SOME SUBMANIFOLDS OF
COMPLEX PROJECTIVE SPACES

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We discuss a generalization of ruled surfaces in \( \mathbb{R}^3 \) to submanifolds in complex projective spaces. A ruled surface is generated by 1-parameter family of lines in \( \mathbb{R}^3 \). Examples are: hyperboloid of one sheet, hyperbolic paraboloid, circular cylinder, circular conic and right helicoid. It is classically known that a ruled surface \( M \) in \( \mathbb{R}^3 \) is minimal if and only if \( M \) is a part of a plane \( \mathbb{R}^2 \) or a right helicoid.

We denote by \( \mathbb{P}^n(\mathbb{C}) \) an \( n \)-dimensional complex projective space with Fubini-Study metric of holomorphic sectional curvature 1 unless otherwise stated. It is known that a totally geodesic submanifold of complex projective space \( \mathbb{P}^n(\mathbb{C}) \) is one of the following:

(a) Kähler submanifold \( \mathbb{P}^k(\mathbb{C}) \) \((k < n)\),
(b) Totally real submanifold \( \mathbb{P}^k(\mathbb{R}) \) \((k \leq n)\).

Now we study the following submanifolds in \( \mathbb{P}^n(\mathbb{C}) \):

(1) Kähler submanifold \( M^{k+r} \) on which there is a holomorphic foliation of complex codimension \( r \) and each leaf is a totally geodesic \( \mathbb{P}^k(\mathbb{C}) \) in \( \mathbb{P}^n(\mathbb{C}) \) (for simplicity, we said that \( M \) is holomorphically \( k \)-ruled).
(2) Totally real (Lagrangian) submanifold \( M^n \) on which there is a (real) foliation of real codimension \( n - k \) and each leaf is a totally geodesic \( \mathbb{P}^k(\mathbb{R}) \) in \( \mathbb{P}^n(\mathbb{C}) \).

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1. The set of \( k \) dimensional totally geodesic complex projective subspaces in \( \mathbb{P}^n(\mathbb{C}) \) is identified with the complex Grassmann manifold \( G_{k+1,n-k}(\mathbb{C}) \) of \( k+1 \) dimensional complex linear subspaces in \( \mathbb{C}^{n+1} \):

\[
\{ \mathbb{P}^k(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C}) \} \cong \{ \mathbb{C}^{k+1} \subset \mathbb{C}^{n+1} \} = G_{k+1,n-k}(\mathbb{C}).
\]

Hence there is a natural correspondence between Kähler submanifolds \( M^{k+r} \) holomorphically foliated by totally geodesic \( \mathbb{P}^k(\mathbb{C}) \) in \( \mathbb{P}^n(\mathbb{C}) \) and Kähler submanifolds \( \Sigma^r \) in the complex Grassmann manifold \( G_{k+1,n-k}(\mathbb{C}) \).

Here we note that a holomorphically \( k \)-ruled Kähler submanifold \( M^{k+r} \) in \( \mathbb{P}^n(\mathbb{C}) \) is totally geodesic if and only if the corresponding Kähler submanifold \( \Sigma^r \leftarrow G_{k+1,n-k}(\mathbb{C}) \) is contained in some totally geodesic \( G_{k+1,r}(\mathbb{C}) \) in \( G_{k+1,n-k}(\mathbb{C}) \) (cf. [CN]).
Examples of holomorphically $k$–ruled submanifolds in $\mathbb{P}^n(\mathbb{C})$.

1. totally geodesic $\mathbb{P}^{k+r}(\mathbb{C})$,
2. Segre imbedding $\mathbb{P}^k(\mathbb{C}) \times \mathbb{P}^r(\mathbb{C}) \hookrightarrow \mathbb{P}^{kr+r}(\mathbb{C})$.

In the case $k = r = 1$, we can see that

**Theorem 1.** Let $M^2$ be a Kähler surface in $\mathbb{P}^n(\mathbb{C})$ holomorphically foliated by totally geodesic projective lines $\mathbb{P}^1(\mathbb{C})$ in $\mathbb{P}^n(\mathbb{C})$. If the scalar curvature of $M^2$ is constant and $M^2$ is not totally geodesic, then the Gauss curvature of the corresponding holomorphic curve $\Sigma^1$ in $G_{2,n-1}(\mathbb{C})$ is constant.

**Remark.** The converse of the above Theorem does not hold. Consider the following holomorphic imbedding:

$$\mathbb{P}^1(\mathbb{C}) \hookrightarrow \mathbb{P}^k(\mathbb{C}) \times \mathbb{P}^l(\mathbb{C}) \hookrightarrow G_{2,k+l}(\mathbb{C}),$$

where the first imbedding is the product of Veronese imbeddings of degree $k$ and $l$, and the second one is the totally geodesic holomorphic imbedding of $\mathbb{P}^k(\mathbb{C}) \times \mathbb{P}^l(\mathbb{C})$ into $G_{2,k+l}(\mathbb{C})$ (cf. [CN]). By direct calculations, we can see that the Gauss curvature of the induced metric on $\mathbb{P}^1(\mathbb{C})$ is $\frac{1}{k+l}$. But the scalar curvature of the corresponding holomorphically 1–ruled Kähler surface $M^2$ in $\mathbb{P}^{k+l+1}(\mathbb{C})$ is constant only when $k = l$. In this case, the Kähler surface $M^2$ is holomorphically congruent to

$$\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1_{1/k}(\mathbb{C}) \hookrightarrow \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^k(\mathbb{C}) \hookrightarrow \mathbb{P}^{2k+1}(\mathbb{C}),$$

where the first one is the product of the identity map and the Veronese imbedding of degree $k$, and the second one is the Segre imbedding. These examples show that contrary to Kähler submanifolds in $\mathbb{P}^n(\mathbb{C})$ (cf. [E]), the rigidity for Kähler submanifolds in complex Grassmann manifolds (of rank $\geq 2$) does not hold (cf. [CZ]).

2. The set $W_{k,n}$ of $k$ dimensional totally geodesic real projective subspaces $\mathbb{P}^k(\mathbb{R})$ in $\mathbb{P}^n(\mathbb{C})$ is:

$$W_{k,n} = \begin{cases} SU(n+1)/SO(n+1), & \text{if } n = k, \\ SU(n+1)/K_{k+1,n-k}, & \text{if } n > k, \end{cases}$$

where

$$K_{k+1,n-k} = \left\{ \begin{pmatrix} e^{i \theta} P & 0 \\ 0 & e^{-i \theta} Q \end{pmatrix} \right\} \quad \theta \in \mathbb{R}, \quad P \in SO(k+1), \quad Q \in SU(n-k).$$

Let $G = SU(n+1)$, $\mathfrak{g} = su(n+1)$, and

$$\mathfrak{e} = \left\{ i \begin{pmatrix} \frac{\theta}{k+1} & 0 \\ 0 & \frac{\theta}{n-k} \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \right\} \quad \theta \in \mathbb{R}, \quad U \in so(k+1), \quad V \in su(n-k).$$
where $E$ denotes identity matrix. Then $\mathfrak{t}$ is the Lie algebra of the Lie group $K = K_{k+1, n-k}$. Put

$$m = \left\{ \begin{pmatrix} iW & Z \\ -Z^* & 0 \end{pmatrix} \mid W \in S_0(k + 1, \mathbb{R}), \ Z \in M(k + 1, n - k, \mathbb{C}) \right\},$$

where $S_0(k + 1, \mathbb{R})$ denotes the set of $(k + 1) \times (k + 1)$ (real) symmetric matrices with trace $= 0$. Then we have $\mathfrak{g} = \mathfrak{t} + m$ (direct sum) and $[\mathfrak{t}, m] \subset m$. Since $K$ is connected, the homogeneous space $G/K$ is reductive. Note that $G/K$ is naturally reductive with respect to some metric.

As §1, we can see that there is a one-to-one correspondence between $k + r$ ($0 < r < 2n - k$) dimensional submanifolds foliated by totally geodesic, totally real $\mathbb{P}^k(\mathbb{R})$ in $\mathbb{P}^n(\mathbb{C})$ and $r$ dimensional submanifolds in $G/K$.

There is a natural fibration on $G/K$ as follows: Let $\pi : G/K \to G/H$ be the projection defined by $gK \mapsto gH$, where $K = K_{k+1, n-k}$ and $H = S(U(k + 1) \times U(n - k))$. Note that $K \subset H$ and $G/H = G_{k+1, n-k}(\mathbb{C})$. Then it can be seen that $\pi$ is a Riemannian submersion with which each fibre $H/K \cong SU(k + 1)/SO(k + 1)$ is totally geodesic in $G/K$.

Since $G/H = G_{k+1, n-k}(\mathbb{C})$ is a Kähler manifold, there is the almost complex structure on the horizontal distribution of $G/K$ compatible to $\pi$ and the canonical complex structure of $G/H$.

We claim

**Proposition 2.** Let $M^n$ be a submanifold in $\mathbb{P}^n(\mathbb{C})$ foliated by totally geodesic $\mathbb{P}^k(\mathbb{R})$. Then $M^n$ is totally real (Lagrangian) if and only if the corresponding submanifold $\Sigma^{n-k}$ in $G/K$ is "horizontal" and "totally real" with respect to the almost complex structure on the horizontal distribution of $G/K$ as above.

Hence if $M^n$ is a totally real submanifold in $\mathbb{P}^n(\mathbb{C})$ foliated by totally geodesic $\mathbb{P}^k(\mathbb{R})$, then there is a corresponding totally real submanifold $\Sigma^{n-k}$ in $G/H = G_{k+1, n-k}(\mathbb{C})$. So we consider the following question: If $\Sigma^{n-k}$ is a (totally real) submanifold in $G/H = G_{k+1, n-k}(\mathbb{C})$, then does there exist a "horizontal lift" of $\Sigma^{n-k}$ in $G/K$ with respect to $\pi$? Note that a totally real submanifold $M^n$ foliated by totally geodesic $\mathbb{P}^k(\mathbb{R})$ in $\mathbb{P}^n(\mathbb{C})$ is totally geodesic if and only if the corresponding submanifold $\Sigma^{n-k}$ in $G/K$ is contained in some $\pi^{-1}(G_{k+1, n-k}(\mathbb{R}))$, where $G_{k+1, n-k}(\mathbb{R})$ is a maximal totally geodesic, totally real submanifold in $G/H = G_{k+1, n-k}(\mathbb{C})$.

Hereafter we assume that: Let $G$ be a linear Lie group, let $H \supset K$ be closed subgroups of $G$ such that $G/K$ is a reductive homogeneous space, $G/H$ is a Riemannian symmetric space and the projection $\pi : G/K \to G/H$ defined by $gK \mapsto gH$ is a Riemannian submersion with which each fibre $H/K$ is totally geodesic in $G/K$. We denote the projections as $\pi_H : G \to G/H$ and $\pi_K : G \to G/K$. Let $\mathfrak{g}, \mathfrak{h}$ and $\mathfrak{t}$ be the Lie algebras of the Lie groups $G, H, K$, respectively, and let $\mathfrak{g} = \mathfrak{t} + \mathfrak{m} = \mathfrak{h} + \mathfrak{p}$ be the canonical decompositions of $\mathfrak{g}$.

Let $f : \Sigma \to G/H$ be an isometric immersion. We would like to find the "condition" for existence of a "horizontal lift" of $f$. By using a local cross
To construct a "horizontal lift" of $f$, we may find the map $\Psi : \Sigma \rightarrow H$ such that $\Phi := \pi_K \circ (\Phi \cdot \Psi) : \Sigma \rightarrow G$ is locally. Here $\Phi \cdot \Psi$ is defined by the product of $\Phi(x) \in G$ and $\Psi(x) \in H \subset G$ for $x \in \Sigma$ as elements of the Lie group $G$.

Let $\alpha := \Phi^{-1} d\Phi$ (resp. $\beta := \Psi^{-1} d\Psi$) be the $\mathfrak{g}$-valued (resp. $\mathfrak{h}$-valued) 1-form on $\Sigma$ which is the pull-back of the Maurer-Cartan form of $G$ (resp. $H$). The following fact is well known (cf. [G]): Let $\omega$ be the Maurer-Cartan form on $G$. Suppose that $\alpha$ is a $\mathfrak{g}$-valued 1-form on a connected and simply connected manifold $\Sigma$. Then there exists a $C^\infty$ map $\Phi : \Sigma \rightarrow G$ with $\Phi^* \omega = \alpha$ if and only if $d\alpha + \frac{1}{2} [\alpha, \alpha] = 0$. Moreover, the resulting map is unique up to left translation.

We see that

\[ F = \pi_K \circ (\Phi \cdot \Psi) : \Sigma \rightarrow G/K \text{ is horizontal}, \]

\[ \iff (\mathfrak{h} \cap \mathfrak{m}) - \text{part of } (\Phi \cdot \Psi)^{-1} d(\pi_K \circ (\Phi \cdot \Psi)) = 0, \]

\[ \iff \beta_m + (\text{ad}(\Psi^{-1})\alpha)_{\mathfrak{h} \cap \mathfrak{m}} = 0. \]

Taking an exterior derivation and using the integrability conditions of $\alpha$ and $\beta$, we get that the condition for existence of a horizontal lift of the isometric immersion $f : \Sigma \rightarrow G/H$ is

\[ [\alpha_p \wedge \alpha_p]_{\mathfrak{h} \cap \mathfrak{m}} = 0. \]

**References**


