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ON INERT EXTENSIONS OF GRADED INTEGRAL DOMAINS

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A torsion-free cancellative commutative semigroup (written additively) $\ni 0$ is called a torsionless grading monoid. By a graded integral domain $\oplus_S R_s$, we mean an integral domain graded by a torsionless grading monoid S with the assumption that each R_s is nonzero.

In [1], Anderson-Anderson investigate different ways to regrade $\oplus_S R_s$.

Let Γ be a torsionless grading monoid, S be a submonoid of Γ and $\oplus_\Gamma R_\alpha$ be a Γ-graded integral domain.

We define an extension of domains $D \subset E$ to be inert if whenever $0 \neq xy \in D$ for some $x, y \in E$, then $xu, yu^{-1} \in D$ for some unit u of E. We define $D \subset E$ to be strongly inert if whenever $0 \neq xy \in D$ for some $x, y \in E$, then $x, y \in D$. We say that S is saturated in Γ if whenever $\alpha + \beta \in S$ for some $\alpha, \beta \in \Gamma$, then $\alpha, \beta \in S$.

Among other theorems, Anderson-Anderson proved the following,

Theorem 1([1,Theorem 3.8]). Let $\oplus_\Gamma R_\alpha$ be a graded integral domain. Then $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is strongly inert if and only if $S \subset \Gamma$ is saturated.

Anderson-Anderson state: It would be very interesting to determine necesary and sufficient conditions on S for $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ to be an inert extension.

The aim of this paper is to determine necessary and sufficient conditions on S for $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ to be an inert extension.

The quotient group $\{s_1 - s_2 \mid s_i \in S\}$ of S is denoted by q(S). q(S) is a subgroup of q(Γ) and q(Γ) is a totally ordered abelian group.

Lemma 2. Assume that $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is inert. Set $\Gamma \cap$ q$(S) = T$.

Then $\oplus_S R_s \subset \oplus_T R_t$ is inert.

Proof. Assume that $0 \neq \varphi_1\varphi_2 \in \oplus_S R_s$ for $\varphi_1, \varphi_2 \in \oplus_T R_t$. Then $\varphi_1 u, \varphi_2 u^{-1} \in \oplus_S R_s$ for some unit $u \in R_{\alpha_1}$ of $\oplus_\Gamma R_\alpha$. It follows that $\alpha_1, -\alpha_2 \in T$.

The submonoid $\{\alpha \in \Gamma \mid \alpha + \alpha' \in S \text{ for some } \alpha' \in \Gamma\}$ of Γ is called saturation of S in Γ.

Lemma 3. Assume that $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is inert. Set $\Gamma \cap \mathrm{q}(S) = T$ and let Γ^* be the saturation of S in Γ. Then $T \subset \Gamma^*$, and $\oplus_T R_t \subset \oplus_{\Gamma^*} R_\beta$ is inert.

Proof. Assume that $0 \neq F_1 F_2 = f \in \oplus_T R_t$ for some $F_1, F_2 \in \oplus_{\Gamma^*} R_\beta$ and $f \in \oplus_T R_t$. Then we have $fa \in \oplus_S R_s$ for some $0 \neq a \in R_{s_1}$ and $s_1 \in S$. Hence we have $F_1 u, F_2 a u^{-1} \in \oplus_S R_s$ for some unit $u \in R_{\alpha_1}$ of $\oplus_\Gamma R_\alpha$. Then we have $\alpha_1, -\alpha_1 \in \Gamma^*$. Therefore $F_2 u^{-1} \in \oplus_T R_t$.

Lemma 4. Let Γ^* be the saturation of S in Γ. Then $\oplus_{\Gamma^*} R_\beta \subset \oplus_\Gamma R_\alpha$ is strongly inert.

Proof. By Theorem 1.

Let $x \in \oplus_S R_s$, with $x = x_1 + \cdots + x_n$, where $0 \neq x_i \in R_{s_i}$ for each i and $s_1 < \cdots < s_n$. Then the subset $\{s_1, \cdots, s_n\}$ of S is called support of x, and is denoted by $\mathrm{Supp}(x)$. Let I_1, I_2 be subsets of $\mathrm{q}(S)$. Then the subset $\{x_1 + x_2 \mid x_i \in I_i \text{ for each } i\}$ of $\mathrm{q}(S)$ is denoted by $I_1 + I_2$. $\underbrace{I + \cdots + I}_{n}$ is denoted by nI for $I \subset \mathrm{q}(S)$. The subset $\{\alpha \in I \mid I_1 + \alpha \subset I_2\}$ is denoted by $(I_2 : I_1)_I$. Next, the subset $\{s \in S \mid R_s \text{ contains a unit of } \oplus_S R_s\}$ is denoted by $S^{(0)}$.

Lemma 5. Assume that $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is inert. Let I_1, I_2 be non-empty finite subsets of Γ such that $I_1 + I_2 \subset S$. Then we have $(S : I_1)_{\Gamma^{(0)}} + (S : I_2)_{\Gamma^{(0)}} \ni 0$.

Proof. We take $F_1, F_2 \in \oplus_\Gamma R_\alpha$ such that $\mathrm{Supp}(F_1) = I_1$, $\mathrm{Supp}(F_2) = I_2$. Then $F_1F_2 \in \oplus_S R_s$. Hence we have $F_1u, F_2u^{-1} \in \oplus_S R_s$ for some unit $u \in R_{\alpha_1}$ of $\oplus_\Gamma R_\alpha$. Then $\alpha_1 \in (S : I_1)_{\Gamma^{(0)}}$ and $-\alpha_1 \in (S : I_2)_{\Gamma^{(0)}}$.

Lemma 6. Assume that $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is inert. Then S is integrally closed in Γ.

Proof. Assume that $0 \neq \alpha_1 \in \Gamma$ is integral over S. We have $n\alpha_1 \in S$ for some $n \in \mathbf{N}$. Take $0 \neq x \in R_{\alpha_1}$. We have $\oplus_S R_s \ni 1 - x^n = (1-x)(1+x+\cdots+x^{n-1})$. Hence there exists a unit $u \in R_{\alpha_2}$ of $\oplus_\Gamma R_\alpha$ such that $(1-x)u, (1+x+\cdots+x^{n-1})u^{-1} \in \oplus_S R_s$. Then it follows $\alpha_1 \in S$.

We say that S satisfies (*) in Γ if whenever $n\alpha \in \mathrm{q}(S)$ for some $\alpha \in \Gamma$ and $n \in \mathbf{N}$, then $\alpha \in S$.

Lemma 7. Assume that $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is inert. Set $\Gamma \cap \mathrm{q}(S) = T$ and let Γ^* be the saturation of S in Γ. Then T satisfies (*) in Γ^*.

Proof. Assume that $n\beta \in \mathrm{q}(T)$ for some $\beta \in \Gamma^*$ and $n \in \mathbf{N}$. Since $\mathrm{q}(T) = \mathrm{q}(S)$, we have $n\beta + s \in S$ for some $s \in S$. Then $n(\beta + s) \in S$. Lemma 6 implies that $\beta + s \in S$. Hence $\beta \in T$.

Let $\alpha_1, \alpha_2 \in \Gamma$. If $n(\alpha_1 - \alpha_2) \in \mathrm{q}(S)$ for some $n \in \mathbf{N}$, we define $\alpha_1 \sim_{(S,\Gamma)} \alpha_2$. Then $\sim_{(S,\Gamma)}$ is an equivalence relation on Γ. The equivalence class of $\alpha \in \Gamma$ is denoted by $\bar{\alpha}$. For $\alpha \in \Gamma$, we define $R_{\bar{\alpha}} = \sum_{\alpha' \in \bar{\alpha}} R_{\alpha'}$.

Lemma 8 ([1,Theorem 3.1]). Assume that S satisfies (*) in Γ. Then the quotient set $\bar{\Gamma}$ of Γ by $\sim_{(S,\Gamma)}$ is a grading monoid. Moreover, $\oplus_\Gamma R_\alpha = \oplus_{\bar{\Gamma}} R_{\bar{\alpha}}$ is a $\bar{\Gamma}$-graded domain and $R_{\bar{0}} = \oplus_S R_s$.

Lemma 9 ([1, Proposition 3.3,(1)]). $R_0 \subset \oplus_S R_s$ is inert if and only if $S^{(0)}$ is the units of S.

Lemma 10. Assume that $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is inert. Assume that S satisfies (*) in Γ and the saturation of S in Γ is Γ. Let $\beta \in \Gamma$. Then there exists $\beta_1 \in \Gamma^{(0)}$ such that $\beta \sim_{(S,\Gamma)} \beta_1$.

Proof. Then the quotient set $\bar{\Gamma}$ is a group. $\oplus_\Gamma R_\alpha = \oplus_{\bar{\Gamma}} R_{\bar{\alpha}}$ is a $\bar{\Gamma}$-graded domain and $R_{\bar{0}} = \oplus_S R_s$ by Lemma 8. By Lemma 9, $R_{\bar{\beta}}$ contains a unit u of $\oplus_{\bar{\Gamma}} R_{\bar{\alpha}}$. Hence there exists $\beta_1 \in \Gamma^{(0)}$ such that $\beta \sim_{(S,\Gamma)} \beta_1$ and $u \in R_{\beta_1}$.

Lemma 11. Assume that $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is inert. Let Γ^* be the saturation of S in Γ. Let $\beta \in \Gamma^*$. Then there exists $\beta_1 \in (\Gamma^*)^{(0)}$ such that $\beta \sim_{(S,\Gamma)} \beta_1$.

Proof. We set $\Gamma \cap \mathrm{q}(S) = T$. T satisfies (*) in Γ^* by Lemma 7. Lemma 3 implies that $\oplus_T R_t \subset \oplus_{\Gamma^*} R_\beta$ is inert. Hence there exists $\beta_1 \in (\Gamma^*)^{(0)}$ such that $\beta \sim_{(T,\Gamma^*)} \beta_1$. Then we have $\beta \sim_{(S,\Gamma)} \beta_1$.

Lamma 12. Assume that S satisfies (*) in Γ and the saturation of S in Γ is Γ. Assume that for each $\alpha \in \Gamma$ there exists $\alpha' \in \Gamma^{(0)}$ such that $\alpha \sim_{(S,\Gamma)} \alpha'$. Then $\oplus_S R_s \subset \oplus_\Gamma \Gamma_\alpha$ is inert.

Proof. The quotient set $\bar{\Gamma}$ is a group. $\oplus_\Gamma R_{\bar{\alpha}} = \oplus_{\bar{\Gamma}} R_{\bar{\alpha}}$ is a $\bar{\Gamma}$-graded domain, and $R_{\bar{0}} = \oplus_S R_s$. Lemma 9 implies that $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is inert.

Lemma 13. Let f, g be non-zero elements of $\oplus_S R_s$ and set $\mid \mathrm{Supp}(g) \mid = k$. Then we have k Supp(f)+Supp$(g) = (k-1)$Supp(f)+Supp(fg).

Proof. The proof is similar with that of [2, 6.2.Proposition].

Lemma 14. Assume that $\Gamma \subset \mathrm{q}(S)$ and S is integrally closed in Γ. Assume that $(S : I_1)_{\Gamma^{(0)}} + (S : I_2)_{\Gamma^{(0)}} \ni 0$ for every non-empty finite subsets I_1, I_2 of Γ such that $I_1 + I_2 \subset S$. Then $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is inert.

Proof. Assume that $0 \neq F_1F_2 \in \oplus_S R_s$ for some $F_1, F_2 \in \oplus_\Gamma R_\alpha$. Lemma 13 implies that $(m+1)$Supp(F_1)+Supp$(F_2) = m$Supp(F_1)+Supp (F_1F_2) for some $m \in \mathbf{N}$. Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be the set of valuation oversemigroups of S. Then $\cap_\lambda V_\lambda$ is the integral closure of S. We have

$(m+1)$Supp(F_1)+ Supp$(F_2) + V_\lambda = m$ Supp(F_1)+Supp$(F_1F_2) + V_\lambda$ for each λ. It follows that Supp(F_1)+Supp$(F_2) + V_\lambda$ =Supp$(F_1F_2) + V_\lambda \subset V_\lambda$ for each λ. Hence Supp(F_1)+Supp(F_2) is contained in the integral closure of S. Then Supp(F_1)+Supp$(F_2) \subset S$. By the assumption, there exists $\alpha_1 \in (S$:Supp$(F_1))_{\Gamma^{(0)}}$ with $-\alpha_1 \in (S$:Supp$(F_2))_{\Gamma^{(0)}}$. We take a unit $u \in R_{\alpha_1}$ of $\oplus_\Gamma R_\alpha$. Then $F_1u_1, F_2u^{-1} \in \oplus_S R_s$.

Theorem 15. Set $\Gamma\cap$ q$(S) = T$ and let Γ^* be the saturation of S in Γ. Then $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is inert if and only if the following (1) $\sim$ (4) hold.

(1) S is integrally closed in Γ.

(2) $(S : I_1)_{T^{(0)}} + (S : I_2)_{T^{(0)}} \ni 0$ for every non-empty finite subsets I_1, I_2 of T such that $I_1 + I_2 \subset S$.

(3) T satisfies (*) in Γ^*.

(4) For each $\beta \in \Gamma^*$, there exists $\beta_1 \in (\Gamma^*)^{(0)}$ with $\beta \sim_{(S,\Gamma)} \beta_1$.

Proof. Assume that $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is inert. Then Lemma 6 implies (1). $\oplus_S R_s \subset \oplus_T R_t$ is inert by Lemma 2. Lemma 5 implies (2). Lemma 7 implies (3). Lemma 11 implies (4). Conversely, assume (1),(2),(3) and (4). Then $\oplus_S R_s \subset \oplus_T R_t$ is inert by Lemma 14. For each $\beta \in \Gamma^*$, there exists $\beta_1 \in (\Gamma^*)^{(0)}$ with $\beta \sim_{(S,\Gamma)} \beta_1$. Then we have $\beta \sim_{(T,\Gamma)} \beta_1$. Then $\oplus_T R_t \subset \oplus_{\Gamma^*} R_\beta$ is inert by Lemma 12. $\oplus_{\Gamma_*} R_\beta \subset \oplus_\Gamma R_\alpha$ is strongly inert by Lemma 4. It follows that $\oplus_S R_s \subset \oplus_\Gamma R_\alpha$ is inert.

Remark 16. Assume that $\oplus_\Gamma R_\alpha$ is the semigroup ring $D[X;\Gamma]$ over a domain D with $R_\alpha = DX^\alpha$ for each α. Then $\Gamma^{(0)}$ is the set of units of Γ. The conditions for $D[X;S] \subset D[X;\Gamma]$ to be inert in Theorem 15 are conditions mearly for $S \subset \Gamma$.

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