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# On a curve in $E^2$ and $S^2$ which is close to a circle or a straight line

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Let $\gamma$ be a closed $C^2$ curve of length $L$ in the 2–dimensional Euclidean space $E^2$. Let $s$ be an arclength parameter of $\gamma$ and $k$ be the signed curvature of $\gamma$. If $n$ denotes the rotational number of $\gamma$, we have

$$\int_\gamma k\, ds = 2\pi n. \tag{1}$$

By Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_\gamma k^2\, ds &\geq \frac{1}{L}\left(\int_\gamma k\, ds\right)^2 \\ &= \frac{(2\pi n)^2}{L}.\end{aligned}$$

When $n \neq 0$, the equality holds if and only if $\gamma$ is the $n$–time covering of a circle of radius $L/2\pi n$. Looking at this, we would like to consider the following question: How close is $\gamma$ to a circle when $\int_\gamma k^2\, ds - (2\pi n)^2/L$ is close to zero ? Our first theorem gives an answer to this question.

**Theorem 1.** Let $\varepsilon$ be any positive constant. Let $\gamma$ be a closed curve of length $L$ in $E^2$ with rotational number $n \neq 0$. If

$$\int_\gamma k^2\, ds < \frac{(2\pi n)^2}{L} + \frac{2\pi^2 n^2 \varepsilon^2}{L^3},$$

then $\gamma$ lies between two concentric circles of radii $r$ and $R$ with $|R - r| < \varepsilon$.

A similar problem can be considered for a simple closed curve in the 2–dimensional unit sphere $S^2$. If $\gamma$ is an oriented simple closed $C^2$ curve in $S^2$ which has length $L \leq 2\pi$ and bounds the region of area $A$, we have

$$\int_\gamma k\, ds = 2\pi - A. \tag{2}$$

By Cauchy–Schwarz inequality and the isoperimetric inequality

$$L^2 - 4\pi A + A^2 \geq 0, \tag{3}$$

we have

$$\begin{aligned} \int_\gamma k^2\, ds &\geq \frac{1}{L}\left(\int_\gamma k\, ds\right)^2 \\ &= \frac{(2\pi - A)^2}{L} \\ &\geq \frac{4\pi^2 - L^2}{L}. \end{aligned}$$

When $L \leq 2\pi$, the equality holds if and only if $\gamma$ is a small circle. When $\int_\gamma k^2\, ds$ is "almost" equal to $(4\pi^2 - L^2)/L$, we have the following theorem.

**Theorem2.** Let $\varepsilon$ be any positive constant less than $\pi/2$. Let $\gamma$ be a simple closed curve of length $L \leq 2\pi$ in $S^2$. If

$$\int_\gamma k^2\, ds < \frac{4\pi^2 - L^2}{L} + \frac{1}{8\pi}\varepsilon^2,$$

then $\gamma$ lies between two concentric small circles of radii $r$ and $R$ with $|R - r| < \varepsilon$.

Now we would like to look at a curve of infinite length in $E^2$. Let $\gamma : x(s)$ be a curve in $E^2$ which is parameterized by arclength $s$ for $-\infty < s < \infty$. If $k(s) = 0$ for all $s$, $\gamma$ is, of course, a straight line. If $k(s)$ decays to zero in a certain order as $s \to \pm\infty$, $\gamma$ has an asymptotic line on each end. In [1] it was shown that if $\gamma$ is properly immersed and there exist positive constants $C_0$ and $\varepsilon$ such that $|k(s)||x(s)|^{2+\varepsilon} \leq C_0$ holds for all $s$, then each end of $C$ has an asymptotic line. Here an asymptotic line means a straight line $\ell : y(t)$ which has a property that the function $h(s) := \inf_t |x(s) - y(t)|$ tends to zero as $s \to +\infty$ or $s \to -\infty$. If $|k(s)||x(s)|^{2+\varepsilon}$ is bounded, we have $\int_{-\infty}^{\infty} |k(s)||x(s)|\, ds < \infty$. In our third theorem, we will show that this condition is sufficient for $\gamma$ to have an asymptotic line on each end.

**Theorem 3.** Let $\gamma : x(s)$ be a curve in $E^2$ which is parameterized by arclength $s$ for $-\infty < s < \infty$. If $\int_{-\infty}^{\infty} |k(s)||x(s)|\, ds < \infty$ , then $\gamma$ is properly immersed and has an asymptotic line on each end.

## §1. Proof of Theorem 1

We have

$$\begin{aligned}
\int_\gamma k^2\,ds - \frac{(2\pi n)^2}{L} &= \int_\gamma k^2\,ds - \frac{1}{L}\left(\int_\gamma k\,ds\right)^2 \\
&= \frac{1}{2L}\int_0^L\int_0^L (k(s)-k(t))^2\,dsdt \\
&\geq \frac{1}{2L^2}\int_0^L\left(\int_0^L (k(s)-k(t))\,dt\right)^2 ds \\
&= \frac{1}{2L^2}\int_0^L\left(L\,k(s) - \int_0^L k(t)\,dt\right)^2 ds \\
&= \frac{1}{2}\int_0^L\left(k(s)-\frac{2n\pi}{L}\right)^2 ds. \qquad (4)
\end{aligned}$$

We set $k_0 = 2n\pi/L$. Since $n \neq 0$, $k_0 \neq 0$. Suppose that

$$\int_\gamma k^2\,ds - \frac{(2\pi n)^2}{L} < \delta. \qquad (5)$$

Then (4) and (5) give

$$\int_0^L (k(s)-k_0)^2\,ds < 2\delta. \qquad (6)$$

(6) means that $k(s)$ is close to $k_0$ except for a set of a small measure. But, in general, this does not imply that $\gamma$ is close to a circle.

Let $e(s)$ be the unit normal vector of $\gamma$ with $\frac{d^2\gamma}{ds^2} = -ke$. We may assume that

$$\gamma(0) = (\frac{1}{k_0}, 0), \qquad \frac{d\gamma}{ds}(0) = (0,1), \qquad e(0) = (\frac{k_0}{|k_0|}, 0).$$

For all $s \in [0, L]$ we have

$$\begin{aligned}
\left|\gamma(s) - \frac{1}{k_0}e(s)\right| &= \left|\int_0^s \left(\frac{d\gamma}{ds} - \frac{1}{k_0}\frac{de}{ds}\right) ds\right| \\
&= \left|\int_0^s \left(1 - \frac{k(s)}{k_0}\right)\frac{d\gamma}{ds}\,ds\right| \\
&\leq \frac{1}{|k_0|}\int_0^s |k(s)-k_0|\,ds \\
&\leq \frac{1}{|k_0|}\int_0^L |k(s)-k_0|\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{|k_0|} L^{1/2} \left( \int_0^L (k(s) - k_0)^2 \, ds \right)^{1/2} \\
&\leq \frac{\sqrt{2\delta L}}{|k_0|} \\
&= \frac{\sqrt{2\delta L^3}}{2\pi |n|}.
\end{aligned}
$$

(7) implies that

$$
\begin{aligned}
|\gamma(s)| &\leq \left| \frac{1}{k_0} e(s) \right| + \left| \gamma(s) - \frac{1}{k_0} e(s) \right| \\
&\leq \frac{L}{2\pi|n|} + \frac{\sqrt{2\delta L^3}}{2\pi|n|} \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
|\gamma(s)| &\geq \left| \frac{1}{k_0} e(s) \right| - \left| \frac{1}{k_0} e(s) - \gamma(s) \right| \\
&\geq \frac{L}{2\pi|n|} - \frac{\sqrt{2\delta L^3}}{2\pi|n|}. \qquad (9)
\end{aligned}
$$

Hence $\gamma$ lies between two concentric circles of radii $\frac{L}{2\pi|n|} - \frac{\sqrt{2\delta L^3}}{2\pi|n|}$ and $\frac{L}{2\pi|n|} + \frac{\sqrt{2\delta L^3}}{2\pi|n|}$. This proves Theorem 1 by setting

$$
\delta = \frac{2\pi^2 n^2 \varepsilon^2}{L^3}.
$$

**Remark.** When $\gamma$ is a simple closed convex plane curve with bounding area $A$, Theorem 1 is obtained from Gage's inequality (**[2]**)

$$
\pi \frac{L}{A} \leq \int_\gamma k^2 \, ds
$$

and the isoperimetric inequality of Bonnesen type

$$
L^2 - 4\pi A \geq \pi^2 (R - r)^2.
$$

Note that Gage's inequality does not hold for nonconvex curves.

## §2. Proof of Theorem 2

Suppose that

$$\int_\gamma k^2\, ds < \frac{4\pi^2 - L^2}{L} + \delta. \tag{10}$$

Since

$$\begin{aligned} \int_\gamma k^2\, ds &\geq \frac{1}{L}\left(\int_\gamma k\, ds\right)^2 \\ &= \frac{(2\pi - A)^2}{L}, \end{aligned}$$

we have

$$\frac{(2\pi - A)^2}{L} < \frac{4\pi^2 - L^2}{L} + \delta,$$

or equivalently,

$$L^2 - 4\pi A + A^2 < L\delta. \tag{11}$$

The isoperimetric inequality given by Osserman ([**3**]) states

$$L^2 - 4\pi A + A^2 \geq (L - \cot\frac{r}{2}A)^2, \tag{12}$$

where $r$ is the radius of the inscribed circle of $\gamma$. It follows from (11) and (12) that

$$(L - \cot\frac{r}{2}A)^2 < L\delta.$$

Hence

$$\frac{L - \sqrt{L\delta}}{A} < \cot\frac{r}{2} < \frac{L + \sqrt{L\delta}}{A}. \tag{13}$$

Another inequality obtained from (11) and (12) is

$$0 \leq L^2 - 4\pi A + A^2 < L\delta. \tag{14}$$

(14) implies that either

$$2\pi - \sqrt{4\pi^2 - L^2 + L\delta} < A \leq 2\pi - \sqrt{4\pi^2 - L^2} \tag{15}$$

or

$$2\pi + \sqrt{4\pi^2 - L^2} \leq A < 2\pi + \sqrt{4\pi^2 - L^2 + L\delta} \tag{16}$$

holds. By reversing the orientation of $\gamma$ if necessary, we may assume that (15) holds.

Now we assume that $\delta < L$. Then, combining (13) and (15), we obtain

$$\frac{L-\sqrt{L\delta}}{2\pi-\sqrt{4\pi^2-L^2}} < \cot\frac{r}{2} < \frac{L+\sqrt{L\delta}}{2\pi-\sqrt{4\pi^2-L^2+L\delta}},$$

or equivalently,

$$2\arctan\left(\frac{2\pi-\sqrt{4\pi^2-L^2+L\delta}}{L+\sqrt{L\delta}}\right) < r < 2\arctan\left(\frac{2\pi-\sqrt{4\pi^2-L^2}}{L-\sqrt{L\delta}}\right). \qquad (17)$$

Let $\gamma_-$ be the curve which is identical to $\gamma$ except for the orintation. Since (16) holds for $\gamma_-$, the radius $r_-$ of the inscribed circle of $\gamma_-$ satisfies

$$2\arctan\left(\frac{2\pi+\sqrt{4\pi^2-L^2}}{L+\sqrt{L\delta}}\right) < r_- < 2\arctan\left(\frac{2\pi+\sqrt{4\pi^2-L^2+L\delta}}{L-\sqrt{L\delta}}\right). \qquad (18)$$

Let $R$ be the radius of the circumscribed circle of $\gamma$. Since $R=\pi-r_-$, we have

$$\begin{aligned}\pi-2\arctan\left(\frac{2\pi+\sqrt{4\pi^2-L^2+L\delta}}{L-\sqrt{L\delta}}\right) &< R \\ &< \pi-2\arctan\left(\frac{2\pi+\sqrt{4\pi^2-L^2}}{L+\sqrt{L\delta}}\right). \qquad (19)\end{aligned}$$

It follows from (17) and (19) that

$$\begin{aligned}R-r &< \pi-2\arctan\left(\frac{2\pi+\sqrt{4\pi^2-L^2}}{L+\sqrt{L\delta}}\right) \\ &\quad -2\arctan\left(\frac{2\pi-\sqrt{4\pi^2-L^2+L\delta}}{L+\sqrt{L\delta}}\right). \qquad (20)\end{aligned}$$

Set

$$\alpha=\frac{2\pi+\sqrt{4\pi^2-L^2}}{L+\sqrt{L\delta}}, \qquad \beta=\frac{2\pi-\sqrt{4\pi^2-L^2+L\delta}}{L+\sqrt{L\delta}}.$$

Then we see that

$$\tan(\pi-2\alpha-2\beta)=2\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)\left(\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)^2-1\right)^{-1}. \qquad (21)$$

Now we assume that $\delta < \frac{L}{16}$. Since

$$\beta \geq \frac{2\pi-\sqrt{4\pi^2-L^2}-\sqrt{L\delta}}{L+\sqrt{L\delta}}, \qquad (22)$$

we have

$$\begin{aligned}
\frac{\alpha+\beta}{1-\alpha\beta} &\geq \frac{(4\pi-\sqrt{L\delta})(L+\sqrt{L\delta})}{(2L+2\pi+\sqrt{4\pi^2-L^2}+\sqrt{L\delta})\sqrt{L\delta}} \\
&\geq \frac{(4\pi-\frac{L}{4})L}{(2L+2\pi+\sqrt{4\pi^2-L^2}+\frac{L}{4})\frac{L}{4}} \\
&= \frac{64\pi-4L}{9L+8\pi+4\sqrt{4\pi^2-L^2}} \\
&> 1. \qquad (23)
\end{aligned}$$

Since $f(y) = \dfrac{2y}{y^2-1}$ is decreasing for $y > 1$, it follows from (23) that

$$\begin{aligned}
&2\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)\left(\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)^2-1\right)^{-1} \\
&< \frac{2(4\pi-\sqrt{L\delta})(L+\sqrt{L\delta})}{(2L+2\pi+\sqrt{4\pi^2-L^2}+\sqrt{L\delta})\sqrt{L\delta}} \\
&\quad \times\left(\left(\frac{(4\pi-\sqrt{L\delta})(L+\sqrt{L\delta})}{(2L+2\pi+\sqrt{4\pi^2-L^2}+\sqrt{L\delta})\sqrt{L\delta}}\right)^2-1\right)^{-1} \\
&= \frac{2(4\pi-\sqrt{L\delta})(L+\sqrt{L\delta})(2L+2\pi+\sqrt{4\pi^2-L^2}+\sqrt{L\delta})\sqrt{L\delta}}{(4\pi-\sqrt{L\delta})^2(L+\sqrt{L\delta})^2-(2L+2\pi+\sqrt{4\pi^2-L^2}+\sqrt{L\delta})^2L\delta} \\
&< \frac{2\cdot 4\pi(L+\frac{L}{4})(2L+2\pi+\sqrt{4\pi^2-L^2}+\frac{L}{4})\sqrt{L\delta}}{(4\pi-\frac{L}{4})^2L^2-(2L+2\pi+\sqrt{4\pi^2-L^2}+\frac{L}{4})^2\frac{L^2}{16}} \\
&= \frac{10\pi(\frac{9}{4}L+2\pi+\sqrt{4\pi^2-L^2})}{(4\pi-\frac{L}{4})^2-\frac{1}{16}(\frac{9}{4}L+2\pi+\sqrt{4\pi^2-L^2})^2}\frac{\sqrt{\delta}}{\sqrt{L}} \\
&< C_1\frac{\sqrt{\delta}}{\sqrt{L}},
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= \frac{10\pi(\frac{\sqrt{97}}{2}\pi+2\pi)}{(4\pi-\frac{2\pi}{4})^2-\frac{1}{16}(\frac{\sqrt{97}}{2}\pi+2\pi)^2} \\
&\approx 7.48\cdots.
\end{aligned}$$

Thus, for any positive constant $\varepsilon < \pi/2$, if

$$\delta \leq \frac{L\varepsilon^2}{C_1^2}, \qquad (24)$$

then $R-r \leq \tan(R-r) < \varepsilon$. Note that the requirement that $\delta < \frac{L}{16}$ is automatically satisfied since $\varepsilon < \frac{\pi}{2}$. This proves Theorem 2 except for curves of length close to zero. For curves with $0 < L < \pi$, we use

$$\beta \geq \frac{2\pi - \sqrt{4\pi^2 - L^2} - \frac{L\delta}{2\sqrt{4\pi^2-L^2}}}{L + \sqrt{L\delta}}, \tag{25}$$

instead of (22). Again we assume that $\delta < \frac{L}{16}$. Then we have

$$\begin{aligned}
\frac{\alpha+\beta}{1-\alpha\beta} &\geq \frac{(4\pi - \frac{L\delta}{2\sqrt{4\pi^2-L^2}})(L+\sqrt{L\delta})}{(2L^{3/2} + \frac{3}{2}L\sqrt{\delta} + \frac{\pi L}{\sqrt{4\pi^2-L^2}}\sqrt{\delta})\sqrt{\delta}} \\
&> \frac{(4\pi - \frac{L^2}{32\sqrt{4\pi^2-L^2}})L}{(2L^{3/2} + \frac{3}{8}L^{3/2} + \frac{\pi L}{\sqrt{4\pi^2-L^2}}\frac{\sqrt{L}}{4})\sqrt{\delta}} \\
&> \frac{4\pi - \frac{\pi^2}{32\sqrt{4\pi^2-\pi^2}}}{(2\sqrt{\pi} + \frac{3}{8}\sqrt{\pi} + \frac{\pi}{\sqrt{4\pi^2-\pi^2}}\frac{\sqrt{\pi}}{4})\sqrt{\delta}} \\
&= \frac{(128\sqrt{3}-1)\sqrt{\pi}}{(76\sqrt{3}+8)\sqrt{\delta}} \\
&> \frac{(128\sqrt{3}-1)\sqrt{\pi}}{(76\sqrt{3}+8)\frac{\sqrt{\pi}}{4}} \\
&> 1.
\end{aligned} \tag{26}$$

Hence

$$\begin{aligned}
&2\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)\left(\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)^2 - 1\right)^{-1} \\
&< \frac{2(128\sqrt{3}-1)\sqrt{\pi}}{(76\sqrt{3}+8)\sqrt{\delta}}\left(\left(\frac{(128\sqrt{3}-1)\sqrt{\pi}}{(76\sqrt{3}+8)\sqrt{\delta}}\right)^2 - 1\right)^{-1} \\
&= \frac{2(128\sqrt{3}-1)\sqrt{\pi}}{76\sqrt{3}+8}\left(\left(\frac{(128\sqrt{3}-1)\sqrt{\pi}}{76\sqrt{3}+8}\right)^2 - \delta\right)^{-1}\sqrt{\delta} \\
&< C_2\sqrt{\delta},
\end{aligned}$$

where

$$\begin{aligned}
C_2 &= \frac{2(128\sqrt{3}-1)\sqrt{\pi}}{76\sqrt{3}+8}\left(\left(\frac{(128\sqrt{3}-1)\sqrt{\pi}}{76\sqrt{3}+8}\right)^2 - \frac{\pi}{16}\right)^{-1} \\
&\approx 0.73\cdots.
\end{aligned}$$

Thus, for any positive constant $\varepsilon < \pi/2$, if we have

$$0 < L < \pi, \quad \delta < \frac{\varepsilon^2}{C_2^2}, \quad \delta < \frac{L}{16}, \tag{27}$$

then $R-r \leq \tan(R-r) < \varepsilon$.

The remaining case is when $0 < L < \pi$ and $\delta \geq \frac{L}{16}$. In this case, we take

$$\delta = \frac{\varepsilon^2}{8\pi}. \tag{28}$$

Then we have

$$L \leq 16\delta < \frac{2\varepsilon^2}{\pi} < \varepsilon.$$

Hence $R-r < \varepsilon$.

Now the proof of Theorem 2 is complete from (24), (27), (28) and

$$\min\left\{\frac{\pi}{C_1^2}, \frac{1}{C_2^2}, \frac{1}{8\pi}\right\} = \frac{1}{8\pi}.$$

## §3. Proof of Theorem 3

First we show that $\gamma$ is properly immersed. Since $|k(s)||x(s)| \geq 0$, $\int_0^\infty |k(s)||x(s)|\,ds$ converges to some positive constant $A$. Set $r(s) = |x(s)|$. We have

$$\begin{aligned}
\frac{dr^2}{ds}(s) - \frac{dr^2}{ds}(0) &= \int_0^s \frac{d^2r^2}{ds^2}\,ds \\
&= \int_0^s \left(2\langle\frac{dx}{ds}, \frac{dx}{ds}\rangle + 2\langle x, \frac{d^2x}{ds^2}\rangle\right) ds \\
&\geq \int_0^s (2 - 2|x(s)|\left|\frac{d^2x}{ds^2}\right|)\,ds \\
&\geq 2s - 2A.
\end{aligned}$$

This shows that $\frac{dr^2}{ds} \to \infty$ as $s \to \infty$. Hence $|x(s)| \to \infty$ as $s \to \infty$. A similar argument shows that $|x(s)| \to \infty$ as $s \to -\infty$. Thus $\gamma$ is properly immersed.

As the second step, we show that $\lim_{s\to\infty} x^\perp(s)$ and $\lim_{s\to-\infty} x^\perp(s)$ exist, where $x^\perp(s) = x(s) - \langle x(s), \frac{dx}{ds}\rangle\frac{dx}{ds}$. It follows from $\int_0^\infty |k(s)||x(s)|\,ds = A$ that, for any $\varepsilon > 0$, there exists $s_0 > 0$ such that $\int_{s_1}^{s_2} |k(s)||x(s)|\,ds < \varepsilon$ for any $s_1, s_2 \geq s_0$. Since

$$\begin{aligned}
\left|\frac{dx^\perp}{ds}\right|^2 &= \left|\frac{dx}{ds} - \langle\frac{dx}{ds}, \frac{dx}{ds}\rangle\frac{dx}{ds} - \langle x(s), \frac{d^2x}{ds^2}\rangle\frac{dx}{ds} - \langle x(s), \frac{dx}{ds}\rangle\frac{d^2x}{ds^2}\right|^2 \\
&= \langle x(s), \frac{d^2x}{ds^2}\rangle^2 + \langle x(s), \frac{dx}{ds}\rangle^2|k(s)|^2 \\
&= |k(s)|^2|x(s)|^2,
\end{aligned}$$

we have

$$\begin{aligned}
|x^\perp(s_2) - x^\perp(s_1)| &= \left|\int_{s_1}^{s_2} \frac{dx^\perp}{ds}\, ds\right| \\
&\le \int_{s_1}^{s_2} \left|\frac{dx^\perp}{ds}\right| ds \\
&= \int_{s_1}^{s_2} |k(s)||x(s)|\, ds \\
&< \varepsilon.
\end{aligned}$$

This shows that $\lim_{s\to\infty} x^\perp(s)$ exists. Similarly, $\lim_{s\to-\infty} x^\perp(s)$ exists.

Set $x^\perp_\infty = \lim_{s\to\infty} x^\perp(s)$ . Now we show that $\lim_{s\to\infty} \langle x^\top(s), x^\perp_\infty\rangle = 0$, where $x^\top(s) = \langle x(s), \frac{dx}{ds}\rangle \frac{dx}{ds}$. Let $\theta(s)$ be the angle between $x^\perp(s)$ and $x^\perp_\infty$. Then we have

$$\left|\frac{d\theta}{ds}\right| = |k(s)|$$

and

$$\begin{aligned}
|\langle x^\top(s), x^\perp_\infty\rangle| &\le |x(s)||x^\perp_\infty||\sin\theta(s)| \\
&\le |x(s)||x^\perp_\infty||\theta(s)|.
\end{aligned}$$

As above, let $s_0$ be a positive constant such that $\int_{s_0}^{\infty} |k(s)||x(s)|\, ds < \varepsilon$. For any $s \ge s_0$ we have

$$\begin{aligned}
|x(s)||\theta(s)| &\le |x(s)| \int_s^\infty \left|\frac{d\theta}{dt}\right| dt \\
&= |x(s)| \int_s^\infty |k(t)|\, dt \\
&\le \int_s^\infty |k(t)|\,|x(t)|\, dt \\
&< \varepsilon.
\end{aligned}$$

This gives

$$\lim_{s\to\infty} |x(s)||\theta(s)| = 0.$$

Hence we have

$$\lim_{s\to\infty} \langle x^\top(s), x^\perp_\infty\rangle = 0,$$

which implies that

$$\lim_{s\to\infty} \langle x(s) - x^\perp_\infty, x^\perp_\infty\rangle = 0.$$

This shows that the straight line $\ell := \{y : \langle y - x_\infty^\perp, x_\infty^\perp\rangle = 0\}$ is an asymptotic line of $\gamma$.

## References


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