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On representations of locally inverse $\ast$-semigroups

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Abstract

The purpose of this paper is to obtain an analogous representation of the Preston-Vagner Representation for locally inverse $\ast$-semigroups which is a generalization of [7].

Firstly, by introducing a concept of a $\pi$-set (which is slightly different from the one in [7]), we shall construct the $\pi$-symmetric locally inverse $\ast$-semigroup $\mathcal{L}\mathcal{I}_{X(\pi')}(\mathcal{M})$ on a $\pi$-set $X(\pi';\omega;\{\sigma_{e,f}\})$, and show that $\mathcal{L}\mathcal{I}_{X(\pi')}(\mathcal{M})$ is a locally inverse $\ast$-semigroup and that any locally inverse $\ast$-semigroup can be embedded up to $\ast$-isomorphism in $\mathcal{L}\mathcal{I}_{X(\pi')}(\mathcal{M})$ on a $\pi$-set $X(\pi';\omega;\{\sigma_{e,f}\})$. Moreover, we shall show that the wreath product (in the sense of Cowan[1]) of locally inverse $\ast$-semigroups is also a locally inverse $\ast$-semigroup.

1 Introduction

A semigroup $S$ with a unary operation $\ast : S \to S$ is called a regular $\ast$-semigroup if it satisfies

(i) \((x^\ast)^\ast = x,\)
(ii) \((xy)^\ast = y^\ast x^\ast,\)
(iii) \(xz^\ast x = x.\)

Let $S$ be a regular $\ast$-semigroup. An idempotent $e$ in $S$ is called a projection if it satisfies $e^\ast = e$. For any subset $A$ of $S$, denote the sets of idempotents and projections of $A$ by $E(A)$ and $P(A)$, respectively. The following result is well-known, and we use it frequently throughout this paper.

\(^1\)This is the abstract and the details will be published elsewhere. The results of § 2 and 4 were obtained after the conference.
Result 1.1 (see [4]). Let $S$ be a regular $\ast$-semigroup. Then we have the followings:

1. $E(S) = P(S)^2$, more precisely, for any $e \in E(S)$, there exist $f, g \in P(S)$ such that $f\Re Lg$ and $e = fg$;
2. for any $a \in S$ and $e \in P(S)$, $a^* e a \in P(S)$;
3. each $\mathcal{L}$-class and each $\mathcal{R}$-class have one and only one projection.

A regular $\ast$-semigroup $S$ is called a locally inverse $\ast$-semigroup if, for any $e \in E(S)$, $eSe$ is an inverse subsemigroup of $S$.

**Lemma 1.2** A regular $\ast$-semigroup $S$ is a locally inverse $\ast$-semigroup if and only if, for each $e \in P(S)$, $eSe$ is an inverse subsemigroup of $S$.

A regular $\ast$-semigroup $S$ is called a generalized inverse $\ast$-semigroup if $E(S)$ forms a normal band, that is, $E(S)$ satisfies the identity $xyzz = zzyx$. It is obvious that a generalized inverse $\ast$-semigroup is a locally inverse $\ast$-semigroup.

**Remark.** It is clear that a regular $\ast$-semigroup $S$ is a generalized inverse $\ast$-semigroup if and only if it satisfies the following condition:

for any $e, f, g, h \in P(S)$, $efgh = egfh$ (in $S$).

However, we remark that even if a locally inverse $\ast$-semigroup $S$ satisfies the condition

for any $e, f, g \in P(S)$, $efge = egfe$ (in $S$),

it is not always a generalized inverse $\ast$-semigroup. The second remark of [6] is its counterexample.

Let $X$ be a set. If $X = \bigcup\{X_i : i \in I\}$ is a partition of $X$, denote it by $X = \Sigma\{X_i : i \in I\}$. For a mapping $\alpha : A \to B$, denote the domain and the range of $\alpha$ by $d(\alpha)$ and $r(\alpha)$, respectively. For a subset $C$ of $A$, $\alpha|_C$ means the restriction of $\alpha$ to $C$.

Let $I_X$ be the symmetric inverse semigroup on a set $X$. For a subset $A$ of $X$, $1_A$ means the identity mapping on $A$. Let $\mathcal{A}$ be an inverse subsemigroup of $I_X$ and $\theta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ a mapping. Denote the image $(\alpha, \beta)\theta$ of an ordered pair $(\alpha, \beta)$ by $\theta_{\alpha,\beta}$. Set $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in \mathcal{A}\}$. If $\mathcal{M}$ satisfies the following conditions:

\begin{enumerate}
\item[(C1)] $\theta_{\alpha,\beta}^{-1} = \theta_{\beta^{-1},\alpha^{-1}}$,
\item[(C2)] $\theta_{\alpha,\alpha^{-1}} = 1_{r(\alpha)}$,
\item[(C3)] $\theta_{1_{d(\alpha)},\alpha} = 1_{d(\alpha)}$,
\item[(C4)] $\theta_{\alpha,\beta}\theta_{\alpha,\beta} \beta = \theta_{\alpha,\beta,\beta,\gamma}(\theta_{\beta,\gamma})$,
\end{enumerate}

we call it the structure sandwich set of $\mathcal{A}$ determined by $\theta$. 
Result 1.3 (see [7]) Let $A$ be an inverse subsemigroup of the symmetric inverse semigroup $I_X$ on a set $X$, and $M$ the structure sandwich set of $A$ determined by a mapping $\theta : A \times A \to A$. Define a multiplication $\circ$ and a unary operation $\ast$ on $A$ as follows:

$$\alpha \circ \beta = \alpha \theta_{\alpha,\beta} \beta,$$

$$\alpha^\ast = \alpha^{-1}$$

Then $A(\circ, \ast)$ becomes a regular $\ast$-semigroup.

Hereafter, we call such a semigroup $A(\circ, \ast)$ a regular $\ast$-semigroup of partial one-to-one mappings determined by the structure sandwich set $M$, and denote it by $A(M)$. The notation and terminology are those of [3] and [4], unless otherwise stated.

In § 2, we shall firstly consider a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ which is a set $X$ with a partition $\pi' : X = \sum\{X_e : e \in \Lambda\}$, a reflexive and symmetric relation $\omega$ on $\Lambda$ and a set of mappings $\{\sigma_{e,f} : (e, f) \in \omega\}$, where $\sigma_{e,f}$ is a bijection of $X_e$ onto $X_f$. We remark that a $\pi$-set, defined in this paper, is slightly different from the one in [7], which is called a strong $\pi$-set in this paper. The set $LI_{X(\pi')}$, say, of all partial one-to-one $\pi$-mappings on $X(\pi'; \omega; \{\sigma_{e,f}\})$ is an inverse subsemigroup of $I_X$. By using $\{\sigma_{e,f} : (e, f) \in \omega\}$, we shall construct a structure sandwich set $M$, and show that $LI_{X(\pi')}(M)$ is a locally inverse $\ast$-semigroup. We call such a semigroup $LI_{X(\pi')}(M)$ the $\pi$-symmetric locally inverse $\ast$-semigroup on a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ with the structure sandwich set $M$.

In § 3, we shall show that any locally inverse $\ast$-semigroup is embedded up to $\ast$-isomorphism in the $\pi$-symmetric locally inverse $\ast$-semigroup $LI_{X(\pi')}(M)$ on a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$.

As a generalization of [2], Cowan [1] gave us the definition of the wreath product $SwrT(X)$ of inverse semigroups $S$ and $T(X)$, where $T(X)$ is an inverse subsemigroup of $I_X$. And he showed that the wreath product $SwrT(X)$ is also an inverse semigroup. In § 4, we shall show that the wreath product of locally inverse $\ast$-semigroups $S$ and $T(X)$ ($\subseteq LI_{X(\pi')}$) is a locally inverse $\ast$-semigroup. Moreover, we shall obtain that the wreath product of generalized inverse $\ast$-semigroups is also a generalized inverse $\ast$-semigroup.

2 $\pi$-Symmetric locally inverse $\ast$-semigroups

Let $X$ be a non-empty set. If there exist a partition $X = \sum\{X_e : e \in \Lambda\}$ and a reflexive and symmetric relation $\omega$ on $\Lambda$ such that

(i) for each $(e, f) \in \omega$, there exists a bijection $\sigma_{e,f} : X_e \to X_f$,

(ii) for all $e \in \Lambda$, $\sigma_{e,e} = 1_{X_e}$,

(iii) for any $(e, f) \in \omega$, $\sigma_{f,e} = \sigma_{e,f}^{-1}$,
then $X$ is called a $\pi$-set with a partition $\pi' : X = \sum \{ X_e : e \in \Lambda \}$, a relation $\omega$ and a set of mappings $\{ \sigma_{e,f} : (e, f) \in \omega \}$, and denote it by $X(\pi'; \omega; \{\sigma_{e,f}\})$, or simply by $X(\pi')$. If a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ satisfies the following two conditions

(v) $\omega$ is transitive, that is, it is an equivalence relation,
(v) for $(e, f), (f, g) \in \omega$, $\sigma_{e,f}\sigma_{f,g} = \sigma_{e,g}$,

it is called a strong $\pi$-set.

Let $X(\pi'; \omega; \{\sigma_{e,f}\})$ be a $\pi$-set. A subset $A$ of $X$ is called a $\pi$-single subset of $X$ if for each $e \in \Lambda$, there exists at most one element $f \in \Lambda$ such that $X_f \cap A \neq \emptyset$ and $(e, f) \in \omega$. We consider the empty set as a $\pi$-single subset. Denote the family of all $\pi$-single subsets of $X(\pi'; \omega; \{\sigma_{e,f}\})$ by $T$.

A mapping $\alpha$ in the symmetric inverse semigroup $I_X$ on $X$ is called a partial one-to-one $\pi$-mapping of $X(\pi'; \omega; \{\sigma_{e,f}\})$ if $d(\alpha)$ and $r(\alpha)$ are $\pi$-single subsets. Let $LI_X(\pi')$ be the set of all partial one-to-one $\pi$-mappings of $X(\pi'; \omega; \{\sigma_{e,f}\})$, that is, $LI_X(\pi') = \{ \alpha \in I_X : d(\alpha), r(\alpha) \in T \}$. The following lemma is clear.

**Lemma 2.1** The set $LI_X(\pi')$, defined above, is an inverse subsemigroup of $I_X$.

For $A, B \in T$, define a mapping $\theta_{A,B}$ as follows:

\[
d(\theta_{A,B}) = \{ x \in A : \text{there exist } e, f \in \Lambda \text{ such that } (e, f) \in \omega, x \in X_e \text{ and } x\sigma_{e,f} \in B \},
\]

\[
(3.1) \quad r(\theta_{A,B}) = \{ y \in B : \text{there exist } e, f \in \Lambda \text{ such that } (e, f) \in \omega, y \in X_f \text{ and } y\sigma_{f,e} \in A \},
\]

\[
x\theta_{A,B} = x\sigma_{e,f} \quad (x \in d(\theta_{A,B}) \cap X_e, (e, f) \in \omega).
\]

For any $\alpha, \beta \in LI_X(\pi')$, define $\theta_{\alpha,\beta} = \theta_{r(\alpha), d(\beta)}$. Since a subset of $\pi$-single subset is also a $\pi$-single subset, we have that $\theta_{\alpha,\beta} \in LI_X(\pi')$ for all $\alpha, \beta \in LI_X(\pi')$. Let $M = \{ \theta_{\alpha,\beta} : \alpha, \beta \in LI_X(\pi') \}$.

**Lemma 2.2** The set $M$, defined above, is the structure sandwich set of $LI_X(\pi')$ determined by a mapping $\theta : LI_X(\pi') \times LI_X(\pi') \rightarrow LI_X(\pi') ((\alpha, \beta) \mapsto \theta_{\alpha,\beta})$. Therefore, $LI_X(\pi')(M)$ is a regular $*$-semigroup.

We call the set $M$, defined above, the structure sandwich set determined by a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$.

It is clear that each projection of $LI_X(\pi')(M)$ is the identity mapping $1_A$ on a $\pi$-single subset $A$. Let $1_A$ be any projection and let $\alpha, \beta$ be any projections of $1_A \circ LI_X(\pi') \circ 1_A$. There exist $B, C \in T$ such that $\alpha = 1_A \circ 1_B \circ 1_A$ and $\beta = 1_A \circ 1_C \circ 1_A$. Then $\alpha = \theta_{A,B}\theta_{A,B}^{-1} = 1d(\theta_{A,B})$ and $\beta = 1d(\theta_{A,C})$. Since $d(\theta_{A,B}) \subseteq A$ and $d(\theta_{A,C}) \subseteq A$, $\theta_{A,B}\theta_{A,B}^{-1} \subseteq \theta_{A,A} = 1_A$. 

\[22\]
Similarly $\theta_{1^{\pi_{A,C}^{*}}}^{A,B} \subseteq 1_{A}$. Then $\alpha \circ \beta = \beta \circ \alpha$. Therefore, $LI_{X(\pi)}(M)$ is a locally inverse $*$-semigroup. We call it the $\pi$-symmetric locally inverse $*$-semigroup on $X(\pi'; \omega; \{\sigma_{e,f}\})$ with the structure sandwich set $M$. Now, we have the following theorem.

**Theorem 2.3** Let $X$ be a $\pi$-set with a partition $\pi' : X = \sum\{X_{e} : e \in \Lambda\}$, a relation $\omega$ on $\Lambda$ and a set of mappings $\{\sigma_{e,f} : (e, f) \in \omega\}$, and let $M$ be the structure sandwich set determined by $X(\pi'; \omega; \{\sigma_{e,f}\})$. Then $LI_{X(\pi')}(M)$ is an inverse subsemigroup of $I_{X}$. Moreover, $LI_{X(\pi')}(M)$ is a locally inverse $*$-semigroup.

Let $X(\pi'; \omega; \{\sigma_{e,f}\})$ be a strong $\pi$-set, where $\pi' = \sum\{X_{e} : e \in \Lambda\}$. Since $\omega$ is an equivalence relation on $\Lambda$, there exists the partition $\Lambda = \sum\{\Lambda_{i} : i \in I\}$ induced by $\omega$. For each $i \in I$, denote the subset $\cup\{X_{e} : e \in \Lambda_{i}\}$ by $X_{i}$.

**Lemma 2.4** A subset $A$ of $X$ is a $\pi$-single subset if and only if it satisfies the condition that for any $i \in I$, $A \cap X_{i} \neq \emptyset$ implies $A \cap X_{e} \subseteq X_{e}$ for some $e \in \Lambda_{i}$.

Let $M$ be the structure sandwich set determined by $X(\pi'; \omega; \{\sigma_{e,f}\})$. By Theorem 3.6 of [7], $LI_{X(\pi')}(M)$ is a generalized inverse $*$-semigroup. We call such a semigroup the $\pi$-symmetric generalized inverse $*$-semigroup and denote it by $GI_{X(\pi')}(M)$ instead of $LI_{X(\pi')}(M)$.

**Corollary 2.5** (Theorem 3.6 [7]) Let $X$ be a strong $\pi$-set with a partition $\pi' : X = \sum\{X_{e} : e \in \Lambda\}$, an equivalence relation $\omega$ on $\Lambda$ and a set of mappings $\{\sigma_{e,f} : (e, f) \in \omega\}$, and let $M$ be the structure sandwich set determined by $X(\pi'; \omega)$. Then $GI_{X(\pi')} (M)$ is a generalized inverse $*$-semigroup.

### 3 Representations

Let $S$ be a locally inverse $*$-semigroup and $I_{S}$ the symmetric inverse semigroup on $S$. In this section, denote $E(S)$ and $P(S)$ simply by $E$ and $P$, respectively. Since each $L$-class has one and only one projection, $\pi' : S = \sum\{L_{e} : e \in P\}$ is a partition of $S$, where $L_{e}$ denotes the $L$-class containing $e$. Let $\omega = \{(e, f) \in P \times P : e \mathcal{R} g \mathcal{L} f \text{ for some } g \in E\}$. It is clear that $\omega$ is a reflexive and symmetric relation on $P$. For $(e, f) \in \omega$, define $\sigma_{e,f} : L_{e} \rightarrow L_{f}$ by $x \sigma_{e,f} = xf$. It follows from Green's Lemma that $S(\pi'; \omega; \{\sigma_{e,f}\})$ is a $\pi$-set. Let $T$ be the set of all $\pi$-single subsets of $S(\pi'; \omega; \{\sigma_{e,f}\})$ and $M$ the structure sandwich set determined by $S(\pi'; \omega; \{\sigma_{e,f}\})$. By Theorem 2.5, $LI_{S(\pi')}(M)$ is a locally inverse $*$-semigroup.

For any $a \in S$, let $\rho_{a} : Sa^{*} \rightarrow Sa$ be a mapping defined by $x \rho_{a} = xa$. 

It is trivial that $\rho_a$ and $\rho_{a^*}$ are mutually inverse mappings of $Sa^*$ and $Sa$ onto each other, and hence $\rho_a \in I_S$. A subset of $S$ is said to be $L$-full if it is a union of some $L$-classes of $S$.

**Lemma 3.1** (1) For any $a \in S$, $\rho_a \in LI_S(\pi')$.  
(2) For any $a, b \in S$, $\theta_{\rho_a, \rho_b} = \rho_{a^*a}b^*b$. Therefore, $\rho_a \circ \rho_b = \rho_a \rho_{a^*a}b^*b \rho_b$.

By the lemma above and Theorem 2.2 of [5], we can easily see the following lemma.

**Lemma 3.2** Define a mapping $\phi : S \rightarrow LI_S(\pi')(M)$ by

$$a\phi = \rho_a.\$$

Then $\phi$ is a $*$-monomorphism.

Now, we have the main theorem.

**Theorem 3.3** A locally inverse $*$-semigroup can be embedded up to $*$-isomorphism in the $\pi$-symmetric locally inverse $*$-semigroup $LI_{X(\pi')}(M)$ on a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ with the structure sandwich set $M$ determined by $X(\pi'; \omega; \{\sigma_{e,f}\})$.

If $S$ is a generalized inverse $*$-semigroup, then a $\pi$-set $S(\pi'; \omega; \{\sigma_{e,f}\})$, constructed above, is a strong $\pi$-set. For, let $(e, f), (f, g) \in \omega$. Then there exist $h, k \in E(S)$ such that $eRhlf$ and $fRkhg$. Since $S$ is a generalized inverse $*$-semigroup, $efg = eg \in E(S)$ and $eRegLg$. In this case, it follows from [7] that $\sigma_{e,f}\sigma_{f,g} = \sigma_{e,g}$, and hence $S(\pi'; \omega; \{\sigma_{e,f}\})$ is a strong $\pi$-set. Then we have the following corollary.

**Corollary 3.4** (Theorem 4.8 [7]). A generalized inverse $*$-semigroup can be embedded up to $*$-isomorphism in $GI_{X(\pi')}$ on a strong $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$.

4 Wreath products

Let $S$ and $T$ be locally inverse $*$-semigroups. By Theorem 3.3, $T$ can be embedded in the $\pi$-symmetric locally inverse $*$-semigroup $LI_{X(\pi')}(M)$ on a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ with the structure sandwich set $M$ determined by $X(\pi'; \omega; \{\sigma_{e,f}\})$. In this case, we can consider $T$ as a locally inverse $*$-subsemigroup of $LI_{X(\pi')}(M)$, and so denote it by $T(X)$.

By $X S$, denote the set of all mappings from the family $T$ of $\pi$-single subsets of $X(\pi'; \omega; \{\sigma_{e,f}\})$ into $S$, and define a multiplication on $XS$ by

$$d(\psi_1 \psi_2) = d(\psi_1) \cap d(\psi_2),$$
$$x(\psi_1 \psi_2) = (x\psi_1)(x\psi_2).$$
For any $\alpha \in \mathcal{L}_{x_\pi}$ and $\psi \in xS$, let us define $^a\psi (\in xS)$ by

$$^a\psi = \alpha \theta_{\alpha, \psi} \psi,$$

where $\theta_{\alpha, \psi} = \theta_{r(\alpha), d(\psi)} \in \mathcal{M}$.

Let $\psi \in xS$ and $\alpha \in \mathcal{L}_{x_\pi}$ such that $d(\psi) = d(\alpha)$. Define a mapping $\psi^*_{\alpha} (\in xS)$ by

$$d(\psi^*_{\alpha}) = d(\alpha^{-1}),$$

$$x \psi^*_{\alpha} = (x \alpha^{-1} \theta_{\alpha^{-1}, \psi} \psi^*)^*.$$

Since $r(\alpha^{-1}) = d(\alpha) = d(\psi)$, $\theta_{\alpha^{-1}, \psi} = 1_{d(\alpha)}$ and hence $x \psi^*_{\alpha} = (x \alpha^{-1} \psi)^*$ for all $x \in d(\psi^*_{\alpha})$.

Now, we define the (right) wreath product $S \wreath T(x)$ of $S$ and $T(x)$ as follows:

$$S \wreath T(x) = \{(\psi, \alpha) \in xS \times T(x) : d(\psi) = d(\alpha)\},$$

$$(\psi, \alpha)(\varphi, \beta) = (\psi^\alpha \varphi, \alpha \circ \beta),$$

$$(\psi, \alpha) = (\psi^*_{\alpha}, \alpha^{-1}).$$

Let $(\psi, \alpha), (\varphi, \beta) \in S \wreath T(x)$. Then

$$x \in d(\psi^\alpha \varphi) \iff x \in d(\psi) \text{ and } x \in d(\alpha^{-1}) = d(\alpha \theta_{\alpha, \psi} \psi),$$

$$x \in d(\alpha), x \alpha \in d(\theta_{\alpha, \psi}) \text{ and } x \alpha \theta_{\alpha, \psi} \in d(\varphi),$$

$$x \in d(\alpha), x \alpha \in d(\theta_{\alpha, \beta}) \text{ and } x \alpha \theta_{\alpha, \beta} \in d(\beta),$$

$$x \in d(\alpha \theta_{\alpha, \beta} \beta) = d(\alpha \circ \beta).$$

Then $(\psi, \alpha)(\varphi, \beta) = (\psi^\alpha \varphi) \in S \wreath T(x)$, and hence $S \wreath T(x)$ is closed under the multiplication. It immediately follows from the definition of $\psi^*_{\alpha}$ that $S \wreath T(x)$ is closed under the unary operation $^*$. 

**Theorem 4.1** Let $S$ and $T(x)$ be locally inverse $^*$-semigroups. Then $S \wreath T(x)$ is a locally inverse $^*$-semigroup. Moreover, we have

$$P(S \wreath T(x)) = \{(\psi, 1_A) \in S \wreath T(x): A \in T \text{ and } r(\psi) \subseteq P(S)\},$$

$$E(S \wreath T(x)) = \{(\psi, \alpha) \in S \wreath T(x): \alpha \in E(T(x)) \text{ and } r(\psi) \subseteq E(S)\}.$$ 

Next, we shall consider wreath products of generalized inverse $^*$-semigroups. Let $S$ and $T(x) (\subseteq \mathcal{G}I_{x_\pi}(\mathcal{M}))$ be generalized inverse $^*$-semigroups. 

**Lemma 4.2** Let $A, B, C$ be a $\pi$-single subsets of a strong $\pi$-set $X(\pi; \omega; \{\sigma_{e,f}\})$, and let $\psi \in xS$ such that $d(\psi) = C$. Then, for any $x \in d(1_{A} \circ 1_{B} \circ 1_{C})$, $x^{1_{A} \circ 1_{B}} \psi = x^{1_{A}} \psi$.

By using the lemma above, we have the following theorem.

**Theorem 4.3** Let $S$ and $T(x) (\subseteq \mathcal{G}I_{x_\pi}(\mathcal{M}))$ be generalized inverse $^*$-semigroups, then $S \wreath T(x)$ is a generalized inverse $^*$-semigroup.
References


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