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# Scattering of dislocation in shallow water

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**Abstract**

Interaction of surface waves in both shallow water and deep water with a vertical vortex is studied analytically. A dislocated wave may exist on the vortex and its strength (degree of dislocation) is characterized by the circulation of the vortex and the frequency and speed of the wave. Using an analogy between Aharonov-Bohm effect in the quantum mechanics and this hydrodynamic system, a scattering problem with the incident dislocated wave is solved and scattering amplitudes are derived.

## 1 Shallow water

We consider a scattering problem of shallow water waves of inviscid incompressible fluid by a vertical vortex. The coordinate system is expressed by $(x, y) = \boldsymbol{x}$ in the horizontal direction and by $z$ in the vertical. The velocity and the surface displacement are denoted by $\boldsymbol{v}(t, \boldsymbol{x}, z) = (\boldsymbol{v}_\perp, w)$ and $\eta(t, \boldsymbol{x})$, respectively.

The equation of motion is

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\rho^{-1}\nabla p - \boldsymbol{g}, \tag{1}$$

where $\rho$ is a density of the fluid, $p$ a pressure, $\boldsymbol{g}$ an acceleration due to gravity, $\partial_t = \partial/\partial t$ and $\nabla$ a three-dimensional gradient. The kinematic boundary condition at the surface is given by

$$w = \partial_t \eta + \boldsymbol{v}_\perp \cdot \nabla_\perp \eta \tag{2}$$

where $\nabla_\perp$ is a horizontal gradient.

In the shallow water case we may write the velocity as

$$\boldsymbol{v} = \sum_n \boldsymbol{v}_n(\boldsymbol{x}, t)(\frac{z}{h})^n. \tag{3}$$

The equation of continuity $\nabla \cdot \boldsymbol{v} = 0$ leads to

$$h\nabla_\perp \cdot \boldsymbol{v}_{\perp n} + (n+1)w_{n+1} = 0, \qquad n = 0, 1, \cdots. \tag{4}$$

At the lowest order, the expressions of the velocity and the pressure become

$$\boldsymbol{v} = (\boldsymbol{v}_\perp(\boldsymbol{x}, t), \frac{z}{h}w(\boldsymbol{x}, t)), \tag{5}$$

$$p = p_0(\boldsymbol{x},t) + \frac{z}{h} p_1(\boldsymbol{x},t). \tag{6}$$

From $z$-component of (1) and the pressure condition at the surface, we have

$$p = \rho g(\eta - z) + p_a, \tag{7}$$

where $p_a$ is the atmospheric pressure. The $\boldsymbol{x}$-component of (1) is written as

$$\partial_t \boldsymbol{v}_\perp + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp = -g\nabla_\perp \eta. \tag{8}$$

The kinematic boundary condition leads to

$$\partial_t \eta + h\nabla_\perp \cdot \boldsymbol{v}_\perp + \nabla_\perp(\eta \boldsymbol{v}_\perp) = 0. \tag{9}$$

The existence of vortices with a non-zero total circulation produce a steady solenoidal background flow $\boldsymbol{U}(\boldsymbol{x})$ and a surface deformation $\eta_0(\boldsymbol{x})$. Substituting $\boldsymbol{v}_\perp = \boldsymbol{U}(\boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{x},t)$ and $\eta = \eta_0(\boldsymbol{x}) + \eta_1(\boldsymbol{x},t)$ into (8) and (9) leads to

$$-g\nabla_\perp \eta_0 = (\boldsymbol{U} \cdot \nabla_\perp)\boldsymbol{U} = \frac{1}{2}\nabla_\perp \boldsymbol{U}^2 - \boldsymbol{U} \times \mathrm{rot}\boldsymbol{U}, \tag{10}$$

$$\partial_t \boldsymbol{u} + (\boldsymbol{U} \cdot \nabla_\perp)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla_\perp)\boldsymbol{U} + g\nabla_\perp \eta_1 = -(\boldsymbol{u} \cdot \nabla_\perp)\boldsymbol{u} \tag{11}$$

and

$$\boldsymbol{U} \cdot \nabla_\perp \eta_0 = 0, \tag{12}$$

$$\partial_t \eta_1 + h\nabla_\perp \cdot \boldsymbol{u} + \eta_0 \nabla_\perp \cdot \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla_\perp)\eta_0 + (\boldsymbol{U} \cdot \nabla_\perp)\eta_1 = -\eta_1 \nabla_\perp \cdot \boldsymbol{u} - \boldsymbol{u} \cdot \nabla_\perp \eta_1. \tag{13}$$

(10) and (12) are equations for the background field. Using (10), we can express $\eta_0$ by $\boldsymbol{U}$. Linearizing (11) and (13) with respect to $\boldsymbol{u}$ leads to

$$\partial_t \boldsymbol{u} + (\boldsymbol{U} \cdot \nabla_\perp)\boldsymbol{u} = -(\boldsymbol{u} \cdot \nabla_\perp)\boldsymbol{U} - g\nabla_\perp \eta_1, \tag{14}$$

$$\partial_t \eta_1 + (\boldsymbol{U} \cdot \nabla_\perp)\eta_1 + h\nabla_\perp \cdot \boldsymbol{u} = -[\eta_0 \nabla_\perp \cdot \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla_\perp)\eta_0]. \tag{15}$$

Taking the divergence of (14), we obtain

$$\partial_t \nabla_\perp \cdot \boldsymbol{u} + g\triangle_\perp \eta_1 = -\nabla_\perp \cdot [(\boldsymbol{U} \cdot \nabla_\perp)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla_\perp)\boldsymbol{U}], \tag{16}$$

where $\triangle_\perp = \nabla_\perp^2$. Using an alternative expression of the right hand side of (16),

$$\nabla_\perp \cdot [(\boldsymbol{U} \cdot \nabla_\perp)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla_\perp)\boldsymbol{U}] = 2(\partial_i U_j)(\partial_j u_i) + (\boldsymbol{U} \cdot \nabla_\perp)(\nabla_\perp \cdot \boldsymbol{u}),$$

we have

$$D_t \nabla_\perp \cdot \boldsymbol{u} + g\triangle_\perp \eta_1 = -2(\partial_i U_j)(\partial_j u_i), \tag{17}$$

where $D_t = \partial_t + \boldsymbol{U}\cdot\nabla_\perp$ Taking $D_t(15) - h \times (17)$, we have

$$D_t^2\eta_1 - c^2\triangle_\perp\eta_1 = -D_t[\eta_0\nabla_\perp\cdot\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla_\perp)\eta_0] + 2h(\partial_i U_j)(\partial_j u_i), \tag{18}$$

where $c = \sqrt{gh}$ is a phase velocity of shallow water waves.

We consider the case that the Mach number $M = U/c$ is much smaller than 1. The square of $M$ is also called Froude number. We denote a typical length scale of vortex by $a$ and a wavelength and a frequency of shallow water waves $\lambda$ and $f$ where $c = \lambda f$. We also assume that $a/\lambda \equiv \beta \gg 1$. Then the right hand side of (18) will be order of $M$ or $\beta^{-1}$ compared with the left hand side. Neglecting them, we have a final equation

$$D_t^2\eta_1 - c^2\triangle_\perp\eta_1 = 0. \tag{19}$$

The localized vortex with the circulation $\Gamma$ produces the background flow $\boldsymbol{U} \approx \Gamma/(2\pi r)\hat{\theta}$. The assumption $\beta \ll 1$ is complementary to the condition for the Born approximation (e.g. Kambe (1982)) to hold.

## 2 Deep water

We may treat the scattering of dislocated waves in deep water as follows. First, we write the velocity, the surface displacement and the pressure as

$$\boldsymbol{v} = (\boldsymbol{U} + \boldsymbol{u}, w), \tag{20}$$

$$\eta = \eta_0 + \eta_1, \tag{21}$$

$$p = P + p_1, \tag{22}$$

where $\boldsymbol{U}$, $\eta_0$ and $P$ denote a steady field due to a vertical vortex and depend only on $\boldsymbol{x}$, and surface-wave components $\boldsymbol{u}$, $w$, $\eta_1$ and $p_1$ are functions of $\boldsymbol{x}, z$, and $t$. The steady flow field satisfies the same equations as (7), (10) and (12).

The linearized equations of motion are given by

$$\partial_t\boldsymbol{u} + (\boldsymbol{U}\cdot\nabla_\perp)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla_\perp)\boldsymbol{U} = -\rho^{-1}\nabla_\perp p_1, \tag{23}$$

$$\partial_t w + \boldsymbol{U}\cdot\nabla_\perp w = -\rho^{-1}\partial_z p_1, \tag{24}$$

and the equation of continuity is

$$\nabla_\perp\boldsymbol{u} + \partial_z w = 0. \tag{25}$$

The boundary condition at the surface is

$$w = \partial_t\eta_1 + \boldsymbol{U}\cdot\nabla_\perp\eta_1 + \boldsymbol{u}\cdot\nabla_\perp\eta_0 \tag{26}$$

and

$$p(z=\eta)=p_a. \tag{27}$$

Here we may assume that the wave components are given in a form of separation of variables as

$$\boldsymbol{u}=\boldsymbol{u}_0(\boldsymbol{x},t)+\hat{\boldsymbol{u}}(\boldsymbol{x},t)\cosh k(z+h), \tag{28}$$

$$w=\hat{w}(\boldsymbol{x},t)\sinh k(z+h), \tag{29}$$

$$p_1=p_0(\boldsymbol{x},t)+\hat{p}(\boldsymbol{x},t)\cosh k(z+h). \tag{30}$$

Substituting these into the equations of motion leads to

$$D_t\boldsymbol{u}_0+(\boldsymbol{u}_0\cdot\nabla_\perp)\boldsymbol{U}=-\rho^{-1}\nabla_\perp p_0, \tag{31}$$

$$D_t\hat{\boldsymbol{u}}+(\hat{\boldsymbol{u}}\cdot\nabla_\perp)\boldsymbol{U}=-\rho^{-1}\nabla_\perp\hat{p}_1, \tag{32}$$

$$D_t\hat{w}=-\rho^{-1}k\hat{p}_1. \tag{33}$$

The continuity equation reduces to

$$w=-k^{-1}\nabla_\perp\cdot\hat{\boldsymbol{u}}, \tag{34}$$

$$\nabla_\perp\cdot\boldsymbol{u}_0=0. \tag{35}$$

Thus the flow $\boldsymbol{u}_0$ is incompressible. The kinematic boundary condition is replaced by

$$\hat{w}(\boldsymbol{x})\sinh kh=D_t\eta_1+(\boldsymbol{u}_0+\hat{\boldsymbol{u}}\cosh kh)\cdot\nabla_\perp\eta_0, \tag{36}$$

where we assumed $k\eta\ll 1$. The pressure condition becomes

$$-\rho g\eta_1+p_0+\hat{p}_1\cosh kh=0. \tag{37}$$

Here we assume $\beta\gg 1$ and $M\ll 1$. Then we can neglect the second term in the left hand side of (31), (32) and in the right hand side of (36). Under these assumption, it can be shown that $p_1$ satisfies the Laplace equation. Then we have $\triangle_\perp p_0=0$. Taking the laplacian of (37) leads to

$$\triangle_\perp\hat{p}_1=(\rho g/\cosh kh)\triangle_\perp\eta_1. \tag{38}$$

Taking the divergence of (32), neglecting $O(\beta^{-1})$ term, and eliminating $\nabla_\perp\cdot\boldsymbol{u}$ by using (34) and (36), we have the same equation as (19) with $c=\sqrt{(g/k)\tanh kh}$, which is a phase velocity of deep water waves.

## 3 Dislocated wave and scattering

As is already pointed out by Berry et al. (1980) and Cerda and Lund (1993), the equation (19) possesses a close analogy to the well-known quantum mechanics of Aharonov-Bohm effect, in which a potential gives physical effects without accessible electromagnetic fields. It is a scattering problem of a beam of particles with charge $q$ and mass $m$ incident normally on a long thin cylinder containing a magnetic field $\boldsymbol{B}(\boldsymbol{x})$ parallel to its axis. The Schrödinger equation in the presence of the magnetic vector potential $\boldsymbol{A}$ due to the magnetic field is given by

$$\frac{1}{2m}(-i\hbar\nabla - q\boldsymbol{A}(\boldsymbol{x}))^2\psi(\boldsymbol{x}) = \frac{\hbar^2 k^2}{2m}\psi(\boldsymbol{x}), \tag{39}$$

where $\hbar$ is a Plank constant, $\boldsymbol{A}(\boldsymbol{x}) = (\Phi/2\pi r)\hat{\theta}$, $\Phi$ is a magnetic flux and $\hat{\theta}$ is an azimuthal unit vector.

Equations (19) and (39) have a solution of a dislocated wave of the form $\exp[-i(\boldsymbol{k}\cdot\boldsymbol{x} + \alpha\theta + \nu t)]$, where $\alpha = \nu\Gamma/(2\pi c^2)$ in the fluid mechanics and $\alpha = -q\Phi/h, (h = 2\pi\hbar)$ in the quantum mechanics. It is noted that, while this dislocated wave is an exact solution in quantum mechanics, it is an approximate one in the water wave problem valid if $M \ll 1$. We are now interested in the case that the effect of dislocation is significant, i.e. $\alpha = \mathrm{O}(1)$. Using $\Gamma = 2\pi U a$, we have a relation $\alpha = 2\pi M\beta$.

As an example, we consider a scattering problem by a circular uniform vortex with vorticity $\omega$ and a radius $a$ surrounded by an irrotational flow. Using polar coordinates $(r, \theta)$, the background flow is given by

$$\boldsymbol{U} = \frac{1}{2}\omega r\hat{\theta}, \qquad r \le a; \qquad \frac{\Gamma}{2\pi r}\hat{\theta}, \qquad r > a; \qquad \Gamma = \pi\omega a^2. \tag{40}$$

Inside the vortex we have from (19)

$$[(\partial_t + (\omega/2)\partial_\theta)^2 - c^2(\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\theta^2)]\eta_1 = 0. \tag{41}$$

Assuming the solution of the form $\eta_1 = \sum_n \tilde{\eta}_{1n} \mathrm{e}^{i(n\theta - \nu t)}$, we obtain

$$(\partial_r^2 + \frac{1}{r}\partial_r - \frac{n^2}{r^2} + k_n^2)\tilde{\eta}_{1n} = 0, \qquad k_n = \frac{|\nu - n\omega/2|}{c}. \tag{42}$$

Equation (42) has solutions of Bessel and Neumann functions. The non-singularity at the origin will exclude the latter. Thus we have

$$\eta_1 = \sum_n a_n J_{|n|}(k_n r)\mathrm{e}^{i(n\theta - \nu t)}. \tag{43}$$

Outside the vortex, $r > a$, the assumption $U^2/c^2 \ll 1$ may reduce (19) into

$$[\partial_t^2 + \frac{\Gamma}{\pi r^2}\partial_\theta\partial_t - c^2(\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\theta^2)]\tilde{\eta}_{1n} = 0. \tag{44}$$

Assuming the above form of solutions, we have

$$(\partial_r^2 + \frac{1}{r}\partial_r - \frac{n^2+2n\alpha}{r^2} + k^2)\tilde{\eta}_{1n} = 0, \qquad k = \frac{\nu}{c}. \tag{45}$$

Since $\partial_r \eta_1 \gg \alpha^2 \eta_1 / r$, we may replace (45) by

$$(\partial_r^2 + \frac{1}{r}\partial_r - \frac{(n+\alpha)^2}{r^2} + k^2)\tilde{\eta}_{1n} = 0. \tag{46}$$

The replacement of (45) by (46) is based on the assumption $n \gg \alpha$. Since $(1/r)\partial_\theta \eta \approx (n/a)\eta \approx k\eta$, the representative value of $n \approx ka = a/\lambda$ is much larger than $\alpha = O(1)$. Then the surface elevation can be given in the form of

$$\eta_1 = \mathrm{Re}(\psi_{AB} + \psi_R), \tag{47}$$

where Re denotes a real part and

$$\psi_{AB} = \sum_n b_n J_m(kr) \mathrm{e}^{i(n\theta - \nu t)}, \qquad m = |n+\alpha|, \tag{48}$$

$$\psi_R = \sum_n c_n H_m^1(kr) \mathrm{e}^{i(n\theta-\nu t)}. \tag{49}$$

In order to obtain the coefficients $a_n$, $b_n$ and $c_n$, we first require the continuity of $\eta$ and $\nabla_\perp \eta$ at $r = a$. It gives two relations:

$$a_n J_{|n|}(k_n a) = b_n J_m(ka) + c_n H_m^1(ka), \tag{50}$$

$$a_n k_n J'_{|n|}(k_n a) = k(b_n J'_m(ka) + c_n H_m^{1'}(ka)). \tag{51}$$

The last condition comes from that the asymptotics of $\psi_{AB}$ should coincide with the dislocated wave, which leads to

$$b_n = (-i)^m = \mathrm{e}^{-i\pi|n+\alpha|/2}. \tag{52}$$

The limit $r \to \infty$ gives the wave function in the form

$$\psi_{AB} \to \mathrm{e}^{i(-kr\cos\theta - \alpha\theta - \nu t)} - \frac{\mathrm{e}^{i(kr-\nu t)}\sin\pi\alpha}{(2\pi i k r)^{1/2}\cos(\theta/2)}(-1)^{[-\alpha]}\mathrm{e}^{i([-\alpha]+1/2)\theta}, \tag{53}$$

$$\psi_R \to [\frac{2}{\pi i k r}]^{1/2} \mathrm{e}^{i(kr-\nu t)} \sum_n c_n \mathrm{e}^{i(n\theta - \pi|n+\alpha|/2)}, \tag{54}$$

where $[x]$ is a notation of Gauss. If $\alpha$ is an integer, the second term of (53) vanishes. Figure 1 shows the dislocated wave given by (48) and (52). This asymptotic is valid except in a narrow sector centered at $\theta = \pi$, where we cannot separate $\psi_{AB}$ into incident and scattered waves. The scattered wave $\psi_S$ can be defined by the sum of the second term of (53) and (54). The general asymptotic form of $\psi_S$ is

$$\psi_S \sim f(\theta) r^{-1/2} \mathrm{e}^{i(kr-\nu t)}, \tag{55}$$

where $f(\theta)$ is a scattering amplitude

$$f(\theta) = \frac{1}{\sqrt{2\pi i k}} \tilde{f}(\theta),$$

$$\tilde{f}(\theta) = -\frac{\sin \pi\alpha}{\cos(\theta/2)}(-1)^{[-\alpha]} \mathrm{e}^{i([-\alpha]+1/2)\theta} + 2\sum_n c_n \mathrm{e}^{i(n\theta-\pi|n+\alpha|/2)}. \tag{56}$$

The coefficients $c_n$ are

$$c_n = b_n[-(k_n/k)J'_{|n|}(k_n a)J_m(ka) + J_{|n|}(k_n a)J'_m(ka)]/\Delta, \tag{57}$$

where

$$\Delta = (k_n/k)J'_{|n|}(k_n a)H^1_m(ka) - J_{|n|}(k_n a)H^{1'}_m(ka). \tag{58}$$

Using a notation $\gamma_n = |1-\delta n|$, with $\delta = \omega/2\nu = M/(2\pi\beta) = \alpha/(4\pi^2\beta^2)$, we may simplify the formula of $c_n$ as

$$c_n = -(-i)^{|n+\alpha|} \frac{\gamma_n J'_{|n|}(\beta\gamma_n)J_m(\beta) - J_{|n|}(\beta\gamma_n)J'_m(\beta)}{\gamma_n J'_{|n|}(\beta\gamma_n)H^1_m(\beta) - J_{|n|}(\beta\gamma_n)H^{1'}_m(\beta)}. \tag{59}$$

The coefficients $c_n$ are parametrized by only two dimensionless numbers $\alpha$ and $\beta$. Figure 2 shows absolute values of $c_n$ versus $n$ for $\alpha = 1$ and $\beta = 0.1, 1, 5$ and 10. We can evaluate the convergence of the sum (49) from this figure.

According to the scattering theory, the differential cross section may be defined by

$$\frac{d\sigma}{d\theta} = |f(\theta)|^2 = \frac{1}{2\pi k}|\tilde{f}(\theta)|^2. \tag{60}$$

Figure 3 shows polar plot of $|\tilde{f}(\theta)|^2$ for $\alpha = 1$ and $\beta = 10$. It is extremely anisotropic; the amplitude is very large in the forward direction ($\theta \approx \pi$) and oscillates in the backward direction.

# 4 References


M. V. Berry, R. G. Chambers, M. D. Large, C. Upstill and J. C. Walmsley, (1980) Wavefront dislocations in the Aharonov-Bohm effect and its water wave analogue, *Eur. J. Phys.* **1** pp. 154-162.

E. Cerda and F. Lund, (1993) Interaction of surface waves with vorticity in shallow water, *Phys. Rev. Lett.* **70** pp. 3896-3899.

T. Kambe, (1982) Scattering of sound waves by vortex systems, (in Japanese) *Nagare* **1** pp. 149-165.

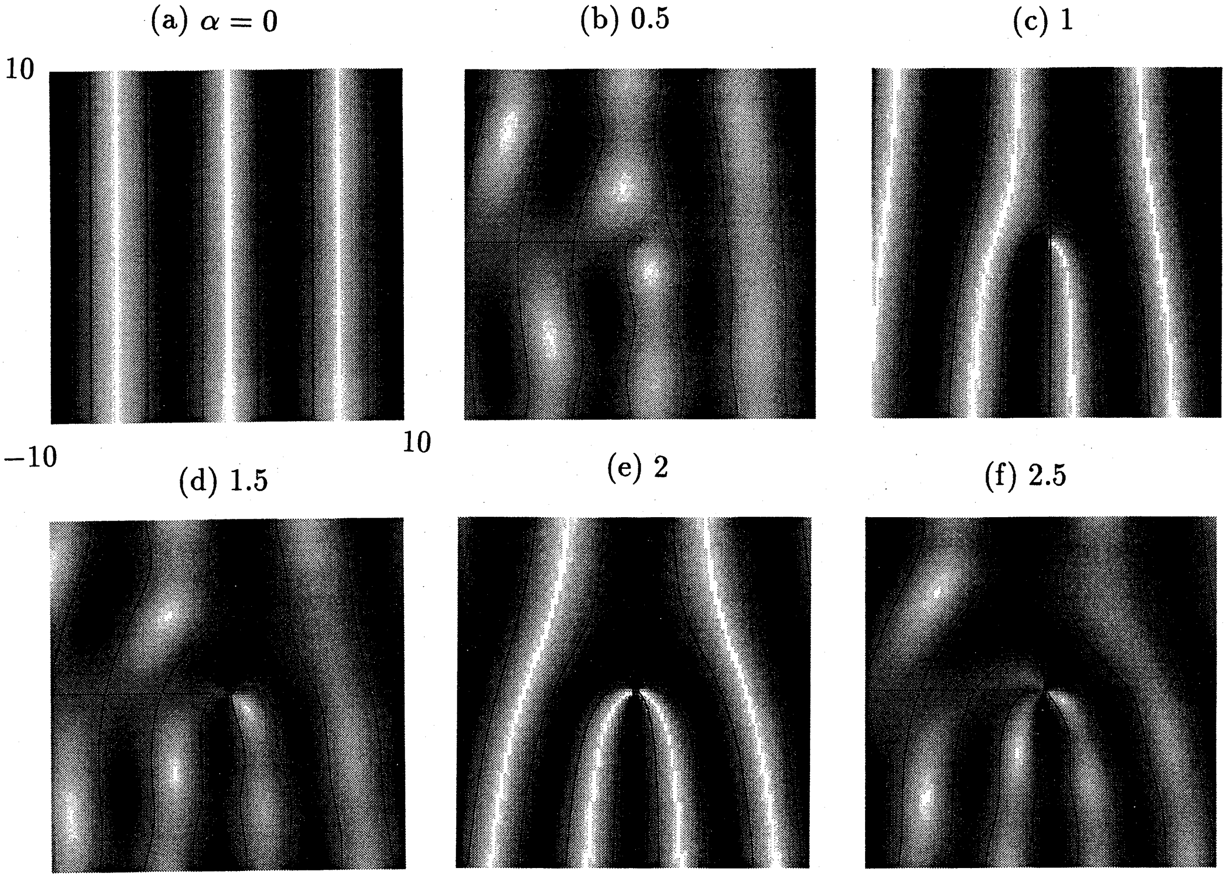


Figure 1. Density plots of incident dislocated waves for $\alpha =$ (a) 0, (b) 0.5, (c) 1, (d) 1.5, (e) 2 and (f) 2.5 The summation in (48) is truncated at $n = \pm 20$. The plotted region is $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$. Lines denote the zero displacement.

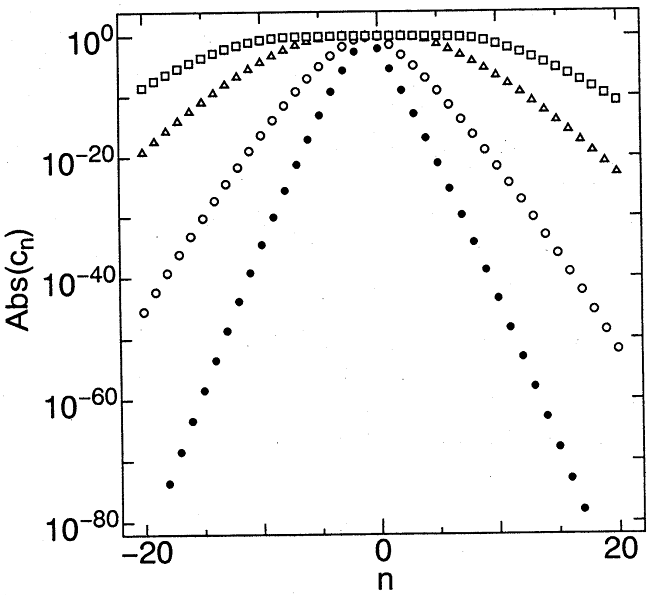


Figure 2. Absolute values of $c_n$ versus $n$ for $\alpha = 1$ and $\beta = 0.1$ (denoted by solid circle), 1 (open circle), 5 (triangle) and 10 (square).

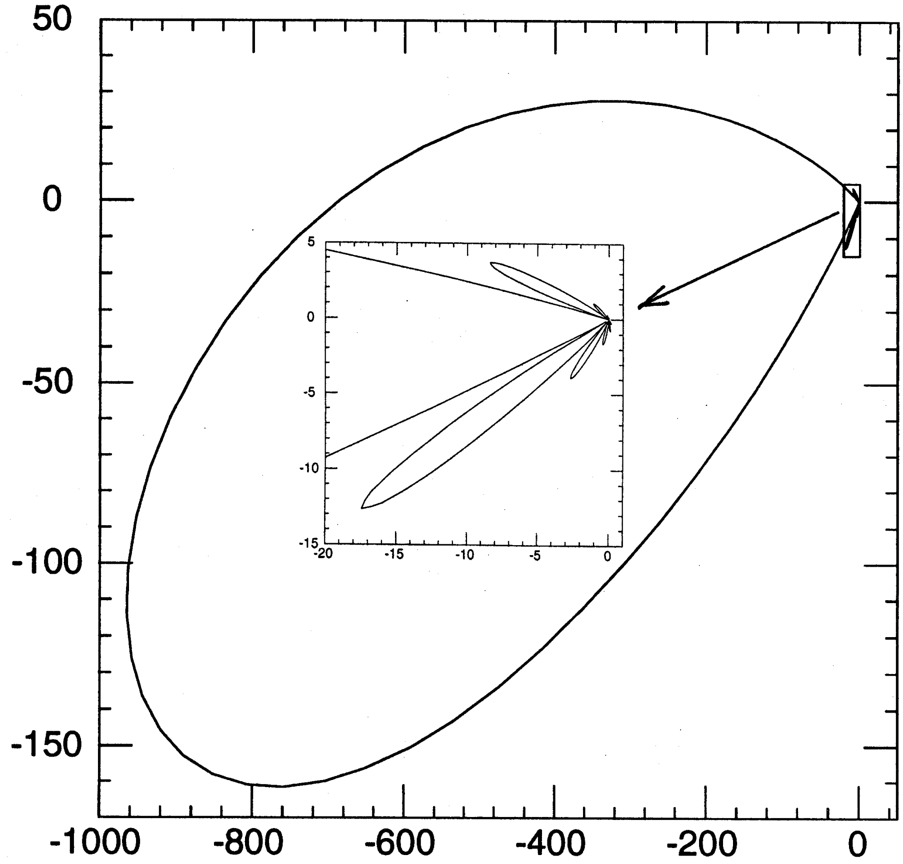


Figure 3. Polar plot of differential cross section for $\alpha = 1$ and $\beta = 10$. The incident wave comes from the right side.