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THE LIMITING ABSORPTION PRINCIPLE FOR ELASTIC WAVE PROPAGATION PROBLEMS IN PERTURBED STRATIFIED MEDIA $\mathbf{R}^3$

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ABSTRACT. We consider the self-adjoint operator governing the propagation of elastic waves in perturbed stratified media $\mathbf{R}^3$ with free boundary-interface conditions. In this paper we establish the limiting absorption principle for this self-adjoint operator in appropriate Hilbert space. The proof of the limiting absorption principle is based on the division theorem which is proved by means of eigenfunction expansions for the self-adjoint operator governing the propagation of elastic waves in unperturbed stratified media $\mathbf{R}^3$.

1. Introduction

In this paper we consider propagation problems of elastic waves in perturbed stratified media $\mathbf{R}^3$ with free boundary-interface conditions.

The object of this work is to establish a limiting absorption principle for the self-adjoint operator governing the propagation of elastic waves. The limiting absorption principle implies some significant spectral properties of the self-adjoint operator and gives a method of selecting steady-state solutions for the propagation problems of elastic waves.

The limiting absorption principle for acoustic wave propagation problems is studied by several authors. For example Ben-Artzi and Dermenjian and Guillot [2], Dermenjian and Guillot [3], [4], Weder [13] for stratified media, Phillips [9], Wilcox [14] for exterior domain.

Concerning elastic wave propagation problems, Dermenjian and Guillot [5] proved the limiting absorption principle in perturbed half space $\mathbf{R}^3_+$ by using so-called division theorem which is one of their main results. In this paper we shall prove the limiting absorption principle for elastic wave propagation problems in perturbed stratified media $\mathbf{R}^3$ using a corresponding division theorem. We prove the division theorem by using the representation of solutions by Lopatinski analysis and the eigenfunction expansion theorem established by [11]. Dermenjian and Guillot used the representation of solutions due to Dunford and Schwartz [6].

We start with the mathematical formulation of the elastic wave propagation problem in perturbed stratified media $\mathbf{R}^3$.

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Let Ω be an exterior domain in $\mathbf{R}^3 = \{x = (x', x_3) = (x_1, x_2, x_3); x_i \in \mathbf{R}\}$ whose boundary $\partial\Omega$ is compact. $\lambda(x)$ and $\mu(x)$ denote Lamé functions in Ω, and $\rho(x)$ denotes a density function in Ω. We assume that

$$(1.1) \qquad 0 < m \leq \lambda(x)(\text{resp. } \mu(x), \rho(x)) \leq M \qquad \text{for } a.e. x \in \bar{\Omega},$$

where

$$\lambda(x) = \begin{cases} \lambda_1(x), \\ \lambda_2(x), \end{cases} \quad \mu(x) = \begin{cases} \mu_1(x), \\ \mu_2(x), \end{cases} \quad \rho(x) = \begin{cases} \rho_1(x) & \text{for } x \in \Omega \cap \mathbf{R}^3_-, \\ \rho_2(x) & \text{for } x \in \Omega \cap \mathbf{R}^3_+, \end{cases}$$

and

$$(1.2) \qquad (\lambda(x), \mu(x), \rho(x)) = \begin{cases} (\lambda_1, \mu_1, \rho_1) & \text{for } x \in \mathbf{R}^3_-, \ |x| > L, \\ (\lambda_2, \mu_2, \rho_2) & \text{for } x \in \mathbf{R}^3_+, \ |x| > L. \end{cases}$$

Here $\mathbf{R}^3_- = \{x \in \mathbf{R}^3, \ x_3 < 0\}$, $\mathbf{R}^3_+ = \{x \in \mathbf{R}^3, \ x_3 > 0\}$, L is a fixed large real number, λ_1, λ_2, μ_1, μ_2 are certain quantities called the Lamé constants and ρ_1, ρ_2 are densities (cf. Figure 1).

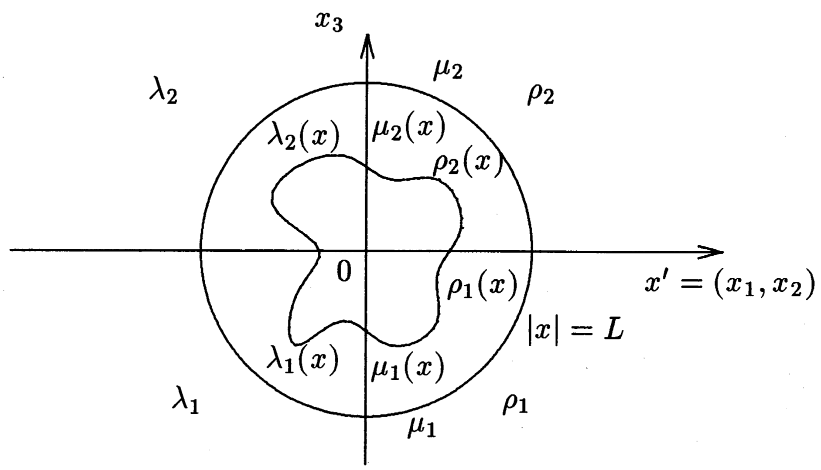

Figure 1 Perturbed Stratified Medium $\mathbf{R}^3$

Let $u(t, x) = {}^t(u_1(t, x), u_2(t, x), u_3(t, x)) \in \mathbf{R}^3$ be the displacement vector at time t and position x. The propagation problem of elastic waves in the perturbed stratified medium $\mathbf{R}^3$ is formulated as the following mixed problem:

$$(1.3) \qquad \frac{\partial^2 u_k}{\partial t^2}(t, x) - \frac{1}{\rho(x)} \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{kj} u(t, x) = 0, \quad x \in \Omega,$$

$$(1.4) \qquad u(t, x)|_{\Omega \cap \{x_3 = -0\}} = u(t, x)|_{\Omega \cap \{x_3 = +0\}},$$

$$(1.5) \qquad \sigma_{k3}(u(t, x))|_{\Omega \cap \{x_3 = -0\}} = \sigma_{k3}(u(t, x))|_{\Omega \cap \{x_3 = +0\}},$$

$$(1.6) \qquad \sum_{j=1}^{3} \sigma_{kj}(u(t, x)) \nu_j |_{\partial\Omega} = 0,$$

$$u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x), \tag{1.7}$$

where

$$\sigma_{kj}(u) = \lambda(\cdot)(\nabla \cdot u)\delta_{kj} + 2\mu(\cdot)\varepsilon_{kj}(u), \tag{1.8}$$

are symmetric stress tensors,

$$\varepsilon_{kj}(u) = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\right), \tag{1.9}$$

are symmetric strain tensors, and $\nu = (\nu_1, \nu_2, \nu_3)$ denotes the exterior normal at point $x \in \partial\Omega$. (1.4) and (1.5) are called free interface conditions, (1.6) is called an free boundary condition, and (1.7) is called an initial condition. Here 'free' means Neumann type, and these free interface and boundary conditions are appeared in practical situations.

Solutions to (1.3)-(1.7) with finite energy are associated with a Hilbert space and a self-adjoint operator as follows. Let

$$(\mathcal{A}u)_k = -\frac{1}{\rho(\cdot)}\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\sigma_{kj}(u). \tag{1.10}$$

Here $(\mathcal{A}u)_k$ has another expression

$$(\mathcal{A}u)_k = -\frac{1}{\rho(\cdot)}\sum_{j=1}^{3}\frac{\partial}{\partial x_j}(c_{kjlh}(\cdot)\varepsilon_{lh}(u)), \tag{1.11}$$

where c_{kjlh} $(k,j,l,h = 1,2,3)$ are called the stress-strain tensors given by

$$c_{kjlh}(\cdot) = \lambda(\cdot)\delta_{kj}\delta_{lh} + \mu(\cdot)\delta_{kh}\delta_{jl}, \tag{1.12}$$

with the properties

$$c_{kjlh}(\cdot) = c_{jklh}(\cdot) = c_{lhkj}(\cdot) \tag{1.13}$$

and δ_{kj} is the Kronecker delta. By the condition (1.1), Lamé functions satisfy the conditions

$$3\lambda(x) + 2\mu(x) > 0, \quad \mu(x) > 0, \quad \text{for } a.e.x \in \bar{\Omega}, \tag{1.14}$$

so we have from Korn's inequality the following stability condition:

$$\sum_{k,j,l,h} c_{kjlh}(\cdot)s_{lh}\overline{s_{kj}} \geq c\sum_{k,j}|s_{kj}|^2, \quad c > 0 \tag{1.15}$$

for all complex symmetric matrices (s_{kj}), $s_{kj} = s_{jk} \in \mathbf{C}$ (cf. [8], [10]).

The Sobolev spaces on Ω are defined by

$$H^m(\Omega, \mathbf{C}^3) = \{u \in \mathbf{C}^3;\ D^\alpha u \in L^2(\Omega, \mathbf{C}^3), \text{ for } |\alpha| \leq m\}, \tag{1.16}$$

where m is a non-negative integer and the multi-index notation is used for derivatives. $H^m(\Omega, \mathbf{C}^3)$ is a Hilbert space with inner product

$$(u,v)_m = \int_\Omega \sum_{|\alpha|\leq m} D^\alpha u(x) \cdot \overline{D^\alpha v(x)}dx, \tag{1.17}$$

where $u \cdot \bar{v}$ denotes the usual scalar product in $\mathbf{C}^3$: $u \cdot \bar{v} = \sum_{k=1}^{3} u_k\overline{v_k}$.

Definition 1.1. *$u \in H^1(\Omega, \mathbf{C}^3) \cap \{\mathcal{A}u \in L^2(\Omega, \mathbf{C}^3)\}$ is said to satisfy the generalized free boundary-interface condition if and only if one has*

$$(1.18) \qquad \int_\Omega (\mathcal{A}u)_k \overline{v_k} \rho(x)dx - \int_\Omega (\lambda(x)(\nabla \cdot u)(\nabla \cdot \bar{v}) + 2\mu(x)\varepsilon_{kj}(u)\varepsilon_{kj}(\bar{v}))dx = 0$$

for every $v \in H^1(\Omega, \mathbf{C}^3)$.

We introduce the Hilbert space

$$(1.19) \qquad \mathcal{H} = L^2(\Omega, \mathbf{C}^3, \rho(x)dx),$$

with inner product

$$(1.20) \qquad (u, v)_{\mathcal{H}} = \int_\Omega u \cdot v \rho(x)dx.$$

Theorem 1.2. *The following operator A in $\mathcal{H}$ with domain:*

(1.21)

$$D(A) = \{u \in H^1(\Omega, \mathbf{C}^3) \cap \{\mathcal{A}u \in L^2(\Omega, \mathbf{C}^3)\};\ u \text{ satisfies the generalized free boundary-interface condition (1.18)}\},$$

and action defined by

$$(1.22) \qquad Au = \mathcal{A}u, \quad u \in D(A)$$

is a non-negative self-adjoint operator.

For a proof of Theorem 1.2 see [11].

Every $u \in D(A)$ satisfies the free interface conditions (1.4) and (1.5), and the free boundary condition (1.6), so the mixed problem (1.3)-(1.7) may be reformulated as the problem of finding a function $u : \mathbf{R} \to \mathcal{H}$ such that

$$(1.23) \qquad \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + Au = 0, \quad \text{for } \forall t \in \mathbf{R},$$

$$(1.24) \qquad u(0) = f, \quad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = g.$$

The operator A is non-negative and the spectral theory for self-adjoint operators implies that (1.23) and (1.24) has a (generalized) solution given by

$$(1.25) \qquad u(t) = \left(\cos tA^{\frac{1}{2}}\right) f + \left(A^{-\frac{1}{2}} \sin tA^{\frac{1}{2}}\right) g, \quad t \in \mathbf{R}.$$

Let $E(u, K, t)$ be the restriction of the energy of u to a measurable subset K of Ω:

(1.26)

$$E(u, K, t) = \frac{1}{2}\left(\sum_{k=1}^{3} \int_K \left|\frac{\partial u_k}{\partial t}\right|^2 \rho(x)dx\right.$$

$$+ \sum_{k,j=1}^{3} \int_K (\lambda(x)|\nabla \cdot u|^2 + 2\mu(x)|\varepsilon_{kj}(u)|^2)dx \Bigg)$$

$$= \left\| \frac{\mathrm{d}u}{\mathrm{d}t} \right\|_{\mathcal{H}}^2 + \|A^{\frac{1}{2}}u\|^2$$

If $f \in D(A^{\frac{1}{2}})$, $g \in \mathcal{H}$, then $u \in D(A^{\frac{1}{2}})$, $\frac{\mathrm{d}u}{\mathrm{d}t} \in \mathcal{H}$ and $E(u, K, t) < \infty$. In this case $u(t)$ is called a solution with finite energy.

In order to state our main theorem, we introduce several function spaces.

Let s_1,s_2 be two real numbers. Let $L^{2;s_1,s_2}(\Omega, \mathbf{C}^3)$ be the space of all measurable $\mathbf{C}^3$ valued functions on Ω defined by

$$L^{2;s_1,s_2}(\Omega, \mathbf{C}^3) = \{u; (1 + x_1^2 + x_2^2)^{\frac{s_1}{2}}(1 + x_3^3)^{\frac{s_2}{2}} u(x) \in L^2(\Omega, \mathbf{C}^3)\}, \tag{1.27}$$

with the norm

$$\|u\|_{0;s_1,s_2}^2 = \int_\Omega (1 + x_1^2 + x_2^2)^{s_1}(1 + x_3^3)^{s_2} u(x) \cdot \overline{u(x)} dx. \tag{1.28}$$

We consider weighted Sobolev spaces $H^{m;s_1,s_2}(\Omega, \mathbf{C}^3)$ defined for any integer $m \geq 0$ by

$$H^{m;s_1,s_2}(\Omega, \mathbf{C}^3) = \{u;\ D^\alpha u \in L^{2;s_1,s_2}(\Omega, \mathbf{C}^3),\ |\alpha| \leq m\}, \tag{1.29}$$

with the norm

$$\|u\|_{m,s_1,s_2}^2 = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{0,s_1,s_2}^2. \tag{1.30}$$

We introduce

$$H^{1;s_1,s_2}(\Omega, \mathcal{A}, \mathbf{C}^3) = \{u \in H^{1;s_1,s_2}(\Omega, \mathbf{C}^3);\ \mathcal{A}u \in L^{2;s_1,s_2}(\Omega, \mathbf{C}^3)\}, \tag{1.31}$$

with the norm

$$\|u\|_{\mathcal{A},s_1,s_2}^2 = \|u\|_{1,s_1,s_2}^2 + \|\mathcal{A}u\|_{0,s_1,s_2}^2. \tag{1.32}$$

Finally let

(1.33)
$$H^{1;-s_1,-s_2}(\Omega, A, \mathbf{C}^3) = \{u \in H^{1;-s_1,-s_2}(\Omega, \mathcal{A}, \mathbf{C}^3);$$
$$u \text{ satisfy the generalized free boundary-interface condition (1.18)}\}.$$

Let $R(z)$ be the resolvent of A. Then the limiting absorption principle which is our main result can be stated as follows:

Main Theorem. *Suppose $s_1 > \frac{1}{2}$ and $s_2 > \frac{1}{2}$. And suppose $\rho_1 = \rho_2$. If $\omega(> 0)$ is not an eigenvalue for A, then the following two limits exist in the uniform operator topology of $B(L^{2;s_1,s_2}(\Omega, \mathbf{C}^3), H^{1;-s_1,-s_2}(\Omega, A, \mathbf{C}^3))$:*

$$R^{\pm}(\omega) = \lim_{\substack{z \to \omega \\ \pm \operatorname{Im} z > 0}} R(z). \tag{1.34}$$

The remainder of this paper is organized as follows. In Section 2, we consider the plane stratified media $\mathbf{R}^3$ with the planer interface $x_3 = 0$, which is defined by

$$(\lambda(x_3), \mu(x_3), \rho(x_3)) = \begin{cases} (\lambda_1, \mu_1, \rho_1) & \text{for} \quad x_3 < 0, \\ (\lambda_2, \mu_2, \rho_2) & \text{for} \quad x_3 > 0. \end{cases}$$

The self-adjoint operator A_0 governing the propagation of elastic waves in this unperturbed media is defined. A is considered as a perturbation of A_0. We recall eigenfunction expansions for A_0 and state the limiting absorption principle for A_0. Section 3 is devoted to the proof of the division theorem for A_0. Finally in Section 4, we prove the limiting absorption principle for A, and give some properties of the spectrum of A.

2. Eigenfunction Expansions and the Limiting Absorption Principle for A_0

In this section, we consider the plane stratified medium $\mathbf{R}^3$ with the planar interface $x_3 = 0$, which is defined by

$$(\lambda(x_3), \mu(x_3), \rho(x_3)) = \begin{cases} (\lambda_1, \mu_1, \rho_1) & \text{for} \quad x_3 < 0, \\ (\lambda_2, \mu_2, \rho_2) & \text{for} \quad x_3 > 0. \end{cases} \tag{2.1}$$

Here λ_1, λ_2, μ_1, μ_2 are certain quantities called the Lamé constants and ρ_1, $\rho_2 > 0$ are the densities.

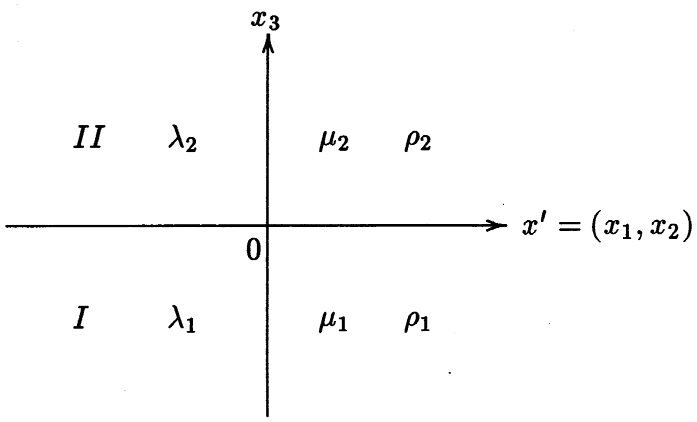

Figure 2 Unperturbed Stratified Medium $\mathbf{R}^3$

The propagation problem of elastic waves in this unperturbed stratified medium is formulated as the following mixed initial and interface value problem:

$$\frac{\partial^2 u}{\partial t^2}(t, x) + A_0 u(t, x) = 0, \tag{2.2}$$

$$u(t,x)|_{x_3=-0} = u(t,x)|_{x_3=+0}, \tag{2.3}$$

$$\sigma_{k3}u(t,x)|_{x_3=-0} = \sigma_{k3}u(t,x)|_{x_3=+0}, \tag{2.4}$$

$$u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x), \tag{2.5}$$

where

$$\mathcal{A}_0 u = -\frac{\lambda(x_3)+\mu(x_3)}{\rho(x_3)}\nabla(\nabla\cdot u) - \frac{\mu(x_3)}{\rho(x_3)}\Delta u. \tag{2.6}$$

We introduce the Hilbert space

$$\mathcal{H}_0 = L^2(\mathbf{R}^3, \mathbf{C}^3, \rho(x_3)\,dx) \tag{2.7}$$

with inner product

$$(u,v)_{\mathcal{H}_0} = \int_{\mathbf{R}^3} u\cdot v\rho(x_3)\,dx.$$

Proposition 2.1. *The following the operator A_0 on $\mathcal{H}_0$ with domain*

$$D(A_0) = \{u \in H^2(\mathbf{R}^3_-, \mathbf{C}^3) \oplus H^2(\mathbf{R}^3_+, \mathbf{C}^3);$$
$$u \text{ satisfies the interface conditions (1.2) and (1.3)}$$
$$\text{in the sense of trace on } x_3 = 0\}$$

and action defined by

$$A_0 u = \mathcal{A}_0 u, \quad u \in D(A_0) \tag{2.8}$$

is a non-negative self-adjoint operator on $\mathcal{H}_0$.

Eigenfunction expansions for A_0 was developed in [11]. We give a brief review of the structure and properties of eigenfunctions and the expansion theorem.

Let $\eta' = (\eta_1, \eta_2) \in \mathbf{R}^2$ be the dual variables of $x' = (x_1, x_2)$ and let $F_{x'}$ denote the partial Fourier transformation with respect to x'. Let

$$\mathrm{U} = \frac{1}{|\eta'|}\begin{pmatrix} \eta_1 & -\eta_2 & 0 \\ \eta_2 & \eta_1 & 0 \\ 0 & 0 & |\eta'| \end{pmatrix}, \quad \mathrm{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.9}$$

where U and C are unitary matrices and $|\eta'| = (\eta_1^2 + \eta_2^2)^{\frac{1}{2}}$ (cf. [5], [7]).

Proposition 2.2. *We have*

$$A_0 u = F_{\eta'}^{-1}\mathrm{UC}(A_0^1(\eta') \oplus A_0^2(\eta'))(\mathrm{UC})^{-1}F_{x'}u \quad \text{for } u \in D(A_0), \tag{2.10}$$

where $A_0^1(\eta')$ and $A_0^2(\eta')$ are non-negative self-adjoint operators in $L^2(\mathbf{R}, \mathbf{C}^2, \rho(x_3)\,dx_3)$ and $L^2(\mathbf{R}, \mathbf{C}, \rho(x_3)\,dx_3)$ defined respectively as follows:

$$D\left(A_0^1(\eta')\right) = \left\{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^2(\mathbf{R}_-, \mathbf{C}^2) \oplus H^2(\mathbf{R}_+, \mathbf{C}^2);\right.$$

$$\left. u|_{x_3=-0} = u|_{x_3=+0}, \; B_0^1\left(\eta', \frac{d}{dx_3}\right)u\bigg|_{x_3=-0} = B_0^1\left(\eta', \frac{d}{dx_3}\right)u\bigg|_{x_3=+0}\right\},$$

$$A_0^1(\eta', \frac{d}{dx_3}) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} -\mu\frac{d^2}{dx_3^2} + (\lambda+2\mu)|\eta'|^2 & -i|\eta'|(\lambda+\mu)\frac{d}{dx_3} \\ -i|\eta'|(\lambda+\mu)\frac{d}{dx_3} & -(\lambda+2\mu)\frac{d^2}{dx_3^2} + \mu|\eta'|^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$$B_0^1\left(\eta', \frac{d}{dx_3}\right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \mu\frac{d}{dx_3} & i|\eta'|\mu \\ i|\eta'|\lambda & (\lambda+2\mu)\frac{d}{dx_3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$$D\left(A_0^2(\eta')\right) = \{u \in H^2(\mathbf{R}_-) \oplus H^2(\mathbf{R}_+);$$
$$u|_{x_3=-0} = u|_{x_3=+0},\ B_0^2\left(\eta', \frac{d}{dx_3}\right)u\Big|_{x_3=-0} = B_0^2\left(\eta', \frac{d}{dx_3}\right)u\Big|_{x_3=+0}\Big\},$$

$$A_0^2\left(\eta', \frac{d}{dx_3}\right)u = -\frac{\mu}{\rho}\frac{d^2u}{dx_3^2} + \frac{\mu}{\rho}|\eta'|^2 u,$$
$$B_0^2\left(\eta', \frac{d}{dx_3}\right)u = \mu\frac{d}{dx_3}u,$$

where $\lambda = \lambda(x_3)$, $\mu = \mu(x_3)$ *and* $\rho = \rho(x_3)$.

The Lopatinski determinant $\Delta(\eta', \zeta)$ for $A_0^1(\eta')$ is given as follows:

$$\Delta(\eta', \zeta) = |\eta'|^6 \mathrm{Dis}(z),$$

where Dis(z) is given in [11 (3.2)] as D(z). The squares of propagation sppeds of shear (SV, SH) and pressure (P) waves are given by

$$\text{(2.11)} \qquad c_{s_i}^2 = \frac{\mu_i}{\rho_i}, \quad c_{p_i}^2 = \frac{\lambda_i + 2\mu_i}{\rho_i}, \quad (i = 1, 2),$$

respectively. Dis(z) has the only one real zero c_{St} when either Dis$(c_{s_1}^2) > 0$ or Dis$(c_{s_1}^2) = 0$ under some restriction if $c_{s_1} < c_{s_2}$ (see [11 Theorem 6.5]). If $c_{s_1} < c_{s_2}$, then we must replace Dis$(c_{s_1}^2)$ by Dis$(c_{s_2}^2)$. Then the zero of $\Delta(\eta', \zeta)$ is $c_{St}^2|\eta'|^2$ and the origin of the Stoneley wave with speed c_{St} propagating along the interface $x_3 = 0$ in the elastic space $\mathbf{R}^3$.

Let $\eta = (\eta_1, \eta_2, \xi) = (\eta', \xi)$. $c_j^2|\eta|^2$ ($j \in M = \{s_1, p_1, s_2, p_2\}$) and $c_k^2|\eta|^2$ ($k \in N = \{s_1, s_2\}$) are the eigenvalues of $A_0^1(\eta')$ and $A_0^2(\eta')$, respectively. We obtain explicit expression of generalized eigenfunctions $\psi_{1j}^{\pm}(x_3, \eta)$, $\psi_{1j}^{St}(x_3, \eta)$ ($j \in M$) for $A_0^1(\eta')$ and $\psi_{2k}^{\pm}(x_3, \eta)$ ($k \in N$) for $A_0^2(\eta')$ (see [11 (4.9)-(4.20), (4.21)-(4.22), (5.8)-(5.13), respectively]).

Using these generalized eigenfunctions for $A_0^1(\eta')$ and $A_0^2(\eta')$, we define generalized eigenfunctions for A_0 as follows:

$$\text{(2.12)} \qquad \psi_{1j}^{\pm}(x, \eta) = \frac{1}{2\pi} e^{i(x_1\eta_1 + x_2\eta_2)} \mathrm{UC}(\psi_{1j}^{\pm}(x_3, \eta) \oplus O_{1\times 1}), \quad j \in M,$$

$$\text{(2.13)} \qquad \psi_{1j}^{St}(x, \eta) = \frac{1}{2\pi} e^{i(x_1\eta_1 + x_2\eta_2)} \mathrm{UC}(\psi_{1j}^{St}(x_3, \eta) \oplus O_{1\times 1}), \quad j \in M,$$

$$\psi_{2k}^{\pm}(x,\eta) = \frac{1}{2\pi} e^{i(x_1\eta_1 + x_2\eta_2)} \mathrm{UC}(O_{2\times 2} \oplus \psi_{2k}^{\pm}(x_3,\eta)), \quad k \in N, \tag{2.14}$$

where $O_{n\times n}$ denotes the $n \times n$ zero matrix.

Now we define the Fourier transform of $f \in \mathcal{H}$ with respect to these generalized eigenfunctions: $f \mapsto (\hat{f}_{1j}^{\pm},\ \hat{f}_{1j}^{St},\ \hat{f}_{2k}^{\pm})$,

$$\hat{f}_{1j}^{\pm}(\eta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|x|\le R} \psi_{1j}^{\pm}(x,\eta)^* f(x)\rho(x_3)\,dx, \quad j \in M, \tag{2.15}$$

$$\hat{f}_{1j}^{St}(\eta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|x|\le R} \psi_{1j}^{St}(x,\eta)^* f(x)\rho(x_3)\,dx, \quad j \in M, \tag{2.16}$$

$$\hat{f}_{2k}^{\pm}(\eta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|x|\le R} \psi_{2k}^{\pm}(x,\eta)^* f(x)\rho(x_3)\,dx, \quad k \in N. \tag{2.17}$$

We then have the eigenfunction expansion theorem.

Theorem 2.3. *We assume that the real zero of* $\Delta(\eta';\zeta)$ *exists.*

(1) For $f,\ g \in \mathcal{H}_0$,

$$(f,g) = \sum_{j\in M}\left(\int_{\mathbf{R}^3} \hat{f}_{1j}^{\pm}(\eta)\cdot\overline{\hat{g}_{1j}^{\pm}(\eta)}\,d\eta + \int_{\mathbf{R}^3} \hat{f}_{1j}^{St}(\eta)\cdot\overline{\hat{g}_{1j}^{St}(\eta)}\,d\eta\right) + \sum_{k\in N}\int_{\mathbf{R}^3} \hat{f}_{2k}^{\pm}(\eta)\cdot\overline{\hat{g}_{2k}^{\pm}(\eta)}\,d\eta. \tag{2.18}$$

(2) For $f \in \mathcal{H}_0$,

$$f(x) = \sum_{j\in M} \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|\eta|\le R} \left(\psi_{1j}^{\pm}(x,\eta)\hat{f}_{1j}^{\pm}(\eta) + \psi_{1j}^{St}(x,\eta)\hat{f}_{1j}^{St}(\eta)\right) d\eta + \sum_{k\in N} \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|\eta|\le R} \psi_{2k}^{\pm}(x,\eta)\hat{f}_{2k}^{\pm}(\eta)\,d\eta. \tag{2.19}$$

(3) For $f \in D(A_0)$,

$$A_0 f(x) = \sum_{j\in M} \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|\eta|\le R} \left(c_j^2|\eta|^2\psi_{1j}^{\pm}(x,\eta)\hat{f}_{1j}^{\pm}(\eta) + c_{St}^2|\eta'|^2\psi_{1j}^{St}(x,\eta)\hat{f}_{1j}^{St}(\eta)\right) d\eta + \sum_{k\in N} \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|\eta|\le R} c_k^2|\eta|^2\psi_{2k}^{\pm}(x,\eta)\hat{f}_{2k}^{\pm}(\eta)\,d\eta. \tag{2.20}$$

(4) We define the mappings by

$\Phi_{1j}^{\pm} : \mathcal{H}_0 \ni f \to \hat{f}_{1j}^{\pm}(\eta) \in L^2(\mathbf{R}_+^3, \mathbf{C}^3)(\xi > 0), \in L^2(\mathbf{R}_-^3, \mathbf{C}^3)(\xi < 0),\ j \in M,$

$\Phi_{1j}^{St} : \mathcal{H}_0 \ni f \to \hat{f}_{1j}^{St}(\eta) \in L^2(\mathbf{R}^3, \mathbf{C}^3),\ j \in M,$

$\Phi_{2k}^{\pm} : \mathcal{H}_0 \ni f \to \hat{f}_{2k}^{\pm}(\eta) \in L^2(\mathbf{R}_+^3, \mathbf{C}^3)(\xi > 0), \in L^2(\mathbf{R}_-^3, \mathbf{C}^3)(\xi < 0),\ k \in N,$

and put

$$\Phi^{\pm} f = \left(\sum_{j\in M} \Phi_{1j}^{\pm} f,\ \sum_{j\in M} \Phi_{1j}^{St} f,\ \sum_{k\in N} \Phi_{2k}^{\pm} f \right).$$

Then we have

$$R(\Phi^{\pm}) = L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3) \oplus L^2(\mathbf{R}^3, \mathbf{C}^3) \oplus L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3). \tag{2.21}$$

This theorem implies that $\Phi^{\pm}$ are unitary operators in $\mathcal{H}_0$, and that the systems of generalized eigenfunctions $\{\psi_{1j}^{+},\ \psi_{1j}^{St},\ \psi_{2k}^{+}\}_{j\in M, k\in N}$ and $\{\psi_{1j}^{-},\ \psi_{1j}^{St},\ \psi_{2k}^{-}\}_{j\in M, k\in N}$ are complete, respectively.

Let $R_0(z)$ be the resolvent of A_0. By using Theorem 2.3 and the operational calculus, we have for f and g in $C_0^\infty(\mathbf{R}^3, \mathbf{C}^3)$ and $z \in \mathbf{C} \setminus [0, \infty)$,

(2.22)

$$\begin{aligned}(R_0(z)f, g)_{\mathcal{H}_0} &= \sum_{j\in M} \left(\int_{\mathbf{R}^3_{\pm}} \frac{1}{c_j^2 |\eta|^2 - z} \hat{f}_{1j}^{\pm}(\eta) \cdot \overline{\hat{g}_{1j}^{\pm}(\eta)}\, d\eta + \int_{\mathbf{R}^3} \frac{1}{c_{St}^2 |\eta'|^2 - z} \hat{f}_{1j}^{St}(\eta) \cdot \overline{\hat{g}_{1j}^{St}(\eta)}\, d\eta \right) \\ &\quad + \sum_{k\in N} \int_{\mathbf{R}^3_{\pm}} \frac{1}{c_k^2 |\eta|^2 - z} \hat{f}_{2k}^{\pm}(\eta) \cdot \overline{\hat{g}_{2k}^{\pm}(\eta)}\, d\eta.\end{aligned}$$

By changing to polar coordinates and using continuity properties of Cauchy type integrals, we get

(2.23)

$$\begin{aligned}\lim_{\substack{z\to\omega \\ \pm \mathrm{Im}\, z>0}} (R_0(z)f, g)_{\mathcal{H}_0} &= \sum_{j\in M} \left(\pm i \frac{\pi}{2\sqrt{\omega}} \int_{|\eta| = \frac{\sqrt{\omega}}{c_j}} \hat{f}_{1j}^{\pm}(\eta) \cdot \overline{\hat{g}_{1j}^{\pm}(\eta)}\, dS_j + \mathrm{p.v.} \int_{\mathbf{R}^3_{\pm}} \frac{\hat{f}_{1j}^{\pm}(\eta) \cdot \overline{\hat{g}_{1j}^{\pm}(\eta)}}{c_j^2 |\eta|^2 - \omega}\, d\eta \right) \\ &\quad + \sum_{j\in M} \left(\pm i \frac{\pi}{2\sqrt{\omega}} \int_{\mathbf{R}} \int_{|\eta'| = \frac{\sqrt{\omega}}{c_{St}}} \hat{f}_{1j}^{St}(\eta) \cdot \overline{\hat{g}_{1j}^{St}(\eta)}\, dS' d\xi + \mathrm{p.v.} \int_{\mathbf{R}^3} \frac{\hat{f}_{1j}^{St}(\eta) \cdot \overline{\hat{g}_{1j}^{St}(\eta)}}{c_{St}^2 |\eta'|^2 - \omega}\, d\eta \right) \\ &\quad + \sum_{k\in N} \left(\pm i \frac{\pi}{2\sqrt{\omega}} \int_{|\eta| = \frac{\sqrt{\omega}}{c_k}} \hat{f}_{2k}^{\pm}(\eta) \cdot \overline{\hat{g}_{2k}^{\pm}(\eta)}\, dS_k + \mathrm{p.v.} \int_{\mathbf{R}^3_{\pm}} \frac{\hat{f}_{2k}^{\pm}(\eta) \cdot \overline{\hat{g}_{2k}^{\pm}(\eta)}}{c_k^2 |\eta|^2 - \omega}\, d\eta \right),\end{aligned}$$

where dS_j, dS', dS_k denote the surface element of the spheres $|\eta| = \frac{\sqrt{\omega}}{c_j}$, $|\eta'| = \frac{\sqrt{\omega}}{c_{St}}$, $|\eta| = \frac{\sqrt{\omega}}{c_k}$, respectively. Now we define generalized trace operators associated with A_0. For any $\omega > 0$, put

$$E_{1j}^{\pm}(\omega) = \left\{ \eta \in \mathbf{R}^3_{\pm},\ |\eta| = \frac{\sqrt{\omega}}{c_j} \right\}, \tag{2.24}$$

$$E_{1j}^{St}(\omega) = \left\{ \eta \in \mathbf{R}^3,\ |\eta'| = \frac{\sqrt{\omega}}{c_{St}},\ \xi \in \mathbf{R} \right\}, \tag{2.25}$$

$$E_{2k}^{\pm}(\omega) = \left\{ \eta \in \mathbf{R}^3_{\pm},\ |\eta| = \frac{\sqrt{\omega}}{c_k} \right\}, \tag{2.26}$$

then we have

Proposition 2.4. *Suppose $s_1 > \frac{1}{2}$ and $s_2 > \frac{1}{2}$. For any $\omega > 0$ there exist generalized trace operators*

$$\tau_{1j}^{\pm}(\omega) : L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3) \to L^2(E_{1j}^{\pm}(\omega)), \tag{2.27}$$

$$\tau_{1j}^{St}(\omega) : L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3) \to L^2(E_{1j}^{St}(\omega)), \tag{2.28}$$

$$\tau_{2k}^{\pm}(\omega) : L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3) \to L^2(E_{2k}^{\pm}(\omega)), \tag{2.29}$$

such that for any $f \in C_0^\infty(\mathbf{R}^3, \mathbf{C}^3)$:

$$\tau_{1j}^{\pm}(\omega)f(\eta) = \hat{f}_{1j}^{\pm}(\eta), \quad |\eta| = \frac{\sqrt{\omega}}{c_j}, \tag{2.30}$$

$$\tau_{1j}^{St}(\omega)f(\eta) = \hat{f}_{1j}^{St}(\eta), \quad |\eta'| = \frac{\sqrt{\omega}}{c_{St}}, \quad \xi \in \mathbf{R}, \tag{2.31}$$

$$\tau_{2k}^{\pm}(\omega)f(\eta) = \hat{f}_{2k}^{\pm}(\eta), \quad |\eta| = \frac{\sqrt{\omega}}{c_k}. \tag{2.32}$$

Furthermore for any $f \in L^2(\mathbf{R}^3, \mathbf{C}^3)$

$$\|\tau_{1j}^{\pm}(\omega)f\|_{L^2(E_{1j}^{\pm}(\omega))} \le M(\omega)\|f\|_{0;s_1,s_2}, \tag{2.33}$$

$$\|\tau_{1j}^{St}(\omega)f\|_{L^2(E_{1j}^{St}(\omega))} \le M(\omega)\|f\|_{0;s_1,s_2}, \tag{2.34}$$

$$\|\tau_{2k}^{\pm}(\omega)f\|_{L^2(E_{2k}^{\pm}(\omega))} \le M(\omega)\|f\|_{0;s_1,s_2}, \tag{2.35}$$

where $M(\omega)$ is a continuous function on $(0, \infty)$.

Then we have the limiting absorption principle for A_0.

Theorem 2.5. *Suppose $s_1 > \frac{1}{2}$ and $s_2 > \frac{1}{2}$. Then for any $\omega > 0$, the following two limits exist in the uniform operator topology of $B(L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3), H^{2;-s_1,-s_2}(\mathbf{R}^3, \mathbf{C}^3))$:*

$$R_0^{\pm}(\omega) = \lim_{\substack{z \to \omega \\ \pm \mathrm{Im}\, z > 0}} R_0(z). \tag{2.36}$$

Finally we conclude this section with the following proposition.

Proposition 2.6. *Suppose $s_1 > \frac{1}{2}$ and $s_2 > \frac{1}{2}$. Let $\omega > 0$ and $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$. Then the following statements are equivalent:*

$$R_0^+(\omega)f = R_0^-(\omega)f, \tag{2.37}$$

$$\mathrm{Im} \int_{\mathbf{R}_{\pm}^3} R_0^+(\omega)f \cdot \bar{f}\rho(x_3)\, dx = 0, \tag{2.38}$$

$$\mathrm{Im} \int_{\mathbf{R}_{\pm}^3} R_0^-(\omega)f \cdot \bar{f}\rho(x_3)\, dx = 0, \tag{2.39}$$

$$\sum_{j \in M} \tau_{1j}^{\pm}(\omega)f = \sum_{j \in M} \tau_{1j}^{St}(\omega)f = \sum_{k \in N} \tau_{2k}^{\pm}(\omega)f = 0, \tag{2.40}$$

$$\sum_{j \in M} \tau_{1j}^{\pm}(\omega)\bar{f} = \sum_{j \in M} \tau_{1j}^{St}(\omega)\bar{f} = \sum_{k \in N} \tau_{2k}^{\pm}(\omega)\bar{f} = 0. \tag{2.41}$$

3. The Division Theorem for A_0

This section is devoted to the division theorem for A_0. This theorem states that if the generalized traces of $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$ vanish on $E^{\pm}_{1j}(\omega)$, $E^{St}_{1j}(\omega)$, and $E^{\pm}_{2k}(\omega)$ then the function $u = R^{\pm}_0(\omega)f$ has a better decay at infinity than is expected from Theorem 2.4. The division theorem plays a role corresponding to radiation condition or uniqueness theorem such as Rellich theorem.

The proof of the division theorem is done along the line of proof by Dermenjian and Guillot [5]. They proved the division theorem for their problem using representations of solutions due to Dunford and Schwartz [6 Theorem XIII. 3.16]. But we prove our division theorem using the integral representation of solutions by means of Lopatinski analysis.

Let us recall (2.10). For any $z \in \mathbf{C} \setminus [0, \infty)$ let

$$R^1_0(z) = (A^1_0(\eta') - z)^{-1}, \qquad R^2_0(z) = (A^2_0(\eta') - z)^{-1}. \tag{3.1}$$

Suppose $s_1 > \frac{1}{2}$, $s_2 > \frac{1}{2}$, and $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$. Let

$$g(\eta', x_3) = {}^t(g_1(\eta', x_3), g_2(\eta', x_3)) = (\mathrm{UC})^{-1} F_{x'} f(\eta', x_3), \tag{3.2}$$

where $g_1(\eta', x_3)$ and $g_2(\eta', x_3)$ are 2×1 and 1×1 vectors, respectively. Thus we have

$$g(\eta', x_3) \in L^{2;0,s_2}(\mathbf{R}^3, \mathbf{C}^3) \tag{3.3}$$

and

$$\left((UC)^{-1} F_{x'} R^{\pm}_0(\omega) f\right)(\eta', x_3) = R^{1\pm}_0(\omega) g_1(\eta', x_3) \oplus R^{2\pm}_0(\omega) g_2(\eta', x_3). \tag{3.4}$$

Then we have the following theorem.

Theorem 3.1. *Suppose $s_1 > \frac{1}{2}$, $s_2 > \frac{1}{2}$, and $\rho_1 = \rho_2$. Let $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$ and ω be a strictly positive number such that*

$$\sum_{j \in M} \tau^{\pm}_{1j}(\omega) f = \sum_{j \in M} \tau^{St}_{1j}(\omega) f = \sum_{k \in N} \tau^{\pm}_{2k}(\omega) f = 0. \tag{3.5}$$

Then we have

$$R^{+}_0(\omega) f = R^{-}_0(\omega) f \in L^{2;s_1-1,\tilde{s_2}}(\mathbf{R}^3_{\pm}, \mathbf{C}^3), \tag{3.6}$$

and

$$\|R^{\pm}_0(\omega) f\|_{0;s_1-1,\tilde{s_2}} \le M(\omega) \|f\|_{0,s_1,s_2}, \tag{3.7}$$

where $M(\cdot)$ is a positive continuous function on $(0, \infty)$ depending only on s_1, s_2, and $\tilde{s_2}$. Here $\tilde{s_2}$ is a real number such that

$$\tilde{s_2} < s_2 - 1. \tag{3.8}$$

This theorem is called division theorem for A_0. The proof of this theorem will be a consequence of (3.4) and Propositions 3.2-3.4 below.

Remark 1. Continuity of $M(\cdot)$ is useful in proving that the positive eigenvalues of A can accumulate only at 0 and $+\infty$.

Remark 2. If $\rho_1 \neq \rho_2$, we have (3.6) with $\tilde{s_2} < -\frac{1}{2}$ by (3.17) below. So we cannot use this result to prove Main Theorem, because $\tilde{s_2}$ is negative.

3.1 The Division Theorem for $A_0^2(\eta')$

Let

$$v_2(\eta', x_3, z) = R_0^2(z) g_2(\eta', x_3). \tag{3.9}$$

$v_2(\eta', x_3, z)$ has a meaning for $z \in \mathbf{C} \setminus [0, \infty)$. $v_2(\eta', x_3, \omega)$ will be defined as the limit of $v_2(\eta', x_3, z)$ as z tends to ω such that $\mathrm{Im} z > 0$; that is,

$$v_2(\eta', x_3, \omega) = R_0^{2+}(\omega) g_2(\eta', x_3). \tag{3.10}$$

Then we have

Proposition 3.2. *Suppose $s_1 > \frac{1}{2}$, $s_2 > \frac{1}{2}$, and $\rho_1 = \rho_2$. Let $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$ and ω be a strictly positive number such that*

$$\sum_{k \in N} \tau_{2k}^+(\omega) \bar{f} = 0 \quad \text{or} \quad \sum_{k \in N} \tau_{2k}^-(\omega) \bar{f} = 0. \tag{3.11}$$

Then

$$v_2(\cdot, \cdot, \omega) = R_0^{2+}(\omega) g_2 = R_0^{2-}(\omega) g_2 \in L^2(\mathbf{R}^3, (1 + x_3^2)^{\delta - \frac{1}{2}} d\eta' dx_3) \tag{3.12}$$

and

$$\|v_2(\cdot, \cdot, \omega)\|_{L^2(\mathbf{R}^3, (1+x_3^2)^{\delta - \frac{1}{2}} d\eta' dx_3)} \leq M(\omega) \|f\|_{0, s_1, s_2}, \tag{3.13}$$

where $M(\cdot)$ is a positive continuous function on $(0, \infty)$ depending only on δ, and δ is a real number such that $\delta < s_2 - \frac{1}{2}$.

Remark. The assertion of the first half of (3.12) follows immediately from Proposition 2.6.

Proof. The explicit integral representation of solution $v_2(\eta', x_3, z)$ is given in [11 (5.4) and (5.5)]. So we have the explicit expression of $v_2(\eta', x_3, \omega)$ by exchanging z, τ_{s_1}, τ_{s_2} to ω,

$$\begin{aligned} \xi_{s_1} &= \lim_{\substack{z \to \omega \\ \mathrm{Im} z > 0}} \tau_{s_1} = \lim_{\substack{z \to \omega \\ \mathrm{Im} z > 0}} \sqrt{\frac{z}{c_{s_1}^2} - |\eta'|^2}, \\ \xi_{s_2} &= \lim_{\substack{z \to \omega \\ \mathrm{Im} z > 0}} \tau_{s_2} = \lim_{\substack{z \to \omega \\ \mathrm{Im} z > 0}} \sqrt{\frac{z}{c_{s_2}^2} - |\eta'|^2}, \end{aligned} \tag{3.14}$$

respectively.

Consider the case where the condition $\sum_{k \in N} \tau_{2k}^+(\omega) \bar{f} = 0$ is satisfied. We also prove (3.12) and (3.13) for $v_2^I(\eta', x_3, \omega)$. The other cases can be handled similarly.

By (2.32), (2.17) and (2.14), the condition $\sum_{k \in N} \tau_{2k}^+(\omega) \bar{f} = 0$ can be rewritten as follows:

$$\sum_{k \in N} \int_{-\infty}^{\infty} \psi_{2k}^+(y_3, \eta) g_2(\eta', y_3) \rho(y_3) \, dy_3 = 0 \quad \text{for } |\eta| = \frac{\sqrt{\omega}}{c_k}. \tag{3.15}$$

In more detail, we have by (5.8)-(5.10) in [11],

$$\left\{\int_{-\infty}^{0} e^{i\xi_{s_1}y_3} - \frac{e^{-i\xi_{s_1}y_3}}{\Delta'(\eta',\omega)}(\rho_1 c_{s_1}^2\xi_{s_1} - \rho_2 c_{s_2}^2\xi_{s_2}) + \int_0^{\infty}\frac{e^{i\xi_{s_2}y_3}}{\Delta'(\eta',\omega)}(2\rho_1 c_{s_1}^2\xi_{s_1})\right\} g_2(\eta',y_3)\,dy_3 = 0. \tag{3.16}$$

By substituting (3.16) multiplied by $e^{-i\xi_{s_1}x_3}$ into $v_2^I(\eta',x_3,\omega)$, we obtain

$$\begin{aligned} v_2^I(\eta',x_3,\omega) &= \frac{i}{2}\frac{1}{c_{s_1}^2\xi_{s_1}}\int_{-\infty}^{x_3}\left(e^{i\xi_{s_1}x_3}e^{-i\xi_{s_1}y_3} - e^{-i\xi_{s_1}x_3}e^{i\xi_{s_1}y_3}\right) g_2(\eta',y_3)dy_3 \\ &\quad + i\frac{\rho_2-\rho_1}{\Delta'(\eta',\omega)}e^{-i\xi_{s_1}x_3}\int_0^{\infty} e^{i\xi_{s_2}y_3}g_2(\eta',y_3)dy_3, \quad x_3<0. \end{aligned} \tag{3.17}$$

If $\rho_1 = \rho_2$, then the second term of the right-hand side of (3.17) is equal to 0, because $\Delta'(\eta',\omega)$ has no zero. Thus we may only estimate the first term of the right-hand side of (3.17).

Let $\chi_1(|\eta'|)$, $\chi_2(|\eta'|)$, $\chi_3(|\eta'|)$ be the characteristic function of $(0,\frac{\sqrt{\omega}}{c_{s_1}})$, $(\frac{\sqrt{\omega}}{c_{s_1}},\frac{\sqrt{2\omega}}{c_{s_1}})$, $(\frac{\sqrt{2\omega}}{c_{s_1}},\infty)$, respectively. Consider the case where $\chi_1(|\eta'|)v^I(\eta',x_3,\omega)$. The other cases can be handled similarly.

In the case where $\chi_1(|\eta'|)v^I(\eta',x_3,\omega)$, we have $0\le|\eta'|\le\frac{\sqrt{\omega}}{c_{s_1}}$, so $\xi_{s_1}\ge 0$. From the inequalities

$$|e^{i\xi_{s_1}x_3} - e^{i\xi_{s_1}y_3}| \le 2|\xi_{s_1}|^{\gamma}|x_3-y_3|^{\gamma} \quad \text{for} \quad 0\le\gamma\le 1 \tag{3.18}$$

and

$$\int_{-\infty}^{x_3}(1+y_3^2)^{\gamma-s_2} \le C(1+x_3^2)^{-\alpha} \quad \text{for} \quad 0<\alpha<s_2-\frac{1}{2}-\gamma, \tag{3.19}$$

it follows that

$$|\chi_1(|\eta'|)v^I(\eta',x_3,\omega)|^2 \le C\xi_{s_1}^{2\gamma-2}(1+x_3^2)^{-\alpha}\|g_2\|_{0;s_2}^2 \tag{3.20}$$

for α such that $0<\alpha<s_2-\frac{1}{2}-\gamma$. Thus

$$\begin{aligned} &\|\chi_1(|\eta'|)v^I(\eta',x_3,\omega)\|^2_{L^2(\mathbf{R}^3_-,(1+x_3^2)^{\delta-\frac{1}{2}}d\eta' dx_3)} \\ &\le C\|g_2\|_{0;s_2}^2\left(\int_{\mathbf{R}_-}(1+x_3^2)^{\delta-\frac{1}{2}-\alpha}dx_3\right)\left(\int_{0\le|\eta'|\le\frac{\sqrt{\omega}}{c_{s_1}}}\frac{1}{\left(\sqrt{\frac{\omega}{c_{s_1}^2}-|\eta'|^2}\right)^{2-2\gamma}}d\eta'\right). \end{aligned} \tag{3.21}$$

Consequently if $\gamma>0$ and $\delta<\alpha<s_2-\frac{1}{2}-\gamma$ then we obtain (3.12) and (3.13) for $0\le|\eta'|\le\frac{\sqrt{\omega}}{c_{s_1}}$.

This completes the proof of proposition 3.2. □

3.2 The Division Theorem for $A_0^1(\eta')$

Let

$$v_1(\eta', x_3, z) = R_0^1(z)g_1(\eta', \cdot)(x_3), \tag{3.22}$$

where $z \in \mathbf{C} \setminus [0, \infty)$. $v_1(\eta', x_3, \omega)$ will be defined as the limit of $v_1(\eta', x_3, z)$ as z tends to ω such that $\mathrm{Im} z > 0$; that is,

$$v_1(\eta', x_3, \omega) = R_0^{1+}(\omega)g_1(\eta', \cdot)(x_3). \tag{3.23}$$

Let $\chi_4(|\eta'|)$ and $\chi_5(|\eta'|)$ be the characteristic functions of $(0,\infty)\setminus \left[\frac{\sqrt{\omega}}{c_{St}} - \varepsilon, \frac{\sqrt{\omega}}{c_{St}} + \varepsilon\right]$ and $\left(\frac{\sqrt{\omega}}{c_{St}} - \varepsilon, \frac{\sqrt{\omega}}{c_{St}} + \varepsilon\right)$, respectively. Then we have the following propositions.

Proposition 3.3. *Suppose $s_1 > \frac{1}{2}$, $s_2 > \frac{1}{2}$, and $\rho_1 = \rho_2$. Let $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$ and ω be a strictly positive number such that*

$$\sum_{j \in M} \tau_{1j}^{+}(\omega)\bar{f} = 0 \quad \text{or} \quad \sum_{j \in M} \tau_{1j}^{-}(\omega)\bar{f} = 0. \tag{3.24}$$

Then we have

(3.25)
$$\chi_4(|\eta'|)v_1(\cdot,\cdot,\omega) = \chi_4(|\eta'|)R_0^{1+}(\omega)g_1 = \chi_4(|\eta'|)R_0^{1-}(\omega)g_1 \in L^2(\mathbf{R}^3, \mathbf{C}^2, (1+x_3^2)^{\delta-\frac{1}{2}}d\eta' dx_3).$$

Proposition 3.4. *Suppose $s_1 > \frac{1}{2}$, $s_2 > \frac{1}{2}$, and $\rho_1 = \rho_2$. Let $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$ and ω be a strictly positive number such that*

$$\sum_{j \in M} \tau_{1j}^{St}(\omega)f = 0. \tag{3.26}$$

Then we obtain

$$\chi_5(|\eta'|)v_1(\cdot,\cdot,\omega) \in H^{s_1-1}(\mathbf{R}^2_{\eta'}, L^{2;\delta-\frac{1}{2}}(\mathbf{R}, \mathbf{C}^2, dx_3)). \tag{3.27}$$

From Propositions 3.2-3.4, we have

$$\chi_5(|\eta'|)(R_0^{1+}(\omega)g_1(\eta',\cdot) \oplus R_0^{2+}(\omega)g_2(\eta',\cdot))(x_3) \in H^{s_1-1}(\mathbf{R}^2_{\eta'}, L^{2;\delta-\frac{1}{2}}(\mathbf{R}, \mathbf{C}^2, dx_3)), \tag{3.28}$$

moreover

(3.29)
$$F_{\eta'}^{-1}(\mathrm{UC})\chi_5(|\eta'|)(R_0^{1+}(\omega)g_1(\eta',\cdot) \oplus R_0^{2+}(\omega)g_1(\eta',\cdot))(x) \in L^{2;s_1-1,\delta-\frac{1}{2}}(\mathbf{R}^3, \mathbf{C}^3).$$

Thus Theorem 3.1 will be a consequence of Propositions 3.2-3.4 and (3.4).

4. The Limiting Absorption Principle for A

In this section we give a proof of the limiting absorption principle for A along the line of proof by Dermenjian and Guillot for their problem [5]. The key part of the proof is the following proposition.

Proposition 4.1. *For every $f \in L^{2;s_1,s_2}(\Omega, \mathbf{C}^3)$ and $z \in J^{\pm}(a,b) \setminus [a,b]$, we have*

$$\|R(z)f\|_{\mathcal{A};-s_1,-s_2} \leq C\|f\|_{0;s_1,s_2}, \tag{4.1}$$

where $[a,b]$ is any compact interval in $(0,\infty)$ which does not contain any eigenvalue of A and

$$J^{\pm}(a,b) = \{z \in \mathbf{C};\ Rez \in [a,b],\ Imz \in [0,1]\}. \tag{4.2}$$

Proof. We prove this proposition by contradiction. Fourth steps are needed.

Step 1. Suppose that (4.1) is false. Then there exist sequences $\{f_n\}_{n\geq 1}$ in $L^{2;s_1,s_2}(\Omega,\ \mathbf{C}^3)$ and $\{z_n\}_{n\geq 1}$ in $J^{\pm}(a,b) \setminus [a,b]$ such that

$$\|f_n\|_{0;s_1,s_2} = 1, \qquad n \geq 1, \tag{4.3}$$

$$\|R(z_n)f_n\|_{\mathcal{A};-s_1,-s_2} > n, \quad n \geq 1. \tag{4.4}$$

It follows that there exists a subsequence such that

$$\lim_{n\to\infty} z_n = \omega \in [a,b], \tag{4.5}$$

we denote it by the same symbol (cf. [14]). Put

$$u_n = \frac{R(z_n)f_n}{\|R(z_n)f_n\|_{\mathcal{A};-s_1,-s_2}}, \quad n \geq 1, \tag{4.6}$$

$$F_n = \frac{f_n}{\|R(z_n)f_n\|_{\mathcal{A};-s_1,-s_2}}, \quad n \geq 1. \tag{4.7}$$

Then we have

$$u_n \in D(A), \qquad n \geq 1, \tag{4.8}$$

$$\|u_n\|_{\mathcal{A};-s_1,-s_2} = 1, \quad n \geq 1, \tag{4.9}$$

$$\|F_n\|_{0;s_1,s_2} < \frac{1}{n}, \qquad n \geq 1, \tag{4.10}$$

$$(\mathcal{A} - z_n)u_n = F_n, \quad n \geq 1. \tag{4.11}$$

From (4.9) and (1.32)

$$\|u_n\|^2_{1;-s_1,-s_2} \leq 1. \tag{4.12}$$

Since $\{u_n\}_{n\geq 1}$ is a bounded sequence in $H^{1;-s_1,-s_2}(\Omega, \mathbf{C}^3)$, by Rellich theorem, we choose a subsequence of $\{u_n\}_{n\geq 1}$ that we denote by the same symbol such that

$\{u_n\}_{n\geq 1}$ converges to a limit, denoted by u in $L^{2;-s_1',-s_2'}(\Omega,\mathbf{C}^3)$, where $s_1' > s_1$ and $s_2' > s_2$. From (4.5), (4.10) and (4.11), it follows that

$$\mathcal{A}u = \omega u \tag{4.13}$$

in the distribution sense. So we get

$$Au \in L^{2;-s_1',-s_2'}(\Omega,\mathbf{C}^3). \tag{4.14}$$

Then we deduce from Korn's inequality

$$u \in H^{1;-s_1',-s_2'}(\Omega, A, \mathbf{C}^3). \tag{4.15}$$

Step 2. In Step 2 and 3, we shall show that u belongs to $D(A)$.

Let $\phi(x)$ be a function in $C^\infty(\mathbf{R}^3)$ such that $\phi(x) = 1$ for $|x| > L+2$ and $= 0$ for $|x| < L+1$. Since $\phi u(x)$ is defined on $\Omega \cap \{|x| > L\}$, we put

$$\mu_1\varepsilon_{13}(\phi u)|_{x_3=-0} = \mu_2\varepsilon_{13}(\phi u)|_{x_3=+0} = h_1, \tag{4.16}$$

$$\mu_1\varepsilon_{23}(\phi u)|_{x_3=-0} = \mu_2\varepsilon_{23}(\phi u)|_{x_3=+0} = h_2, \tag{4.17}$$

$$\sigma_{33}(\phi u)|_{x_3=-0} = \sigma_{33}(\phi u)|_{x_3=+0} = h_3, \tag{4.18}$$

where

$$h = {}^t(h_1,h_2,h_3) \in H^{\frac{1}{2}}(\mathbf{R}^2,\mathbf{C}^3), \quad \mathrm{supp} h \subset \{x \in \mathbf{R}^3; L+1 < |x| < L+2\}. \tag{4.19}$$

It follows from Lemma 5.1 in Dermenjian and Guillot [5] that there exists an extension $\tilde{u}$ of h belongs to $H^2(\mathbf{R}^3,\mathbf{C}^3)$ such that

$$\mu_1\varepsilon_{13}(\tilde{u})|_{x_3=-0} = \mu_2\varepsilon_{13}(\tilde{u})|_{x_3=+0} = h_1, \tag{4.20}$$

$$\mu_1\varepsilon_{23}(\tilde{u})|_{x_3=-0} = \mu_2\varepsilon_{23}(\tilde{u})|_{x_3=+0} = h_2, \tag{4.21}$$

$$\sigma_{33}(\tilde{u})|_{x_3=-0} = \sigma_{33}(\tilde{u})|_{x_3=+0} = h_3, \tag{4.22}$$

and

$$\mathrm{supp}\tilde{u} \subset \{x \in \mathbf{R}^3; L < |x| < L+3\}. \tag{4.23}$$

Putting

$$u' = \phi u - \tilde{u}, \tag{4.24}$$

the support of u' is contained in $\{x \in \mathbf{R}^3; |x| > L\}$. u' in $H^{1;-s_1',-s_2'}(\Omega,\mathcal{A},\mathbf{C}^3)$ satisfies generalized free boundary-interface condition (1.18) for every v in $H^{1;s_1',s_2'}(\Omega,\mathbf{C}^3)$. We have

$$u' = R_0(z_n)(\mathcal{A}_0 - z_n)u' \quad \text{for } n \geq 1, \tag{4.25}$$

From (4.24), (4.5), (4.10), (4.11) and (4.13),

$$(4.26)\qquad \begin{aligned}&(\mathcal{A}_0 - z_n)u'_n - (\mathcal{A}_0 - \omega)u'\\ &= \phi(\mathcal{A}_0 - z_n)u_n + C\nabla\phi\cdot\nabla u_n + u_n\mathcal{A}_0\phi - (\mathcal{A}_0 - z_n)\tilde{u}\\ &- \phi(\mathcal{A}_0 - \omega)u - C\nabla\phi\cdot\nabla u - u\mathcal{A}_0\phi + (\mathcal{A}_0 - \omega)\tilde{u}\\ &= \phi F_n + C\nabla\phi\cdot\nabla(u_n - u) + (u_n - u)\mathcal{A}_0\phi + (z_n - \omega)\tilde{u}\end{aligned}$$

converges to 0 as $n \to \infty$ in $L^{2;s'_1,s'_2}(\mathbf{R}^3, \mathbf{C}^3)$ because the supports of $\nabla\phi$, $\mathcal{A}_0\phi$ and $\tilde{u}$ are compact.

From the sequence $\{z_n\}_{n\geq 1}$ there exists a subsequence we denote by the same symbol such that either $\mathrm{Im}z_n > 0$ or $\mathrm{Im}z_n < 0$. Suppose that $\mathrm{Im}z_n > 0$. It follows from (4.25) and (4.26) that

$$(4.27)\qquad u' = R_0^+(\omega)(\mathcal{A}_0 - \omega)u'$$

in $H^{2;-s'_1,-s'_2}(\mathbf{R}^3, \mathbf{C}^3)$ by Theorem 2.4.

Step 3. We shall show

$$(4.28)\qquad \begin{aligned}\sum_{j\in M} \tau_{1j}^{\pm}(\omega)[(\mathcal{A}_0 - \omega)u'] &= \sum_{j\in M} \tau_{1j}^{St}(\omega)[(\mathcal{A}_0 - \omega)u']\\ &= \sum_{k\in N} \tau_{2k}^{\pm}(\omega)[(\mathcal{A}_0 - \omega)u'] = 0.\end{aligned}$$

Then it follows from Theorem 4.1 that $u' \in L^2(\mathbf{R}^3, \mathbf{C}^3)$ taking $s'_1 > 1$ and $s'_2 > 1$. Thus u belongs to $L^2(\Omega, \mathbf{C}^3, \rho(x)dx)$.

We denote by $<\cdot,\cdot>_\rho$ the duality between $L^{2;-s'_1,-s'_2}(\Omega, \mathbf{C}^3, \rho(x)dx)$ and $L^{2;s'_1,s'_2}(\Omega, \mathbf{C}^3, \rho(x)dx)$. From Proposition 2.6 and (4.27) it is sufficient to show that

$$(4.29)\qquad I = \left\langle \overline{R_0^+[(\mathcal{A}_0 - \omega)u']}, (\mathcal{A}_0 - \omega)u' \right\rangle_\rho = \langle \bar{u}', (\mathcal{A}_0 - \omega)u' \rangle_\rho$$

is a real number. Remark that the support of $(\mathcal{A}_0 - \omega)u'$ is contained in $|x| < L+2$. Let χ be a function $\chi(x) \in C_0^\infty(\mathbf{R}^3, \mathbf{R})$ such that $\chi(x) = 1$ for $|x| < L+2$. Then we have

$$(4.30)\qquad I = <\chi\bar{u}', \mathcal{A}_0 u'>_\rho - \omega|\chi u'|^2.$$

Thus it is sufficient to show that $<\chi\bar{u}', \mathcal{A}_0 u'>_\rho$ is real. Since u' satisfies the generalized free boundary-interface condition, we have

$$\begin{aligned}(4.31)\qquad I &= \int_{\mathbf{R}^3_-} \left(\lambda_1(\nabla\cdot u')(\bar{u}', \nabla\chi) + \mu_1 \sum_{k,j=1}^{3} \varepsilon_{kj}(u') \left(\bar{u}'_k \frac{\partial\chi}{\partial x_j} + \bar{u}'_j \frac{\partial\chi}{\partial x_k} \right) \right) dx\\ &+ \int_{\mathbf{R}^3_+} \left(\lambda_2(\nabla\cdot u')(\bar{u}', \nabla\chi) + \mu_2 \sum_{k,j=1}^{3} \varepsilon_{kj}(u') \left(\bar{u}'_k \frac{\partial\chi}{\partial x_j} + \bar{u}'_j \frac{\partial\chi}{\partial x_k} \right) \right) dx\\ &+ \int_{\mathbf{R}^3_-} \left(\lambda_1|\chi(\nabla\cdot u')|^2 + 2\mu_1 \sum_{k,j=1}^{3} |\chi\varepsilon_{kj}(u')|^2 \right) dx\\ &+ \int_{\mathbf{R}^3_+} \left(\lambda_2|\chi(\nabla\cdot u')|^2 + 2\mu_2 \sum_{k,j=1}^{3} |\chi\varepsilon_{kj}(u')|^2 \right) dx,\end{aligned}$$

where the third and fourth terms of the right-hand side of (4.31) are real numbers. Note that the first and second terms of the right-hand side of (4.31) are integrated on

$$\text{supp}\nabla\chi \in \{x \in \mathbf{R}^3; L+2 < |x| < L+3\}.$$

Consider $< \chi\bar{u}, \mathcal{A}u >_\rho$. From (4.13) and (4.15), we have

$$J =< \chi\bar{u}, \mathcal{A}u >_\rho =< \chi\bar{u}, \omega u >_\rho = \omega|\chi u|^2.$$

On the other hand

$$J = \left(\int_{\Omega_-} + \int_{\Omega_+}\right)\left(\lambda_(x)(\nabla\cdot u)(\nabla\cdot\chi\bar{u}) + 2\mu(x)\sum_{k,j=1}^{3}\varepsilon_{kj}(u)\varepsilon_{kj}(\chi\bar{u})\right)dx$$

$$= \int_{\mathbf{R}^3}\left(\lambda(x)(\nabla\cdot u)(\bar{u},\nabla\chi) + \mu(x)\sum_{k,j=1}^{3}\varepsilon_{kj}(u)\left(\bar{u}_k\frac{\partial\chi}{\partial x_j} + \bar{u}_j\frac{\partial\chi}{\partial x_k}\right)\right)dx + G,$$

where G is a real number. On the support of $\nabla\chi$, we have

$$\lambda(x) = \begin{cases}\lambda_1, \\ \lambda_2,\end{cases} \quad \mu(x) = \begin{cases}\mu_1 & x \in \text{supp}\nabla\chi \cap \mathbf{R}^3_-, \\ \mu_2 & x \in \text{supp}\nabla\chi \cap \mathbf{R}^3_+.\end{cases}$$

So we obtain I is real, because J is real.

Step 4. Finally we prove

$$\|u\|_{\mathcal{A};-s_1,-s_2} = 1.$$

On the sequence

$$u_n = \phi u_n - \tilde{u} + (1-\phi)u_n + \tilde{u},$$

$(1-\phi)u_n$ converges to $(1-\phi)u$ in $H^{1;-s_1,-s_2}(\Omega, A, \mathbf{C}^3)$, and $\phi u_n - \tilde{u}$ converges to $\phi u - \tilde{u}$ in $H^{2;-s_1,-s_2}(\Omega, \mathbf{C}^3)$. So we have

$$\begin{aligned}\|u_n - u_m\|_{\mathcal{A};-s_1,-s_2} &\le \|(\phi u_n - \tilde{u}) - (\phi u_m - \tilde{u})\|_{\mathcal{A};-s_1,-s_2} \\ &\quad + \|(1-\phi)(u_n - u_m)\|_{\mathcal{A};-s_1,-s_2} \\ &\to 0 \quad \text{as} \quad n, m \to \infty.\end{aligned}$$

Thus

$$\|u\|_{\mathcal{A};-s_1,-s_2} = \lim_{n\to\infty}\|u_n\|_{\mathcal{A};-s_1,-s_2} = 1.$$

This completes the proof of Proposition 4.1. □

Proposition 4.2. *Let $f \in L^{2;s_1,s_2}(\Omega, \mathbf{C}^3)$ and $z \in J^{\pm}(a,b) \setminus [a,b]$. Then the mapping*

$$T : z \to R(z)f$$

is uniformly continuous in $H^{1;-s_1,-s_2}(\Omega, A, \mathbf{C}^3)$.

This proposition can be proved similarly in [4, Section 3 Proposition 2].

Proof of Main Theorem. The mapping

$$T : z \to R(z)f$$

is extended from $J^{\pm}(a,b) \setminus [a,b]$ to $J^{\pm}(a,b)$, because of the completeness of $H^{1;-s_1,-s_2}(\Omega, A, \mathbf{C}^3)$ and of Proposition 4.2.

Therefore we prove Main Theorem. □

Finally by Theorem 3.1 and Main Theorem, we have some properties of the spectrum of A.

Theorem 4.3.
1. A has no continuous singular spectrum.
2. If $[a, b]$ is a compact interval contained in $(0, \infty)$, A can only have a finite number of eigenvalues in $[a, b]$, and each of these eigenvalues has a finite multiplicity.

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