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Inverse semigroups and permutation properties

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The purpose of this talk is to introduction a theorem on permutation properties of inverse semigroups, appearing in Okniński's book[2] and to give a comment on the proof of a lemma for the theorem.

Definition. A semigroup S has the permutation property $\mathcal{B}_n$ if there exists an integer $n \geq 2$ such that $w_1 w_2 \ldots w_n = w_{\sigma(1)} w_{\sigma(2)} \ldots w_{\sigma(n)}$ for some $\sigma \neq 1 \in S_n$.

Definition. A ring R satifies a polynomial identity $p(x_1, x_2, \ldots x_n)$ if all coefficients of $p(x_1, x_2, \ldots x_n)$ are $\pm$ 1 and $p(r_1, r_2, \ldots r_n) = 0$ for all $r_i \in R$.
In this case, R is called a PI-ring.

Problem [3, Restivo and Reutenauer].

Does the semigroup ring $k[S]$ of a semigroup S has the permutation property satisfy a polynomial identity?

Theorem ([2, Theorem 23]). Let S be an inverse semigroup. Then the following are eqivalent :

(1) S is finitely generated and satisfies the permutation property.
(2) S has finitely many idempotents, and all subgroups of S are finitely generated and abelian-by-finite.
(3) $K[S]$ is a left and right noetherian PI-algebra.

A proof of the theorem is based on Shirshov's results concerning combinatorics on words, Blyth's results concerning groups with the permutation property, and structure theorems in semigroup ring theory.

We shall give a semigroup theoretical proof of the following lemma used for the proof of the theorem above.

Lemma ([2, Lemma 22]). Let S be a finitely generated inverse semigroup. If S has the permutation property, then S has finitely many idempotents.

Proof. Let $a \in S$. Then the principal factor semigroup $S_a = \mathcal{J}_a/I(a)$ is a 0-simle (or simple) semigroup (see [1]). By [2, Theorem 17], S_a is a completely 0-simple (or simple) semigroup. Here we assume that S has $\mathcal{B}_n$, where n is a positive integer. By [2, Proposition 19], the number of $\mathcal{R}$ $[\mathcal{L}]$ of S_a is less than $m = \frac{n}{2}$. Thus, each $\mathcal{D}$ of S has at most m $\mathcal{R}$ $[\mathcal{L}]$-classes. By the way of Shüzenberger representation, S is embedded in the direct product of row-monomial or column-monomial matrix semigroups S_i of less than m over groups G_i with zero. Then each S has idempotent-separating congruence ρ such that S/ρ is embedded in the direct product of row-monomial or column-monomial matrix semigroups S_i of rank less than m over a single-element groups $\{e\}$ with zero. Then for any $s \in S$, $s^{m!}$ is an idempotent. So, S/ρ is periodic. Thus, by Restivo and Reutenauer's result, S/ρ has only finitely many idempotents, and so does S.

Remark. In the proof above, row-monomial matrix semigroups of rank m over a single-element group are the partial transformation semigroups of a set of m elements.

references

[1] Clifford, A.H. & G.B. Preston, Algebraic thoery of semigroups, Vol.I, Mathematical Survey, No.7, Amer. Math. Soc. Providence, R. I., 1961.

[2] Okniński, Semigroup algebras, Marcel Dekker, 1991.

[3] Restivo A. and C. Reutenauer, On the Burnside problem for semigroups, J. Algebra 89(1984), 102-104.