

Title	Monoids with finite (regular) complete presentation(Semigroups, Formal Languages and Combinatorics on Words)
Author(s)	Kobayashi, Yuji; Katura, Masashi
Citation	数理解析研究所講究録 (1995), 910: 40-43
Issue Date	1995-05
URL	<a href="http://hdl.handle.net/2433/59540">http://hdl.handle.net/2433/59540</a>
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

### Monoids with finite (regular) complete presentation

東邦大理 小林美治 (Yuji Kobayashi)

京都産業大理 勝良昌司 (Masashi Katura)

If a monoid is defined by a finite complete rewriting system, then it has some good properties such as solvability of the word problem (see [1]). So, it is an important problem to find good conditions for monoids to have finite complete presentation.

A rewriting system over a (finite) alphabet  $\Sigma$  is a subset  $R$  of  $\Sigma^* \times \Sigma^*$ . An element  $(u, v)$  of  $R$  is called a rule and written  $u \rightarrow v$ . A word  $x \in \Sigma^*$  is rewritten to  $y \in \Sigma^*$  under  $R$ , if  $x = x_1 u x_2$ ,  $y = x_1 v x_2$  for some  $x_1, x_2 \in \Sigma^*$  and some  $u \rightarrow v \in R$ . In this case we write  $x \rightarrow_R y$ . The reflexive transitive closure and the reflexive symmetric transitive closure of the relation  $\rightarrow_R$  is denoted by  $\rightarrow_R^*$  and  $\leftrightarrow_R^*$ , respectively. The quotient monoid  $M = \Sigma^* / \leftrightarrow_R^*$  is a monoid presented by  $(\Sigma, R)$ .

A subset  $S$  of  $\Sigma^*$  is called s-closed, if any subword of an element of  $S$  is also in  $S$ , equivalently, if the complement of  $S$  forms an ideal of  $\Sigma^*$ .  $S$  is s-closed if and only if  $S$  is expressed as

$$S = \Sigma^* - \Sigma^* T \Sigma^* \tag{1}$$

with a subset  $T$  of  $\Sigma^*$ . A set  $S$  is finitely s-closed if  $S$  is expressed as (1) with a finite set  $T$ , or equivalently,  $S$  is the complement of a finitely generated ideal.

Proposition 1. For an s-closed subset  $S$  of  $\Sigma^*$  the following are equivalent.

- (1)  $S$  is finitely s-closed.
- (2) There is a positive  $n$  such that any subword of length  $\leq n$  of  $x \in \Sigma^*$  is in  $S$ , then  $x$  is in  $S$ .

Corollary. A finite s-closed set is finitely s-closed.

Let  $R \subset \Sigma^* \times \Sigma^*$  and let  $M = M(\Sigma, R)$  be the monoid presented by  $(\Sigma, R)$ . A subset  $S$  of  $\Sigma^*$  is a transversal for  $M$  (or for  $(\Sigma, R)$ ), if  $S$  forms a complete set of representatives for  $M$ , that is, for any  $x \in \Sigma^*$  there is a unique  $\hat{x} \in S$  with  $x \leftrightarrow^* \hat{x}$ .

$M$  has a complete presentation, if  $M$  is presented by  $(\Sigma, R)$  such that  $R$  is a complete (i.e. noetherian and confluent) rewriting system.

Proposition 2. If a monoid has a finite complete presentation, then it has a finitely s-closed transversal.

We are interested in the converse problem of this results.

Problem 1. If a monoid has a finitely s-closed transversal, does it admit a finite complete presentation ?

Proposition 3. Let  $S = \Sigma^* - \Sigma^*T\Sigma^*$  be an s-closed transversal for  $M$ , and let  $R(T) = \{u \rightarrow \hat{u} \mid u \in T\}$ . If  $R(T)$  is noetherian, then  $R$  is a complete system defining  $M$ .

Unfortunately, the system  $R(T)$  given above is not always noetherian even when  $S$  is finitely s-closed and  $T$  is chosen minimally.

Example 1. Let  $\Sigma = \{a, b\}$  and  $M = M(\Sigma, E)$ , where

$$E = \{(baab, aba)\} \cup \{(x, y) \in \Sigma^* \times \Sigma^* \mid |x| = |y| = 4\}.$$

Then  $M$  is a finite monoid with finite s-closed transversal

$$S = \{x \in \Sigma^* \mid |x| \leq 3, x \neq aba\} \cup \{baab\}.$$

A set  $T$  with which  $S$  is expressed as (1) must contain the rule

aba. Hence,  $R = R(T)$  contains the rule  $aba \rightarrow baab$ . Thus

$$abaa \rightarrow_R baaba \rightarrow_R babaab \rightarrow_R bbaabab \rightarrow_R bbabaabb \rightarrow_R \dots,$$

and  $R$  is not noetherian.

If  $M$  has a transversal  $S = \Sigma^* - \Sigma^*T^*$  with  $T$  a singleton, then we expect that the system  $R(T)$  would be noetherian, and  $M$  would admit a complete one-rule system.

**Problem 2.** Suppose  $M$  has a transversal  $S$  of the form

$$S = \Sigma^* - \Sigma^*v\Sigma^*, \quad v \in \Sigma^*.$$

Is the one-rule system  $\{v \rightarrow \hat{v}\}$  noetherian ?

A system  $R$  is regular, if the left hand sides of the rules from  $R$  forms a regular set (see [2]).

**Theorem 1.** Suppose  $M$  is presented by a regular complete system  $(\Sigma, R)$ . Then,

- (1)  $M$  has a regular sets of representative.
- (2)  $\text{Irr}(R)$  grows either exponentially or polynomially.
- (3) If  $\text{Irr}(R)$  grows exponentially, then the monoid  $M = M(\Sigma, R)$  also grows exponentially and contains a free submonoid of rank 2.

**Problem 3.** Does  $M$  have a regular complete presentation, provided  $M$  has a regular  $s$ -closed transversal ?

A system  $(\Sigma, R)$  is polynomially mild, if there is a polynomial  $f$  such that  $|y| \leq f(|x|)$  for any  $x \in \Sigma^*$  and  $y$  such that  $x \rightarrow^* y$ .

**Theorem 2.** If  $(\Sigma, R)$  is a polynomially mild regular complete system such that  $\text{Irr}(R)$  grows polynomially, then the monoid  $M = M(\Sigma, R)$  grows polynomially.

In theorem 2, the mildness of  $R$  cannot be removed. In fact,  $M$  can grows exponentially in general, even if  $\text{Irr}(R)$

grows polynomially.

Example 2. Let  $\Sigma = \{a, b\}$  and  $R_1 = \{abb \rightarrow ba\}$ ,  $R_2 = \{ba \rightarrow abb\}$ . Then, the both  $R_1$  and  $R_2$  are complete and define the same monoid. However,  $\text{Irr}(R_1)$  grows exponentially, while  $\text{Irr}(R_2)$  grows polynomially. So, the system  $R_2$  is not mild.

There is a monoid with a very simple presentation which never admits a finite complete presentation.

Example 3. Let  $\Sigma = \{a, b\}$  and  $R = \{b^2a \rightarrow ab^2, ab^3 \rightarrow b^3, b^3a \rightarrow b^3\}$ , then  $M = M(\Sigma, R)$  has no finitely  $s$ -closed transversal. So,  $M$  has no finite complete presentation.

#### References

- [1] K. Madlener and F. Otto, About the descriptive power of certain classes of finite string-rewriting systems, Theoret. Comp. Sci. **67**, (1989), 143-172.
- [2] Y. Kobayashi, A finitely presented monoid which has solvable word problem but has no regular complete presentation, to appear in Theoret. Comp. Sci.