and extension property.*

Then the Verlinde algebra for V is a commutative associative algebra with unit.

This theorem is an easy consequence of the associativity of the tensor product theory for modules for a vertex operator algebra developed by Lepowsky and the author [HL1] [HL4]–[HL6] [Hu6].

Fusion rules and Verlinde algebras are very important concepts and tools in the study of conformal field theory. One of the most interesting results in the mathematical study of conformal field theory is that the fusion rules and their higher-genus generalizations for the WZNW conformal field theory can be expressed in terms of elementary functions (actually, the sine functions) [Ve]. On the other hand, these fusion rules and generalizations can also be shown to be equal to the dimensions of the space of "generalized theta functions" on the moduli spaces of semistable principal bundles on smooth projective irreducible algebraic curves [KNR]. Thus one obtains a simple and beautiful formula for these dimensions. These are the so called Verlinde formulas. Mathematical proofs of these formulas have been obtained in [TUY] and [Fa].

3. Geometric interpretation of vertex operator algebras

We give a brief description of the geometric interpretation of vertex operator algebras in this section. The geometric interpretations of vertex operators, their duality properties and their transformation properties under the projective transformations were first given by Frenkel [Fr] using the geometry of $\mathbb{C}\cup\{\infty\}$ with some discs deleted. The complete geometric interpretation is obtained in [Hu1] and [Hu8]. The formulation using operads is given in [HL2] and [HL3]. See [Hu1]–[Hu4], [Hu8], [HL2] and [HL3] for details and other expositions.

In classical algebraic theories we study mostly algebraic structures defined by binary operations. These binary operations can always be described by one-dimensional geometric objects. For example, Lie algebras can be described by binary trees. A Lie algebra can be defined to be a "linear representation" of the moduli space of binary trees with a "welding operation," satisfying certain "conservation" and "orientation" properties [Hu1] [Hu5]. Any associative binary operation, for example, the multiplication for a group or an algebra, can be described using the moduli space of circles with punctures and local coordinates [HL2] [HL3]. The general philosophy behind the geometric interpretation of vertex operator algebras is to study certain two-dimensional analogues of the classical binary operations, that is, to study operations described by two-dimensional analogues of binary trees or circles with punctures and local coordinates.

The two-dimensional analogues, used to describe vertex operator algebras, of both binary trees and circles with punctures and local coordinates are spheres with analytically parametrized boundaries, where by spheres we mean one-dimensional compact connected genus-zero complex manifolds. These spheres with boundaries are in some sense equivalent to spheres with ordered points (which are called punctures), one negatively oriented and others positively oriented, and local coordinates vanishing at these points, as is explained in [Hu1] and [Hu8]. We will use the the index 0 to denote the negatively oriented puncture on such a sphere with punctures and local coordinates. Let S_1 and S_2 be two such spheres with punctures and local coordinates, p_j, $j=0,\dots,m$, the punctures of S_1, q_k, $k=0,\dots,n$, the punctures of S_2, (U_j,φ_j), $j=0,\dots,m$, the local coordinates vanishing at p_j and (V_k,ψ_k), $k=0,\dots,n$, the local coordinates vanishing at q_k. For any integer i satisfying $0<i\leq n$, we would like to sew S_1 and S_2 through the i-th puncture of S_1 and the 0-th puncture of S_2 to obtain a new spheres with punctures and local coordinates. Assume that there exists a positive number r such that $\varphi_i(U_i)$ contains the closed disc $\bar{B}_0^r$ centered at 0 with radius r and $\psi_0(V_0)$ contains the closed disc $\bar{B}_0^{1/r}$ centered at 0 with radius $1/r$. Assume also that p_i and q_0 are the only punctures in $\varphi_i^{-1}(\bar{B}_0^r)$ and $\psi_0^{-1}(\bar{B}_0^{1/r})$,

respectively. In this case we say that *the i-th puncture of S_1 can be sewn with the 0-th puncture of S_2*. In this case, we obtain a sphere with $n+m+1$ punctures and local coordinates by cutting $\varphi_i^{-1}(B_0^r)$ and $\psi_0^{-1}(B_0^{1/r})$ from S_1 and S_2, respectively, and then identifying the boundaries of the resulting surfaces using the map $\varphi_i \circ \gamma \circ \psi_0^{-1}$ where γ is the map from $\mathbb{C} \setminus \{0\}$ to itself defined by $\gamma(z) = 1/z$. The punctures (with ordering) of this sphere with punctures and local coordinates are $p_0, \ldots, p_{i-1}, q_1, \ldots, q_n, p_{i+1}, \ldots, p_m$. The local coordinates vanishing at these punctures are given in the obvious way. Thus we have a partial operation. Given two such spheres with punctures and local coordinates, S_1 and S_2, with the same number of punctures, if there is a analytic isomorphism from the underlying sphere of S_1 to the underlying sphere of S_2 such that the ordered punctures of S_1 are mapped to the ordered punctures of S_2 and the germs containing the pull-backs of the local coordinates of S_2 are the same as the germs containing the local coordinates of S_1, we say that S_1 and S_2 are *conformally equivalent*. This is an equivalence relation. The space of conformal equivalence classes of such spheres with punctures and local coordinates is called the *moduli space of spheres with punctures and local coordinates.*

The moduli space of spheres with $n+1$ punctures and local coordinates $(n \geq 1)$ can be identified with $K(n) = M^{n-1} \times H \times H_c^n$ where H is the set of all sequences A of complex numbers such that $\exp(\sum_{j=1}^{\infty} A_j x^{j+1} \frac{d}{dx}) \cdot x$ is a convergent power series in some neighborhood of 0, $H_c = \mathbb{C}^\times \times H$, and M^{n-1} is the subset of elements in $\mathbb{C}^{n-1}$ with nonzero and distinct components. The moduli space of spheres with one punctures and local coordinates can be identified with $K(0) = \{B \in H \mid B_1 = 0\}$. Then the moduli space of spheres with punctures and local coordinates can be identified with $\cup_{n=1}^{\infty} K(n)$. From now on we will refer to $K(n)$, $n \in \mathbb{N}$ as the moduli space of spheres with $n+1$ punctures and local coordinates. The sewing operation for spheres with punctures and local coordinates induces a partial operation on $\cup_{n=1}^{\infty} K(n)$. It is still called the *sewing operation* and is denoted ${}_i\infty_0$. Note that there is an obvious action of S_n on $K(n)$ by permuting the ordering of the n positively oriented punctures and local coordinates.

Now we have a sequences of sets $K = \{K_n\}_{n=1}^{\infty}$ together with partial operations ${}_i\infty_0 : K(j) \times K(k) \to K(j+k-1)$, $j \in \mathbb{Z}_+$, $k \in \mathbb{N}$, $i \in \mathbb{Z}_+$ and actions of S_n on $K(n)$, $n \in \mathbb{Z}_+$, respectively. It is easy to show that the sew operations satisfy the following conditions when the sewing operations appear in the equations below exist:

(1) For any $j \in \mathbb{Z}_+$, $k, l \in \mathbb{N}$, i_1, $1 \leq i_1 \leq j$, i_2, $1 \leq i_2 \leq j+k-1$, $Q_1 \in K(j)$, $Q_2 \in K(k)$, $Q_3 \in K(l)$,

$$(Q_1\, {}_{i_1}\infty_0\, Q_2)\, {}_{i_2}\infty_0\, Q_3 = \begin{cases} (Q_1\, {}_{i_2}\infty_0\, Q_3)\, {}_{l+i_1-1}\infty_0\, Q_2, & i_2 < i_1, \\ Q_1\, {}_{i_1}\infty_0 (Q_2\, {}_{i_2-i_1+1}\infty_0\, Q_3), & i_1 \leq i_2 < i_1+k, \\ (Q_1\, {}_{i_2-j+1}\infty_0\, Q_3)\, {}_{i_1}\infty_0\, Q_2, & i_1+k \leq i_2. \end{cases} \tag{3.1}$$

(2) For any $j \in \mathbb{Z}_+$, $k \in \mathbb{N}$, i, $1 \leq i \leq k$, $Q_1 \in K(j)$, $Q_2 \in K(k)$, $\sigma \in S_j$ and $\tau \in S_k$,

$$\sigma(Q_1)\,{}_i\infty_0\, Q_2 = \sigma(\overbrace{1, \dots, 1}^{j-1}, k, \overbrace{1, \dots, 1}^{j-k})(Q_1\, {}_{\sigma(i)}\infty_0\, Q_2), \tag{3.2}$$

$$Q_1\, {}_i\infty_0\, \tau(Q_2) = (\overbrace{1 \oplus \cdots \oplus 1}^{k-1} \oplus \tau \oplus \overbrace{1 \oplus \cdots \oplus 1}^{j-k})(Q_1\, {}_i\infty_0\, Q_2). \tag{3.3}$$

(3) Let $I = (\mathbf{0}, (1, \mathbf{0})) \in H \times (\mathbb{C}^\times \times H) = K(1)$. Then for any $k \in \mathbb{N}$, i, $1 \leq i \leq k$, $Q \in K(k)$,

$$Q\, {}_i\infty_0\, I = I\, {}_1\infty_0\, Q = Q. \tag{3.4}$$

A sequence $\{\mathcal{X}(j)\}_{j \in \mathbb{N}}$ of sets equipped with $\circ_i : \mathcal{X}(j) \times \mathcal{X}(k) \to \mathcal{X}(j+k-1)$, $j \in \mathbb{Z}_+$, $k \in \mathbb{N}$, $1 \leq i \leq k$, actions of S_n on $\mathcal{X}(n)$, $n \in \mathbb{Z}_+$, respectively, and $I \in \mathcal{X}(1)$ satisfying the conditions (1)–(3) above with $K(n)$, $n \in \mathbb{N}$, replaced by $\mathcal{X}(n)$, ${}_i\infty_0$ by $\circ_i$, is called an *operad* [M1]. If the operations $\circ_i$ are only partial and conditions (1)–(3) are satisfied when the operations in the equations in (1)–(3) exist, it is called a *partial operad* [M2] [HL2] [HL3]. Thus we see that K is a partial operad. We can also give a topological structure and a complex analytic structure to K such that the sewing operations ${}_i\infty_0$ are continuous and complex analytic.

We shall define a (geometric) vertex operator algebra to be a "linear projective representation" of this partial operad satisfying some additional conditions. In the representation theory of groups, a linear projective representation of a group is a linear representation of a central extension of the group. For K, we also have certain extensions which are analogues of central extensions of groups. These extensions are constructed using determinant lines over spheres with analytically parametrized boundary.

We describe briefly Segal's work on determinant lines over Riemann surfaces with analytically parametrized boundary here. For details, see [S]. Let Σ be a compact Riemann surface with analytically parametrized and oriented boundary components. We have the Cauchy-Riemann operator $\bar{\partial}$ from the space $\Omega^0(\Sigma)$ of smooth functions on the surface to the space $\Omega^{0,1}(\Sigma)$ of $(0,1)$-forms on the surface. The boundary of Σ can be decomposed as $\partial\Sigma = \cup_{i=1}^k C_i^{\epsilon_i}$ where for any i, $1 \leq i \leq k$, $C_i^{\epsilon_i}$ is a connected component of $\partial\Sigma$ and thus is parametrized by an analytic map from the circle S^1 to C_i^{ϵ} and where $\epsilon_i = \pm$ indicates the orientation of the component. Any smooth function on $C_i^{\epsilon_i}$ can be decomposed as the sum of two smooth functions, one of which, as a function on S^1, has a Fourier expansion of the form $\sum_{n \geq 0} a_n e^{2\pi n\theta i}$ (θ is the usual parametrization of the circle by angles) and the other of which, as a function on S^1, has a Fourier expansion of the form $\sum_{n<0} a_n e^{2\pi n\theta i}$. If $\epsilon_i = +$ ($\epsilon_i = -$), that is, this component is positively (negatively) oriented, we denote by $\Omega_+^0(C_i^{\epsilon_i})$ the space of all smooth functions on $C_i^{\epsilon_i}$ which as functions on S^1 have

Fourier expansions of the form $\sum_{n\geq 0} a_n e^{2\pi n\theta i}$ ($\sum_{n<0} a_n e^{2\pi n\theta i}$) and by $\Omega^0_-(C_i^{\epsilon_i})$ the space of smooth functions on $C_i^{\epsilon_i}$ which as functions on S^1 have Fourier expansions of the form $\sum_{n<0} a_n e^{2\pi n\theta i}$ ($\sum_{n\geq 0} a_n e^{2\pi n\theta i}$). Thus the space $\Omega^0(\partial\Sigma)$ of all smooth functions on $\partial\Sigma$ can be decomposed as $\oplus_{i=1}^k(\Omega^0_+(C_i^{\epsilon_i}) \oplus \Omega^0_-(C_i^{\epsilon_i}))$. Let

$$\Omega^0_+(\partial\Sigma) = \oplus_{i=1}^k \Omega^0_{-\epsilon_i}(C_i^{\epsilon_i}) \subset \Omega^0(\partial\Sigma). \tag{3.5}$$

Let pr be the composition of the restriction from $\Omega^0(\Sigma)$ to $\Omega^0(\partial\Sigma)$ and the projection from $\Omega^0(\partial\Sigma)$ to $\Omega^0_+(\partial\Sigma)$. We have an operator

$$\overline{\partial} \oplus \mathrm{pr} : \Omega^0(\Sigma) \to \Omega^{0,1}(\Sigma) \oplus \Omega^0_+(\partial\Sigma). \tag{3.6}$$

Using the theory of elliptic boundary problems on manifolds with boundaries (see, for example, [Hö]), we can show that $\overline{\partial} \oplus \mathrm{pr}$ can be extended to Fredholm operators from suitable generalizations of Sobolev spaces on Σ to closed subspaces of suitable Sobolev spaces on $\partial\Sigma$. In addition, the kernels of these extensions are equal to the kernel of $\overline{\partial} \oplus \mathrm{pr}$ and the orthogonal complements of the images of these extensions are in $\Omega^{0,1}(\Sigma) \oplus \Omega^0_+(\partial\Sigma)$. Thus we can regard the kernel and cokernel of $\overline{\partial} \oplus \mathrm{pr}$ as the kernels and cokernels of its extensions. Since these extensions are Fredholm, the kernel and cokernel of $\overline{\partial} \oplus \mathrm{pr}$ are finite-dimensional. The determinant line over Σ is defined as

$$\mathrm{Det}_\Sigma = \mathrm{Det}\ (\mathrm{Ker}\ (\overline{\partial} \oplus \mathrm{pr}))^* \otimes \mathrm{Det}\ \mathrm{Coker}\ (\overline{\partial} \oplus \mathrm{pr}) \tag{3.7}$$

where $\mathrm{Det}\ (\mathrm{Ker}\ (\overline{\partial} \oplus \mathrm{pr}))^*$ and $\mathrm{Det}\ \mathrm{Coker}\ (\overline{\partial} \oplus \mathrm{pr})$ are the highest nonzero exterior powers of $(\mathrm{Ker}\ (\overline{\partial} \oplus \mathrm{pr}))^*$ and $\mathrm{Coker}\ (\overline{\partial} \oplus \mathrm{pr})$, respectively. The main property of determinant lines over Riemann surfaces with analytically parametrized and oriented boundary components is that if we sew two such Riemann surfaces, Σ_1 and Σ_2, by identifying certain boundary components on Σ_1 to certain boundary components with opposite orientations on Σ_2 using the given analytic parametrizations to obtain another such, denoted by $\Sigma_1\infty\Sigma_2$, then there exists a canonical isomorphism

$$\ell_{\Sigma_1,\Sigma_2} : \mathrm{Det}_{\Sigma_1} \otimes \mathrm{Det}_{\Sigma_2} \to \mathrm{Det}_{\Sigma_1\infty\Sigma_2}. \tag{3.8}$$

These determinant lines give a holomorphic line bundle over the moduli space of Riemann surfaces with oriented and analytically parametrized boundaries, and there is a canonical connection on this line bundle. See [S] for more details.

Now we want to use Segal's work described above to define the determinant line for an element Q of K. We need to find a sphere with analytically parametrized and oriented boundary Σ_Q determined uniquely by Q. For any $Q \in K$, there is a unique sphere with punctures and local coordinates in Q such that its underlying sphere is $\mathbb{C} \cup \{\infty\}$, the negatively oriented puncture is ∞, the last positively oriented puncture is 0, the value at ∞ of the derivative of the local coordinate map at ∞ is 1 and all the local coordinate neighborhoods at the punctures are the preimages under the local coordinate maps of the maximal open disks (possibly with infinite radius) centered at 0 on which the inverses of local coordinate maps have well-defined analytic extensions. For any positive real number r and any puncture, consider the closed disk of radius equal to r times the minimum of 1 and half of the radius of the maximal disk above at the puncture. (To avoid closed disks with infinite radius, we choose the minimum of 1 and half of the radius of the maximal disk instead of half of the radius of the maximal disk.) For a fixed r, a closed disks above is called a *closed disks associated to* r. Let X be the set of all positive real numbers such that if $r \in X$, then at any puncture the closed disk associated to r are contained in the maximal open disk above and preimages under local coordinate maps of closed disks associated to r at different punctures do not intersect with each other. Let $r_0 = \sup X$ and $r_1 = \min(1, \frac{r_0}{2})$. (To make sure that r_1 is not ∞, we define r_1 to be $\min(1, \frac{r_0}{2})$ instead of $\frac{r_0}{2}$.) We obtain a Riemann surface with oriented and analytically parametrized boundary components Σ_Q by cutting the preimages of the closed disks associated to r_1 and giving its boundary components the obvious orientations and analytic parametrizations. We define

$$\mathrm{Det}_Q = \mathrm{Det}_{\Sigma_Q}. \tag{3.9}$$

For $m, n \in \mathbb{N}$, $Q_1 \in K(m)$ and $Q_2 \in K(n)$ such that $Q_1 {}_i\infty_0 Q_2$ exists, we also have a canonical isomorphism

$$\ell^i_{Q_1,Q_2} : \mathrm{Det}_{Q_1} \otimes \mathrm{Det}_{Q_2} \to \mathrm{Det}_{Q_1 {}_i\infty_0 Q_2} \tag{3.10}$$

induced from the canonical isomorphism for spheres with analytically parametrized and oriented boundary. Let

$$\begin{aligned} \tilde{K}(n) &= \cup_{Q\in K(n)}\mathrm{Det}_Q, \quad n \in \mathbb{N}, & (3.11)\\ \tilde{K} &= \{\tilde{K}(n)\}_{n\in\mathbb{N}}. & (3.12)\end{aligned}$$

Then $\tilde{K}(n)$, $n \in \mathbb{N}$, are holomorphic line bundles (in a suitable sense) over $K(n)$. There are also operations in $\tilde{K}$ obtained from the sewing operations in K and the canonical isomorphisms for determinant lines defined as follows: Let $m, n \in \mathbb{N}$, i an integer satisfying $1 \le i \le m$, $Q_1 \in K(m)$, $Q_2 \in K(n)$, $\tilde{Q}_1 \in \mathrm{Det}_{Q_1} \subset \tilde{K}(m)$ and $\tilde{Q}_2 \in \mathrm{Det}_{Q_2} \subset \tilde{K}(n)$, such that $Q_1 {}_i\infty_0 Q_2$ exists. We define

$$\tilde{Q}_1 {}_i\widetilde{\infty}_0 \tilde{Q}_2 = \ell^i_{Q_1,Q_2}(\tilde{Q}_1 \otimes \tilde{Q}_2) \in \mathrm{Det}_{Q_1 {}_i\infty_0 Q_2} \subset \tilde{K}(m+n-1). \tag{3.13}$$

Thus we obtain a partial operation ${}_i\widetilde{\infty}_0 : \tilde{K}(m) \times \tilde{K}(n) \to \tilde{K}(m+n-1)$ for any $m, n \in \mathbb{N}$ and any integer i satisfying $1 \le i \le m$. Note that the definition of determinant line over an element $Q \in K(n)$ for any $n \in \mathbb{N}$ does not use the ordering of the positively oriented puncture of Q. Thus for any $\sigma \in S_n$, Det_Q is canonically isomorphic to $\mathrm{Det}_{\sigma(Q)}$. We denote this canonical isomorphism by φ^σ_Q. For any $\tilde{Q} \in \mathrm{Det}_Q \subset \tilde{K}(n)$, we define

$$\sigma(\tilde{Q}) = \varphi^\sigma_Q(\tilde{Q}) \in \mathrm{Det}_{\sigma(Q)} \subset \tilde{K}(n). \tag{3.14}$$

We obtain an action of S_n on $\tilde{K}(n)$. Let $\tilde{I}$ be the unique element of Det_I satisfying $\ell^1_{I,I}(\tilde{I}\otimes\tilde{I})=\tilde{I}$. Then the sequence $\tilde{K}$ together with the operations

$$
{}_i\widetilde{\infty}_0 : \tilde{K}(m)\times\tilde{K}(n)\to\tilde{K}(m+n-1),
$$

$m,n\in\mathbb{N}$, $1\leq i\leq m$, the actions of the symmetric groups and $\tilde{I}$ is a partial operad. Also the operations ${}_i\widetilde{\infty}_0$, $m,n\in\mathbb{N}$, $1\leq i\leq m$, are all continuous and analytic with respect to the topological and analytic structures on the holomorphic line bundles $\tilde{K}(n)$ over $K(n)$, $n\in\mathbb{N}$.

For any $n\in\mathbb{N}$, there is a canonical connection on the determinant line bundle $\tilde{K}(n)$. Using this connection, we can prove that the determinant line bundle $\tilde{K}(n)$ is trivial. Thus for any complex number c, the c-th power of determinant line bundle $\tilde{K}(n)$ is well defined. Note that the c-th power of $\tilde{K}(n)$ is the line bundle whose fibers are the same as those of $\tilde{K}(n)$ and whose transition functions are equal to the c-th powers of the transition functions of $\tilde{K}(n)$. The existence of the c-power of $\tilde{K}(n)$ means that the the c-powers of the transition functions of $\tilde{K}(n)$ can be chosen consistently so that they also give a holomorphic line bundle, the c-th power of $\tilde{K}(n)$. So we see that because $\tilde{K}(n)$ is trivial, the c-th power of $\tilde{K}(n)$ is in fact canonically isomorphic to $\tilde{K}(n)$. We denote the c-th power of $\tilde{K}(n)$ by $\tilde{K}^c(n)$. Since, as a line bundle over $K(n)$, $\tilde{K}^c(n)$ is canonically isomorphic to $\tilde{K}(n)$, we shall not distinguish between the elements of $\tilde{K}(n)$ and the elements of $\tilde{K}^c(n)$. In particular, for any element $\tilde{Q}$ of $\tilde{K}^c(n)$ there is $Q\in K(n)$ such that $\tilde{Q}$ is in Det_Q. The difference between $\tilde{K}^c(n)$ and $\tilde{K}(n)$ is that the canonical isomorphisms in them are different. We can prove that we can choose $\ell^i_{Q_1,Q_2}$ and φ^σ_Q raised to the complex power c (denoted by $(\ell^i_{Q_1,Q_2})^c$ and $(\varphi^\sigma_Q)^c$, respectively) consistently for $m,n\in\mathbb{N}$, $1\leq i\leq m$, $Q_1\in K(m)$, $Q,Q_2\in K(n)$ and $\sigma\in S_n$, such that $\tilde{K}^c=\{\tilde{K}^c(n)\}_{n\in\mathbb{N}}$ together with the operations ${}_i\widetilde{\infty}^c_0$ defined in the same way as that for ${}_i\widetilde{\infty}_0$ except that $\ell^i_{Q_1,Q_2}$ is replaced by $(\ell^i_{Q_1,Q_2})^c$, the actions of the symmetric groups defined using $(\varphi^\sigma_Q)^c$ and $\tilde{I}\in\tilde{K}^c(1)$, is also a partial operad. The canonical connection on $\tilde{K}(n)$ gives a canonical connection on $\tilde{K}^c(n)$. Beginning with $\tilde{I}$, we obtain a section ψ_1 of $\tilde{K}^c(1)$ by parallel transport (this section is in fact not continuous when $c\neq 0$). Let $J\in K(0)$ be the conformal equivalence class containing the sphere $\mathbb{C}\cup\{\infty\}$ with the negative oriented puncture ∞ and the standard local coordinate $w\to w^{-1}$ vanishing at ∞ and let $\tilde{J}$ be a fixed element of Det_J. Then beginning with $\tilde{J}$, we obtain a section ψ_0 of $\tilde{K}^c(1)$ by parallel transport. Let $P(1)\in K(2)$ be the conformal equivalent class containing the sphere $\mathbb{C}\cup\{\infty\}$ with the negatively oriented puncture ∞, the positively oriented puncture 1 and 0, the standard local coordinate $w\to w^{-1}$ vanishing at ∞, the standard local coordinate $w\to w-1$ vanishing at 1 and the standard local coordinate $w\to w$ vanishing at 0. Let $\tilde{P}(1)$ be the unique element of $\mathrm{Det}_{P(1)}$ such that $(\ell^1_{P(1),J})^c(\tilde{P}(1)\otimes\tilde{J})=\tilde{I}$. Beginning with $\tilde{P}(1)\in\tilde{K}^c(2)$ we obtain a section ψ_2 of $\tilde{K}^c(2)$ by parallel transport. Since K is generated by $K(0)$, $K(1)$ and $K(2)$ (which means that any element in

$K(n)$ for any $n \in \mathbb{N}$ can be obtained by sewing elements in $K(0)$, $K(1)$ and $K(2)$), we obtain sections ψ_n of $\tilde{K}^c(n)$, $n \in \mathbb{N}$. Then we have $\{\psi_n\}_{n\in\mathbb{N}}$ which is a section of $\tilde{K}^c$.

To define a "linear representation" of $\tilde{K}^c$, we first have to construct a partial operad from a vector space. Given a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ such that $\dim V_{(n)} < \infty$, we can construct a partial operad in the following way (see [HL2] [HL3]): Let

$$\mathcal{H}_V(n) = \mathrm{Hom}(V^n, \overline{V}), \tag{3.15}$$
$$\mathcal{H}_V = \{\mathcal{H}_V(n)\}_{n=1}^{\infty} \tag{3.16}$$

where $\overline{V} = \prod_{n\in\mathbb{Z}} V_{(n)}$. Let P_n, $n \in \mathbb{Z}$, be the projection from $\overline{V}$ to $V_{(n)}$. For $f \in \mathcal{H}_V(m)$, $g \in \mathcal{H}_V(n)$ and $0 \le i \le m$, if for any $v' \in V'$, $v_1, \dots, v_{m+n-1} \in V$ the series

$$\sum_{n\in\mathbb{Z}} \langle v', f(v_1, \dots, v_{i-1}, P_n(g(v_i, \dots, v_{i+n-1})), v_{i+n}, \dots, v_{m+n-1})\rangle \tag{3.17}$$

(where $\langle \cdot, \cdot \rangle$ denotes the pairing between V' and $\overline{V}$) converges, we say that *the contraction* $f\, {}_i{*}_0\, g$ *exists* and define the *contraction* $f\, {}_i{*}_0\, g \in \mathcal{H}_V(m+n-1)$ using the values of these series. Note that contractions are partial operations. The permutation group S_n also acts on $\mathcal{H}_V(n)$ in the obvious way. We also have the inclusion map $I_V \in \mathcal{H}_V(1) = \mathrm{Hom}(V, \overline{V})$. The sequence $\mathcal{H}_V$ together with the contractions, the actions of the symmetric groups and the inclusion map I_V, is a partial operad, called the *endomorphism partial operad of* V.

Roughly speaking, a "geometric vertex operator algebra" (or a "vertex associative algebra") is a $\mathbb{Z}$-graded vector space V equipped with a "homomorphism" from the partial operad $\tilde{K}^c$ to the partial operad $\mathcal{H}_V$ satisfying some additional natural axioms. Precisely, we have the following:

Definition 3.1. A *geometric vertex operator algebra of central charge c* is a $\mathbb{Z}$-graded vector space V and a map $\Phi : \tilde{K}^c \longrightarrow \mathcal{H}_V$ such that $\Phi(\tilde{K}^c(n)) \subset \mathcal{H}_V(n)$ satisfying:

(1) The positive energy axiom: $V_{(n)} = 0$ for n sufficiently small.

(2) The grading axiom: Let $Q(a) = (\mathbf{0}, (a, \mathbf{0})) \in H \times (\mathbb{C}^{\times} \times H) = K(1)$ (the conformal equivalence class containing the sphere $\mathbb{C} \cup \{\infty\}$ with the negatively oriented puncture ∞, the positively oriented puncture 0, the standard local coordinate $w \to w^{-1}$ vanishing at ∞ and the local coordinate $w \to aw$ vanishing at 0). Then for any $n \in \mathbb{Z}$, $v \in V_{(n)}$, $v' \in V'$,

$$\langle v', \Phi(\psi_2(Q(a)))(v)\rangle_V = a^{-n}\langle \cdot, \cdot \rangle_{V_{(n)}} \tag{3.18}$$

where $\langle \cdot, \cdot \rangle_n$ is the pairing between V' and $V_{(n)}$ induced from the pairing $\langle \cdot, \cdot \rangle_V$ between V' and $\overline{V}$.

(3) The permutation axiom: For any $n \in \mathbb{N}$, $\sigma \in S_n$ and $\tilde{Q} \in \tilde{K}^c(n)$,

$$\Phi(\sigma(\tilde{Q})) = \sigma(\Phi(\tilde{Q})). \tag{3.19}$$

(4) The analyticity axiom: For any $n \in \mathbb{N}$, let

$$\nu_n = \Phi \circ \psi_n : K \to \mathcal{H}_V. \tag{3.20}$$

Then for any $v' \in V'$, $v_1, \dots, v_n \in V$, $\langle v', \nu_n(\cdot)(v_1 \otimes \cdots \otimes v_n)\rangle$ as a function on $K(n)$ is meromorphic on M^{n-1} with $z_i = 0$ and $z_i = z_j$, $i, j = 1, \dots, n-1$, $i \neq j$, as the only possible poles, and is a Laurent polynomial in the components belonging to $\mathbb{C}^\times$ of $K(n)$ and is a polynomial in the components belonging to H of $K(n)$. In addition, for fixed i, j, $1 \leq i < j \leq\leq n$, and $v_i, v_j \in V$ there is an upper bound, independent of v_k, $k \neq i, j$, for the order of the pole $z_i - z_j$ of the function $\langle v', \nu_n(\cdot)(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n)\rangle$.

(5) The sewing axiom: For any $m, n \in \mathbb{N}$, $\tilde{Q}_1 \in \tilde{K}^c(m)$ and $\tilde{Q}_2 \in \tilde{K}^c(n)$ such that $\tilde{Q}_1 {}_i\widetilde{\infty}_0 \tilde{Q}_2$ exists, $\Phi(\tilde{Q}_1) {}_i{*}_0\, \Phi(\tilde{Q}_2)$ also exists and

$$\Phi(\tilde{Q}_1 {}_i\widetilde{\infty}_0 \tilde{Q}_2) = \Phi(\tilde{Q}_1) {}_i{*}_0\, \Phi(\tilde{Q}_2). \tag{3.21}$$

The definition of homomorphisms from one geometric vertex operator algebra to another of the same rank is clear. The following theorem (see [Hu1]–[Hu4] [Hu8]) establishes the equivalence between vertex operator algebras and geometric vertex operator algebras:

Theorem 3.1. *The category of geometric vertex operator algebras of rank c is isomorphic to the category of vertex operator algebras of rank c.*

The map ν in the definition above can also be constructed algebraically.

4. Vertex operator algebras and conformal field theories

The rapidly-evolving theory of vertex operator algebras has been beginning to show its power in the study of many problems related to conformal field theories. It is expected that in the future this theory will play a more important role in the study of conformal field theories and related mathematical problems.

Basically, there are two approaches to conformal field theories. One is the geometric approach. In physics, many models of conformal field theories are studied using the path integral method. Starting from the work of Friedan and Shenker [FS], physicists have realized the importance of the moduli space of Riemann surfaces with punctures in the study of conformal field theories. The basic mathematical work in the geometric approach is Segal's definition of conformal field theory using Riemann surfaces with oriented and analytically parametrized boundary components [S]. Motivated by the operator formalism for the theory of free bosons and free fermions, another closely related formulation of conformal field theories is given by Vafa [Va] using Riemann surfaces with punctures and local coordinates vanishing at these punctures, on a physical level of rigor. The geometric approach has the advantage that it gives conceptually satisfactory definitions and it also allows one to derive many important results using geometric intuition. But the main difficulty that the geometric approach encountered is that it is very difficult to construct nontrivial examples satisfying all these geometric axioms and thus also difficult to discover subtle structures that a conformal field theory might have. On the other hand, beginning with the seminal work of Belavin, Polyakov and Zamolodchikov [BPZ] in physics and the works of Borcherds [B], Frenkel, Lepowsky and Meurman [FLM2] in mathematics, another approach, the algebraic one, provides a practical way for both physicists and mathematicians to construct concrete examples of conformal field theories. There are already many examples of conformal field theories (in the algebraic formulation) constructed from Lie algebras, lattices, Jordan algebras, $\mathcal{W}$-algebras (certain associative algebras similar to the universal enveloping algebra of a Lie algebra). There are also algebraic methods, for example, methods to construct orbifold theories and coset models, which give more examples from the known ones. But the algebraic approach has the disadvantage that it mostly constructs and studies only the genus-zero and genus-one theory. Also the axioms in the algebraic formulations may seem unfamiliar or complicated at first (although they are indeed completely canonical). It is therefore necessary and important to establish rigorously the relationship between the algebraic and geometric approaches. One of the main ingredient in a conformal field theory is its "chiral algebra" which is a vertex operator algebra. The geometric interpretation of vertex operator algebras described in the preceding section can be viewed as a first step of the project of establishing the equivalence between the two approaches and thus obtaining examples satisfying the geometric axioms from the known examples satisfying the algebraic axioms. Another step in this direction is Zhu's work [Z] in which he constructed certain genus-one correlation functions from a vertex operator algebra and its irreducible modules, assuming that the vertex operator algebra satisfies certain conditions.

Let me end this exposition with the following picture describing the program of studying conformal field theories and related mathematical problems using the representation theory of vertex operator algebras:

Elementary mathematical data (lattices, Lie algebras, Jordan algebras, $\mathcal{W}$-algebras, etc.)

$\Downarrow$

Vertex operator algebras, modules, intertwining operators

$\Downarrow$

Modular functors and conformal field theories (in the sense of Segal)

$\Downarrow$

Consequences (Verlinde formulas, modular tensor categories, knot invariants and three-manifold invariants, monstrous moonshine, etc.)

REFERENCES

[BPZ] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite conformal symmetries in two-dimensional quantum field theory, *Nucl. Phys.* **B241** (1984), 333–380.

[B] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068–3071.

[D1] C. Dong, Representations of the moonshine module vertex operator algebra, in: *Proc. 1992 Joint Summer Research Conference on Conformal Field Theory, Topological Field Theory and Quantum Groups, Mount Holyoke, 1992*, ed. P. Sally, M. Flato, J. Lepowsky, N. Reshetikhin, G. Zuckerman, Contemporary Math., Vol. 175, Amer. Math. Soc., Providence, 1994.

[D2] C. Dong, Introduction to vertex operator algebras, I, in this volume.

[DGM] L. Dolan, P. Goddard and P. Montague, Conformal field theory of twisted vertex operators, *Nucl. Phys.* **B338** (1990), 529–601.

[Fa] G. Faltings, A proof for the Verlinde formula, to appear.

[Fr] I. B. Frenkel, talk presented at the Institute for Advanced Study, 1988; and private communications.

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs Amer. Math. Soc.* **104**, Number 494, 1993.

[FLM1] I. B. Frenkel, J. Lepowsky and A. Meurman, A natural representation of the Fischer-Griess Monster with the modular function J as character, *Proc. Natl. Acad. Sci. USA* **81** (1984), 3256–3260.

[FLM2] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.

[FS] D. Friedan and S. Shenker, The analytic geometry of two-dimensional conformal field theory, *Nucl. Phys.* **B281** (1987), 509–545.

[Hö] L. Hörmander, The Analysis of Linear Partial Differential Operators III, Grundlehren Der Mathematischen Wissenschaften, Vol. 274, Springer-Verlag, Berlin, 1985.

[Hu1] Y.-Z. Huang, *On the geometric interpretation of vertex operator algebras*, Ph.D. thesis, Rutgers University, 1990.

[Hu2] Y.-Z. Huang, Geometric interpretation of vertex operator algebras, *Proc. Natl. Acad. Sci. USA* **88** (1991), 9964–9968.

[Hu3] Y.-Z. Huang, Applications of the geometric interpretation of vertex operator algebras, in: *Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991*, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, Vol. 1, 333–343.

[Hu4] Y.-Z. Huang, Vertex operator algebras and conformal field theory, *Intl. J. Mod. Phys.* **A7** (1992), 2109–2151.

[Hu5] Y.-Z. Huang, Binary trees and finite-dimensional Lie algebras, in *Proc. AMS Summer Research Institute on Algebraic Groups and Their Generalizations, Pennsylvania State University, 1991,* ed. W. J. Haboush and B. J. Parshall, American Mathematical Society, Providence, 1994, Vol. 2, 337–348.

[Hu6] Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, to appear.

[Hu7] Y.-Z. Huang, A nonmeromorphic extension of the moonshine module vertex operator algebra, to appear.

[Hu8] Y.-Z. Huang, Operads and the geometric interpretation of vertex operator algebras, to appear.

[HL1] Y.-Z. Huang and J. Lepowsky, Toward a theory of tensor products for representations of a vertex operator algebra, in: *Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991*, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, Vol. 1, 344–354.

[HL2] Y.-Z. Huang and J. Lepowsky, Vertex operator algebras and operads, in: *The Gelfand Mathematical Seminars, 1990–1992*, ed. L. Corwin, I. Gelfand and J. Lepowsky, Birkhäuser, Boston, 1993, 145–161.

[HL3] Y.-Z. Huang and J. Lepowsky, Operadic formulation of the notion of vertex operator algebra, in: *Proc. 1992 Joint Summer Research Conference on Conformal Field Theory, Topological Field Theory and Quantum Groups, Mount Holyoke, 1992*, ed. P. Sally, M. Flato, J. Lepowsky, N. Reshetikhin, G. Zuckerman, Contemporary Math., Vol. 175, Amer. Math. Soc., Providence, 1994, 131–148.

[HL4] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, to appear.

[HL5] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, to appear.

[HL6] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, to appear.

[KNR] S. Kumar, M. S. Narasimhan and A. Ramanathan, Infinite Grassmannians and moduli spaces of G-bundles, *Math. Ann.* **300** (1994), 41–75.

[L] H. Li, Introduction to vertex operator algebras, II, in this volume.

[M1] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Math. **271**, Springer-Verlag, 1972.

[M2] J. P. May, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Mathematics, No. 577, Springer-Verlag, 1977.

[S] G. Segal, The definition of conformal field theory, preprint, 1988.

[TUY] A. Tsuchiya, K. Ueno and Y. Yamada, Conformal field theory on universal family of stable curves with gauge symmetries, in: *Advanced Studies in Pure Math.*, Vol. 19, Kinokuniya Company Ltd., Tokyo, 1989, 459–565.

[Va] C. Vafa, Conformal theories and punctured surfaces, *Phys. Lett.* **B199** (1987), 195–202.

[Ve] E. Verlinde, Fusion rules and modular transformations in 2D conformal field theory, *Nucl. Phys.* **B300** (1988), 360–376.

[Z] Y. Zhu, Vertex operators, elliptic functions and modular forms, Ph.D. Thesis, Yale University, 1990.

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903
E-mail address: yzhuang@math.rutgers.edu