REMARKS ON ISOMORPHISMS OF REGRESSIVE
TRANSFORMATION SEMIGROUPS

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For a (finite or infinite) set $X$, let $T(X)$ be the full transformation semigroup on $X$, i.e. the set of all maps from $X$ to $X$, the semigroup operation being composition of maps.

When $X$ is a partially ordered set, we let
\[ T_{\mathrm{RE}}(X) = \{ f \in T(X) \mid f(x) \leq x \text{ for all } x \in X \}, \]
\[ T_{\mathrm{OP}}(X) = \{ f \in T(X) \mid f(x) \leq f(y) \text{ if } x \leq y \text{ for } x, y \in X \}. \]

Then, both of them are subsemigroups of $T(X)$ with the identity $\text{id}_{T(X)}$. We call $T_{\mathrm{RE}}(X)$ the full regressive transformation semigroup on $X$, and $T_{\mathrm{OP}}(X)$ the full order-preserving transformation semigroup on $X$.

Recently, some interesting results on $T_{\mathrm{RE}}(X)$ have been obtained (cf. [1], [4], [5]).

It is known that, for partially ordered sets $X$, $Y$, if $T_{\mathrm{OP}}(X)$ and $T_{\mathrm{OP}}(Y)$ are isomorphic as semigroups, then $X$ and $Y$ are isomorphic or anti-isomorphic as ordered sets (see [3], Theorem V.8.9).

It is natural to ask whether the above result holds or not for regressive transformation semigroups. In general, it does not hold. However, we obtain a necessary and sufficient condition on partially ordered sets $X$ and $Y$ for $T_{\mathrm{RE}}(X)$ and $T_{\mathrm{RE}}(Y)$ to be isomorphic.

Umar showed in [6] that, when $X$ and $Y$ are totally ordered sets, any idempotent in $T_{\mathrm{RE}}(X)$ whose image is an order-ideal is mapped to an idempotent in $T_{\mathrm{RE}}(Y)$ with the same property by isomorphisms from $T_{\mathrm{RE}}(X)$ to $T_{\mathrm{RE}}(Y)$, and he considered the above problem through this result. If the result holds even if "an order-ideal" in it is replaced by "a principal order-ideal", then it can be shown that if $T_{\mathrm{RE}}(X) \cong T_{\mathrm{RE}}(Y)$ as semigroups then $X \cong Y$ as ordered sets. At the present time, this is unsolved.

In here, we achieve our purpose by showing that any idempotent of defect 1 in $T_{\mathrm{RE}}(X)$ is mapped to an idempotent of defect 1 in $T_{\mathrm{RE}}(Y)$ by isomorphisms from $T_{\mathrm{RE}}(X)$ to $T_{\mathrm{RE}}(Y)$, where the defect of $\alpha$ in $T_{\mathrm{RE}}(X)$ means the cardinality of the set of idempotents in $X$ which do not belong to the image of $\alpha$.

For partially ordered set $X$, an element in $X$ is said to be isolated if it is incomparable with every element in $X$ except itself. Let $\text{Is}(X)$ be the set of all isolated elements in $X$. Then, it is easy to see that $T_{\mathrm{RE}}(X)$ and $T_{\mathrm{RE}}(X \setminus \text{Is}(X))$ are isomorphic. Therefore, we may assume that every partially ordered set, treated in this paper, does not contain any isolated elements.

Let $X$ be a partially ordered set under the order relation $\leq$.

For $a \in X$, the set of (resp. strict) upper bounds of $a$ is denoted by $U(a)$ (resp. $SU(a)$), i.e.
\[ U(a) = \{ x \in X \mid x \geq a \} \quad \text{and} \quad SU(a) = \{ x \in X \mid x > a \}, \]
and the set of all minimal elements in $X$ is denoted by $\text{Min}(X)$.

This is an abstract and the details will be published in Semigroup Forum.
Lemma 2. (1) $k(a, b) = k(c, d)$ if and only if $b = d$.
(2) If $a < c < b$, then $k(a, c) = j(c, b)$ and $j(a, b) = j(a, c)$.
(3) $j(a, b) = j(c, d)$ if and only if $a = c$.

Proof. (1) It is easy to see that
$$b = d \iff \lambda^b c \circ \lambda^b a = \lambda^b a \iff \lambda^{k(c, d)} j(c, d) \circ \lambda^{k(a, b)} j(a, b) = \lambda^{k(a, b)} j(a, b) \iff k(a, b) = k(c, d).$$
This assertion means that $k(a, b)$ depends only on $b$.

(2) The proof is omitted.

(3) To show the assertion, we need that $X$ and $Y$ are adjusted. Let $a = c$. If $a$ is not minimal in $X$, then $e < a$ for some $e \in X$. From (2), we have that $j(a, b) = k(e, a) = j(a, d) = j(c, d)$.

If $a$ is maximal in $X$, then $b$ and $d$ are connected in $SU(a)$, since $X$ is adjusted, so that there exist $e_1, e_2, \ldots, e_n \in SU(a)$ such that $b = e_1 \leq e_2 \leq \ldots \leq e_n = d$. Since $e_i$ and $e_{i+1}$ are comparable, by (2) we have that $j(a, e_i) = j(a, e_{i+1}) (i = 1, 2, \ldots, e_{n-1})$. Thus, we have that $j(a, b) = j(c, d) = j(a, d)$.

Let $j(a, b) = j(c, d)$. If we apply the above fact to $j'$, then we have that $a = j'(j(a, b), k(a, b)) = j'(j(c, d), k(c, d)) = c$.

This assertion means that $j(a, b)$ depends only on $a$.

We write $j(a, b) = j(a)$ and $k(a, b) = k(b)$ for $a, b \in X$ with $a < b$. In this case, if $a$ is maximal in $X$, then $j(a)$ is undefined, and if $b$ is minimal in $X$, then $k(b)$ is undefined. Since $j(a) < k(b)$ if $a < b$, we have that if $a$ is not maximal in $X$, then neither is $j(a)$ in $Y$. By (2) of Lemma 2, if $c$ is neither maximal nor minimal in $X$, then $j(c) = k(c)$.

Similarly, we write $j'(a', b') = j'(a')$ and $k'(a', b') = k'(b')$ for $a', b' \in Y$ with $a' < b'$. Then, we have that $j'(j(a)) = a', k'(k(b)) = b, j'(j'(a')) = a'$ and $k'(k'(b')) = b'$.

Let $a$ be maximal in $X$. Then, we can show that $k(a)$ is maximal in $Y$.

Define a map $h : X \to Y$ by $h(a) = j(a)$ if $a$ is not maximal in $X$, and $h(a) = k(a)$ if $a$ is maximal in $X$. Then, we can show that the $h$ is an order-isomorphism of $X$ onto $Y$.

Since any totally ordered set is clearly adjusted, we obtain:

Corollary 3.
Let $X$ and $Y$ be totally ordered sets. Then, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic as semigroups if and only if $X$ and $Y$ are isomorphic as ordered sets.

Let $X, Y$ be partially ordered sets. From Theorem 1 and Theorem 2, we have that
$$T_{RE}(X) \cong T_{RE}(Y) \iff T_{RE}(A(X)) \cong T_{RE}(A(Y)) \iff A(X) \cong A(Y).$$
Thus, we obtain the following main theorem:

Theorem 4.
Let $X$ and $Y$ be partially ordered sets. Then, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic as semigroups if and only if their adjusted sets $A(X)$ and $A(Y)$ are isomorphic as ordered sets.
and the set of all minimal elements in $X$ is denoted by $\text{Min}(X)$.

Let $\preceq$ be the symmetric relation generated by $\leq$, i.e., $a \preceq b$ if and only if $a \leq b$ or $b \leq a$, and let $\simeq$ be the equivalent relation generated by $\leq$, i.e., $a \simeq b$ if and only if there exist $c_1, c_2, \ldots, c_n \in X$ such that $a = c_1 \simeq c_2 \simeq \ldots \simeq c_n = b$ (see [2], I). In this case, we say that $a$ and $b$ are connected in $X$. A subset $Y$ of $X$ is connected if every $a, b \in Y$ are connected in $Y$, i.e., there exist $c_1, c_2, \ldots, c_n \in Y$ such that $a = c_1 \simeq c_2 \simeq \ldots \simeq c_n = b$.

A partially ordered set $X$ is said to be adjusted if it does not contain any minimal elements, or for every $m \in \text{Min}(X)$, $SU(m)$ is connected.

**Theorem 1.**

Let $X$ be a partially ordered set. Then, there exists an adjusted partially ordered set $A$ such that $T_{RE}(A)$ is isomorphic to $T_{RE}(X)$ as semigroups.

We can construct an adjusted partially ordered set $A(X)$ from $X$ such that $T_{RE}(A(X))$ is isomorphic to $T_{RE}(X)$. In this case, the $A(X)$ is called the adjusted partially ordered set of $X$.

**Theorem 2.**

Let $X, Y$ be adjusted partially ordered sets. Then, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic as semigroups if and only if $X$ and $Y$ are isomorphic as ordered sets.

Suppose that $X$ and $Y$ are isomorphic. Let $h$ be an isomorphism from $X$ onto $Y$. Then, it is easy to show that the map $i : T_{RE}(X) \rightarrow T_{RE}(Y), f \rightarrow i(f)$ defined by $i(f)(h(x)) = h(y)$ if $f(x) = y$, is an isomorphism.

To show the only if-part, we need two lemmas (Lemmas 1 and 2).

For each pair $a, b \in Z$ with $a < b$, where $Z$ is a partially ordered set, we define $\lambda_{a}^{b}$ in $T_{RE}(Z)$ by

$$\lambda_{a}^{b}(b) = a, \lambda_{a}^{b}(x) = x \text{ if } x \neq b.$$

From now until the end of the proof of Theorem 2, $X$ and $Y$ will denote adjusted partially ordered sets, and $i$ will denote an isomorphism from $T_{RE}(X)$ onto $T_{RE}(Y)$.

**Lemma 1.** For each pair $a, b \in X$ with $a < b$, there exist $a', b' \in Y$ such that $i(\lambda_{a}^{b}) = \lambda_{a'}^{b'}$.

The assertion can be easily shown by using the following facts:

For $g \in T_{RE}(X),$

$$\lambda_{a}^{b} \circ g = \lambda_{a}^{b} \text{ if and only if } g = id_{T_{RE}(X)} \text{ or } g = \lambda_{a}^{b},$$

$$g \circ \lambda_{a}^{b} = g \text{ if and only if } g(a) = g(b) \text{ and } a < b.$$ 

For each pair $a, b \in X$ with $a < b$, the pair $a', b'$ in Lemma 1 is clearly unique. So we write $a' = j(a, b)$ and $b' = k(a, b)$, namely $i(\lambda_{a}^{b}) = \lambda_{j(a, b)}^{k(a, b)}$.

We similarly have that for each pair $a', b'$ in $Y$ with $a' < b'$, there exist unique elements $j'(a', b'), k'(a', b')$ in $X$ such that $i(j'(a', b')) = \lambda_{j'(a', b')}^{k'(a', b')}$. Then, for each $a, b$ in $X$ with $a <
Corollary 5.
Let $X$ and $A$ be as in Theorem 1. Then $A$ is uniquely determined by $X$ up to isomorphisms.

We next aim to refine Theorem 2 to the following:

Theorem 6.
Let $X$ and $Y$ be as in Theorem 2, and let $i$ be a semigroup isomorphism from $T_{RE}(X)$ onto $T_{RE}(Y)$. Then, there exists an order isomorphism $h$ from $X$ onto $Y$ such that $h(f(a)) = i(f(h(a)))$ for all $f \in T_{RE}(X)$ and all $a \in X$.

Let $h$ be an isomorphism from $X$ onto $Y$ determined by $i$ in Theorem 6 as in the proof of Theorem 2. Thus, $i(\lambda^b_a) = \lambda^h(\theta h(a))$ for each $a, b \in X$ with $a < b$. We show that this $h$ serves as a desired $h$ in Theorem 6. To show the theorem, again we need two lemmas (Lemmas 3 and 4).

For each $f \in T_{RE}(X)$ and each $a \in X$, we define $f^a$ and $f_a$, as follows:

$f^a(x) = x$ if $x \geq a$, $f^a(x) = f(x)$ otherwise, and $f_a(x) = f(x)$ if $x > a$, $f_a(x) = x$ otherwise.

Then, it is easy to check that $f = f_a \circ \lambda^a_{f(a)} \circ f^a$ for all $a \in X$, where $\lambda^a_{f(a)} = \text{id}_{T(X)}$ if $a = f(a)$.

Lemma 3. For any $f \in T_{RE}(X)$ and any $b, c \in X$ with $c \leq b$,

1. $f(b) = f(c)$ if and only if $i(f(h(b))) = i(f(h(c)))$. In particular,
2. if $a \leq b$, then $i(f^a(h(b))) = i(f^a(h(c)))$ implies that $h(b) = h(c)$, and
3. if $b \vDash a$, then $i(f_a(h(b))) = i(f_a(h(c)))$ implies that $h(b) = h(c)$, where $b \vDash a$ means that $b \leq a$ or $a$ and $b$ are incomparable.

From Lemma 3, we have:

Lemma 4. For every $a, b \in X$,

1. if $h(b) \geq h(a)$, then $i(f^a(h(b))) = h(b)$,
2. if $h(b) \vDash h(a)$, then $i(f_a(h(b))) = h(b)$.

Since $f = f_a \circ \lambda^a_{f(a)} \circ f^a$ for all $a \in X$, and since $h(f(a)) \leq h(a)$, we have that

$i(f)(h(a)) = i(f_a) \circ i(\lambda^a_{f(a)}) \circ i(f^a)(h(a)) = i(f_a) \circ \lambda^h(\theta h(a))(h(a))$

$= i(f_a) \circ \lambda^h(\theta h(a))(h(a)) = i(f_a)(h(f(a))) = h(f(a))$.

References


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