STABILITY OF SOLITARY WAVES FOR THE ZAKHAROV EQUATIONS IN ONE SPACE DIMENSION

OHTA, Masahito

数理解析研究所講究録 (1995), 908: 148-158

1995-05

http://hdl.handle.net/2433/59494

Departmental Bulletin Paper

Kyoto University
STABILITY OF SOLITARY WAVES FOR THE ZAKHAROV EQUATIONS IN ONE SPACE DIMENSION

東京大学大学院数理科学研究科
太田 雅人（Masahito OHTA）

1. INTRODUCTION AND RESULT

We consider here the stability of solitary waves for the Zakharov equations:

\[ \begin{align*}
  i \frac{\partial}{\partial t} u + \frac{\partial^2}{\partial x^2} u &= nu, & t > 0, & x \in \mathbb{R}, \\
  \frac{\partial}{\partial t} n + \frac{\partial}{\partial x} v &= 0, & t > 0, & x \in \mathbb{R}, \\
  \frac{\partial}{\partial t} v + \frac{\partial}{\partial x} n &= -\frac{\partial}{\partial x}|u|^2, & t > 0, & x \in \mathbb{R}
\end{align*} \] (1.1-1.3)

where \( u, n \) and \( v \) are functions on the time-space \( \mathbb{R} \times \mathbb{R} \) with values in \( \mathbb{C}, \mathbb{R} \) and \( \mathbb{R} \), respectively. From (1.2) and (1.3), we have

\[ \frac{\partial^2}{\partial t^2} n - \frac{\partial^2}{\partial x^2} n = \frac{\partial^2}{\partial x^2}|u|^2. \] (1.4)

The system of equations (1.1) and (1.4) was first obtained by Zakharov [20] as a model which describes the propagation of Langmuir turbulence in a plasma. In this system, \( u \) denotes the envelope of the electric field and \( n \) is the deviation of the ion density from its equilibrium. On the other hand, (1.1)–(1.3) was given by Gibbons, Thornhill, Wardrop and ter Haar [4] from a Lagrangian formalism.
It is well known that (1.1)–(1.3) has a two parameter family of solitary wave solutions:

\[
\begin{align*}
u_{\omega,c}(t, x) &= \sqrt{2\omega(1-C^2)} \text{sech} \sqrt{\omega}(x-ct) \cdot \exp i \left( \frac{c}{2} x - \frac{c^2}{4} t + \omega t \right), \\
n_{\omega,c}(t, x) &= -2\omega \text{sech}^2 \sqrt{\omega}(x-Ct), \\
v_{\omega,c}(t, x) &= -2c\omega \text{sech}(\sqrt{\omega}(x-Ct)), 
\end{align*}
\]

where \(\omega > 0\) and \(-1 < c < 1\). Our purpose in this paper is to show the stability of the solitary wave solution given by (1.5)–(1.7) of (1.1)–(1.3) for any \(\omega > 0\) and \(-1 < c < 1\).

There are a large amount of papers concerning the stability and instability of solitary waves for the nonlinear Schrödinger equations (see, e.g., [2, 3, 7, 14, 15, 16, 18, 19]). However, to our knowledge, there are only a few results concerning the stability of solitary waves for coupled systems of Schrödinger equations and other wave equations, except the abstract theory by Grillakis, Shatah and Strauss [8] and our recent results for the coupled nonlinear Schrödinger equations [10] and for the coupled Klein–Gordon–Schrödinger equations [11].

We now state our main result.

**Theorem 1.1.** For any \(\omega > 0\) and \(-1 < c < 1\), the solitary wave solution \((u_{\omega,c}(t), n_{\omega,c}(t), v_{\omega,c}(t))\) of (1.1)–(1.3) is stable in the following sense: for any \(\varepsilon > 0\) there exists a \(\delta > 0\) such that if \((u_0, n_0, v_0) \in X\) verifies

\[
\| (u_0, n_0, v_0) - (u_{\omega,c}(0), n_{\omega,c}(0), v_{\omega,c}(0)) \|_X < \delta,
\]

then the solution \((u(t), n(t), v(t))\) of (1.1)–(1.3) with \((u(0), n(0), v(0)) = (u_0, n_0, v_0)\) satisfies

\[
\inf_{\theta, y \in \mathbb{R}} \| (u(t), n(t), v(t)) - (e^{i\theta} u_{\omega,c}(t, \cdot + y), n_{\omega,c}(t, \cdot + y), v_{\omega,c}(t, \cdot + y)) \|_X < \varepsilon
\]
for any $t \geq 0$, where $X = H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

**Remark 1.2.** For any $(u_0, n_0, v_0) \in X$, there exists a weak solution $(u(\cdot), n(\cdot), v(\cdot)) \in L^\infty([0, \infty); X)$ of (1.1)-(1.3) with $(u(0), n(0), v(0)) = (u_0, n_0, v_0)$ (see C. Sulem and P.L. Sulem [17]). We do not necessarily have the uniqueness and the energy identity. However, by using the method in Ginibre and Velo [5], we can find a weak solution satisfying

$$H(u(t), n(t), v(t)) \leq H(u_0, n_0, v_0), \quad t \geq 0,$$

$$N(u(t)) = N(u_0), \quad t \geq 0,$$

$$P(u(t), n(t), v(t)) = P(u_0, n_0, v_0), \quad t \geq 0,$$

where

$$H(u, n, v) = \int_{-\infty}^{\infty} \left( |\frac{\partial}{\partial x} u|^2 + n|u|^2 + \frac{1}{2} n^2 + \frac{1}{2} v^2 \right) dx,$$

$$N(u) = \int_{-\infty}^{\infty} |u|^2 dx,$$

$$P(u, n, v) = \int_{-\infty}^{\infty} \left( i\bar{u} \frac{\partial}{\partial x} u - nv \right) dx.$$

For the Cauchy problem of the Zakharov equations, see also [1], [12] and [13].

**Remark 1.3.** Recently, Glangetas and Merle [6] proved the strong instability (instability by blow-up) of standing waves of the Zakharov equations in two space dimensions.

In the next section, we give the proof of Theorem 1.1. We apply the variational method introduced by Cazenave and Lions [3] to the coupled
system of the Schrödinger equation and the wave equations as well as in our previous papers [10] and [11]. In [3] they proved the stability of standing waves for some nonlinear Schrödinger equations. By a simple inequality in Lemma 2.3 below, we reduce our problem for the Zakharov equations to the case of the single nonlinear Schrödinger equation.

2. Proof of Theorem 1.1

In what follows, we fix the parameter $c \in (-1, 1)$. First, we briefly recall the proof by Cazenave and Lions [3] for the stability of standing wave solution $u(t, x) = e^{i\omega t} \varphi_{\omega, c}(x)$ of the nonlinear Schrödinger equation:

$$i \frac{\partial}{\partial t} u + \frac{\partial^2}{\partial x^2} u + \frac{1}{1 - c^2} |u|^2 u = 0, \quad t > 0, \quad x \in \mathbb{R},$$

(2.1)

where $\varphi_{\omega, c}(x) = \sqrt{2\omega(1-c^2)} \sech \sqrt{\omega} x$. We consider the minimization problem:

$$I^1(\mu) = \inf \{E^1(u) : u \in H^1(\mathbb{R}), \ N(u) = \mu \},$$

(2.2)

$$E^1(u) = \int_{-\infty}^{\infty} \left( |\frac{\partial}{\partial x} u|^2 - \frac{1}{2(1-c^2)} |u|^4 \right) dx,$$

$$\Sigma^1(\mu) = \{u \in H^1(\mathbb{R}) : E^1(u) = I^1(\mu), \ N(u) = \mu \}.$$

We note that $E^1(u)$ and $N(u)$ are the conserved quantities of (2.1). The following two lemmas are crucial parts to prove the stability of the standing wave of (2.1). We use them in the proof of Theorem 1.1 later.

Lemma 2.1. For any $\omega > 0$, we have

$$\Sigma^1(\mu(\omega)) = \{e^{i\theta} \varphi_{\omega, c}(\cdot + y) : \theta, y \in \mathbb{R} \},$$

where $\varphi_{\omega, c}(x) = \sqrt{2\omega(1-c^2)} \sech \sqrt{\omega} x$ and

$$\mu(\omega) = N(\varphi_{\omega, c}) = 4(1-c^2) \sqrt{\omega}. $$
Lemma 2.2. Let $\mu > 0$. If $\{u_j\} \subset H^1(\mathbb{R})$ satisfies $E^1(u_j) \to I^1(\mu)$ and $N(u_j) \to \mu$, then there exists $\{y_j\} \subset \mathbb{R}$ such that $\{u_j(\cdot + y_j)\}$ is relatively compact in $H^1(\mathbb{R})$.

Lemma 2.2 is proved by using the concentration compactness method introduced by Lions [9]. For the proofs of Lemmas 2.1 and 2.2, see [3]. From the conservation laws of (2.1) and the compactness of any minimizing sequence of (2.2), Lemma 2.2, one can easily show the stability of the set of minimizers $\Sigma^1(\mu)$ for any $\mu > 0$. Moreover, the characterization of the set of minimizers, Lemma 2.1, concludes the stability of the standing wave of (2.1) (for details, see [3]).

Following Cazenave and Lions [3], we consider the following minimization problem:

$$I(\mu) = \inf \{E(u, n, v) : (u, n, v) \in X, \ N(u) = \mu\}, \quad (2.3)$$

$$E(u, n, v) = \int_{-\infty}^{\infty} \left( |\frac{\partial}{\partial x} u|^2 + n|u|^2 + \frac{1}{2} n^2 + \frac{1}{2} v^2 - cnv \right) dx,$$

$$\Sigma(\mu) = \{(u, n, v) \in X : E(u, n, v) = I(\mu), \ N(u) = \mu\},$$

where $X = H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$. We note that

$$E(e^{-icx/2} u, n, v) = H(u, n, v) + cP(u, n, v) + \frac{c^2}{4} N(u). \quad (2.4)$$

The following lemma plays an essential role in the proof of Theorem 1.1.

Lemma 2.3. For any $(u, n, v) \in X$, we have $E^1(u) \leq E(u, n, v)$. Moreover, the equality holds if and only if $n = -(1/(1 - c^2))|u|^2$ and $v = cn$. 
Proof. Since
\[ 0 \leq |u|^2 + (1 - c^2)n = |u|^2 + 2(1 - c^2)n|u|^2 + (1 - c^2)^2n^2, \quad (2.5) \]
we have
\[ E(u, n, v) \geq \int_{-\infty}^{\infty} \left( |\frac{\partial}{\partial x}u|^2 - \frac{1}{2(1-c^2)}|u|^4 + \frac{1}{2}v^2 - cnv \right) dx \]
\[ = \int_{-\infty}^{\infty} \left( |\frac{\partial}{\partial x}u|^2 - \frac{1}{2(1-c^2)}|u|^4 + \frac{1}{2}(cn - v)^2 \right) dx \]
\[ \geq E^1(u). \quad (2.6) \]
From (2.5) and (2.6), we see that the equality holds if and only if
\[ n = -\frac{1}{1-c^2}|u|^2 \quad \text{and} \quad v = cn. \quad \square \]

The following lemma follows immediately from Lemma 2.3.

Lemma 2.4. For any \( \mu > 0 \), we have \( I(\mu) = I^1(\mu) \) and
\[ \Sigma(\mu) = \left\{ (u, n, v) : u \in \Sigma^1(\mu), n = -\frac{1}{1-c^2}|u|^2, v = cn \right\}. \]

Proof. We set
\[ \Sigma^0(\mu) = \left\{ (u, n, v) : u \in \Sigma^1(\mu), n = -\frac{1}{1-c^2}|u|^2, v = cn \right\}. \]
For \( u \in \Sigma^1(\mu) \), we have from Lemma 2.3
\[ I(\mu) \leq E \left( u, -\frac{1}{1-c^2}|u|^2, -\frac{c}{1-c^2}|u|^2 \right) = E^1(u) = I^1(\mu) \leq I(\mu). \]
Thus, we have \( I(\mu) = I^1(\mu) \) and \( \Sigma^0(\mu) \subset \Sigma(\mu) \).
Moreover, for \((u, n, v) \in \Sigma(\mu)\), we have

\[
I(\mu) = I^1(\mu) \leq E^1(u) \leq E(u, n, v) = I(\mu),
\]

which implies that \(u \in \Sigma^1(\mu)\) and \(E(u, n, v) = E^1(u)\). Thus, it follows from Lemma 2.3 that \(\Sigma(\mu) \subset \Sigma^0(\mu)\). Hence, we have \(\Sigma(\mu) = \Sigma^0(\mu)\). □

We note that from (1.5)–(1.7) and Lemma 2.1, we have

\[
e^{-icx/2}u_{\omega,c}(t) \in \Sigma^1(\mu(\omega)),
\]

\[
n_{\omega,c}(t) = -\frac{1}{1-c^2}|u_{\omega,c}(t)|^2, \quad v_{\omega,c}(t) = cn_{\omega,c}(t)
\]

for any \(t \in \mathbb{R}\). Therefore, from Lemma 2.4, in order to show Theorem 1.1, we have only to prove the following proposition.

**Proposition 2.5.** For any \(\mu > 0\), the set

\[
A = \{(e^{icx/2}u, n, v) : (u, n, v) \in \Sigma(\mu)\}
\]

is stable in the following sense: for any \(\varepsilon > 0\) there exists a \(\delta > 0\) such that if \((u_0, n_0, v_0) \in X\) verifies \(\text{dist}((u_0, n_0, v_0), A) < \delta\) then the solution \((u(t), n(t), v(t))\) of (1.1)–(1.3) with \((u(0), n(0), v(0)) = (u_0, n_0, v_0)\) satisfies \(\text{dist}((u(t), n(t), v(t)), A) < \varepsilon\) for any \(t \geq 0\), where

\[
\text{dist}((u, n, v), A) = \inf\{\|(u, n, v) - (u^0, n^0, v^0)\|_X : (u^0, n^0, v^0) \in A\}.
\]

In order to prove Proposition 2.5, we need one lemma concerning the compactness of any minimizing sequence of (2.3).

**Lemma 2.6.** Let \(\mu > 0\). If \(\{(u_j, n_j, v_j)\} \subset X\) satisfies \(E(u_j, n_j, v_j) \to I(\mu)\) and \(N(u_j) \to \mu\), then there exists \(\{y_j\} \subset \mathbb{R}\) such that \(\{(u_j(\cdot + y_j), n_j(\cdot + y_j), v_j(\cdot + y_j))\}\) is relatively compact in \(X\).
Proof. From Lemma 2.3 and our assumption, we have $E^1(u_j) \to I(\mu) = I^1(\mu)$. Thus, from Lemma 2.2, there exists $\{y_j\} \subset \mathbb{R}$ such that 
$\{u_j(\cdot + y_j)\}$ is relatively compact in $H^1(\mathbb{R})$. Moreover, if we put $u_j^0 = u_j(\cdot + y_j)$, $n_j^0 = n_j(\cdot + y_j)$, $v_j^0 = v_j(\cdot + y_j)$, then $\{(u_j^0, n_j^0, v_j^0)\}$ is bounded in $X$. Therefore, for some subsequence (still denoted by the same letter), we have 

$$(u_j^0, n_j^0, v_j^0) \to (u^0, n^0, v^0) \quad \text{weakly in } X,\n$$

$$u_j^0 \to u^0 \quad \text{in } H^1(\mathbb{R}).$$

Since $n^2 + v^2 - 2cnv = (1 - |c|)(n^2 + v^2) + |c|(n - (c/|c|)u)^2$ and $|c| < 1$, we obtain

$$I(\mu) \leq E(u^0, n^0, v^0) \leq \liminf_{j \to \infty} E(u_j^0, n_j^0, v_j^0) = I(\mu),$$

from which it follows that

$$(u_j^0, n_j^0, v_j^0) \to (u^0, n^0, v^0) \quad \text{in } X,$$

and $(u^0, n^0, v^0) \in \Sigma(\mu)$. □

Proof of Proposition 2.5. In what follows, we often extract subsequences without explicitly mentioning this fact. We prove by contradiction. If $\mathcal{A}$ is not stable, then there exist a positive constant $\varepsilon_0$ and sequences $\{(u_{0j}, n_{0j}, v_{0j})\} \subset X$ and $\{t_j\} \subset \mathbb{R}$ such that

$$\text{dist}((u_{0j}, n_{0j}, v_{0j}), \mathcal{A}) \to 0. \quad (2.7)$$

$$\text{dist}((u_j(t_j), n_j(t_j), v_j(t_j)), \mathcal{A}) \geq \varepsilon_0, \quad (2.8)$$
where \((u_j(t), n_j(t), v_j(t))\) is a solution of (1.1)–(1.3) with 
\((u_j(0), n_j(0), v_j(0)) = (u_{0j}, n_{0j}, v_{0j})\). From the conservation laws (1.8)–
(1.10), (2.4) and (2.7), we have
\[
E(e^{-icx/2}u_j(t_j), n_j(t_j), v_j(t_j)) \leq E(e^{-icx/2}u_{0j}, n_{0j}, v_{0j}) \rightarrow I(\mu),
\]
(2.9)
\[
N(e^{-icx/2}u_j(t_j)) = N(u_j(t_j)) = N(u_{0j}) = N(e^{-icx/2}u_{0j}) \rightarrow \mu.
\]
(2.10)
From (2.9), (2.10) and the definition of \(I(\mu)\), we have
\[
E(e^{-icx/2}u_j(t_j), n_j(t_j), v_j(t_j)) \rightarrow I(\mu).
\]
(2.11)
If we put \(u_j^1(x) = e^{-icx/2}u_j(t_j, x), n_j^1(x) = n_j(t_j, x), v_j^1(x) = v_j(t_j, x),\)
then from (2.10), (2.11) and Lemma 2.6, there exists \(\{y_j\} \subset \mathbb{R}\) such that
\[
(u_j^1(\cdot + y_j), n_j^1(\cdot + y_j), v_j^1(\cdot + y_j)) \rightarrow (u^1, n^1, v^1) \text{ in } X
\]
(2.12)
for some \((u^1, n^1, v^1) \in \Sigma(\mu)\). Since we have
\[
u_j^1(x + y_j) = e^{-icx/2}e^{-icy_j/2}u_j(t_j, x + y_j).
\]
it follows from (2.12) that
\[
\text{dist}((u_j(t_j), n_j(t_j), v_j(t_j)), A) \rightarrow 0,
\]
which contradicts (2.8).
Hence, \(A\) is stable. This completes the proof. \(\square\)

3. 追記

この研究集会で講演した後しばらくして、北海道大学理学部数学教室の
小澤 徹先生より Zakharov 方程式の孤立波解の安定性に関して類似の結
果がすでに Y. Wu [21] により発表されていることを教えて頂きました。これ
に記して、小澤先生に感謝いたします。方法としては、線形化作用素
のスペクトル解析を行い、Grillakis, Shatah and Strauss [8] による抽象的
理論を Zakharov 方程式に適用している。証明を比較すると今回上で紹介
したような変分的方法の方が直接的であり、簡単であるようにと思われる。
REFERENCES


