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On representations of locally inverse $*$-semigroups

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Abstract

The purpose of this paper is to obtain an analogous representation of the Preston-Vagner Representation for locally inverse $*$-semigroups which is a generalization of [7].

Firstly, by introducing a concept of a $\pi$-set (which is slightly different from the one in [7]), we shall construct the $\pi$-symmetric locally inverse $*$-semigroup $\mathcal{L}I_{X(\pi')}(\mathcal{M})$ on a $\pi$-set $X(\pi';\omega;\{\sigma_{e,f}\})$, and show that $\mathcal{L}I_{X(\pi')}(\mathcal{M})$ is a locally inverse $*$-semigroup and that any locally inverse $*$-semigroup can be embedded up to $*$-isomorphism in $\mathcal{L}I_{X(\pi')}(\mathcal{M})$ on a $\pi$-set $X(\pi';\omega;\{\sigma_{e,f}\})$. Moreover, we shall show that the wreath product (in the sense of Cowan[1]) of locally inverse $*$-semigroups is also a locally inverse $*$-semigroup.

1 Introduction

A semigroup $S$ with a unary operation $*:S\rightarrow S$ is called a regular $*$-semigroup if it satisfies

(i) $(x^*)^* = x$,
(ii) $(xy)^* = y^*x^*$,
(iii) $x^*x = x$.

Let $S$ be a regular $*$-semigroup. An idempotent $e$ in $S$ is called a projection if it satisfies $e^* = e$. For any subset $A$ of $S$, denote the sets of idempotents and projections of $A$ by $E(A)$ and $P(A)$, respectively. The following result is well-known, and we use it frequently throughout this paper.

1This is the abstract and the details will be published elsewhere. The results of § 2 and 4 were obtained after the conference.
Result 1.1 (see [4]). Let $S$ be a regular $*$-semigroup. Then we have the followings:

1. $E(S) = P(S)^2$, more precisely, for any $e \in E(S)$, there exist $f, g \in P(S)$ such that $f \mathrm{Re}La$ and $e = fg$;
2. for any $a \in S$ and $e \in P(S)$, $a^*e \in P(S)$;
3. each $L$-class and each $R$-class have one and only one projection.

A regular $*$-semigroup $S$ is called a locally inverse $*$-semigroup if, for any $e \in E(S)$, $eSe$ is an inverse subsemigroup of $S$.

Lemma 1.2 A regular $*$-semigroup $S$ is a locally inverse $*$-semigroup if and only if, for each $e \in P(S), eSe$ is an inverse subsemigroup of $S$.

A regular $*$-semigroup $S$ is called a generalized inverse $*$-semigroup if $E(S)$ forms a normal band, that is, $E(S)$ satisfies the identity $xyzz = zzyx$. It is obvious that a generalized inverse $*$-semigroup is a locally inverse $*$-semigroup.

Remark. It is clear that a regular $*$-semigroup $S$ is a generalized inverse $*$-semigroup if and only if it satisfies the following condition:

for any $e, f, g, h \in P(S), efgh = egfh$ (in $S$).

However, we remark that even if a locally inverse $*$-semigroup $S$ satisfies the condition

for any $e, f, g \in P(S), efge = egfe$ (in $S$),

it is not always a generalized inverse $*$-semigroup. The second remark of [6] is its counterexample.

Let $X$ be a set. If $X = \bigcup\{X_i : i \in I\}$ is a partition of $X$, denote it by $X = \sum\{X_i : i \in I\}$. For a mapping $\alpha : A \to B$, denote the domain and the range of $\alpha$ by $d(\alpha)$ and $r(\alpha)$, respectively. For a subset $C$ of $A$, $\alpha|_C$ means the restriction of $\alpha$ to $C$.

Let $I_X$ be the symmetric inverse semigroup on a set $X$. For a subset $A$ of $X$, $1_A$ means the identity mapping on $A$. Let $\mathcal{A}$ be an inverse subsemigroup of $I_X$ and $\theta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ a mapping. Denote the image $(\alpha, \beta)\theta$ of an ordered pair $(\alpha, \beta)$ by $\theta_{\alpha, \beta}$. Set $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in A\}$. If $\mathcal{M}$ satisfies the following conditions:

(C1) $\theta_{\alpha, \beta}^{-1} = \theta_{\beta^{-1}, \alpha^{-1}}$,
(C2) $\theta_{\alpha, \alpha^{-1}} = 1_r(\alpha)$,
(C3) $\theta_{1_{d(\alpha)}}, \alpha = 1_{d(\alpha)}$,
(C4) $\theta_{\alpha, \beta} \theta_{\alpha, \beta} = \theta_{\alpha, \beta \gamma \beta \gamma}$,

we call it the structure sandwich set of $\mathcal{A}$ determined by $\theta$. 
Result 1.3 (see [7]) Let \( A \) be an inverse subsemigroup of the symmetric inverse semigroup \( \mathcal{I}_X \) on a set \( X \), and \( \mathcal{M} \) the structure sandwich set of \( A \) determined by a mapping \( \theta : A \times A \rightarrow A \). Define a multiplication \( \circ \) and a unary operation \( * \) on \( A \) as follows:

\[
\alpha \circ \beta = \alpha \theta_{\alpha,\beta} \beta, \\
\alpha^* = \alpha^{-1}
\]

Then \( A(\circ, *) \) becomes a regular \( * \)-semigroup.

Hereafter, we call such a semigroup \( A(\circ, *) \) a regular \( * \)-semigroup of partial one-to-one mappings determined by the structure sandwich set \( \mathcal{M} \), and denote it by \( A(\mathcal{M}) \). The notation and terminology are those of [3] and [4], unless otherwise stated.

In §2, we shall firstly consider a \( \pi \)-set \( X(\pi' ; \omega ; \{\sigma_{e,f}\}) \) which is a set \( X \) with a partition \( \pi' : X = \sum \{X_e : e \in \Lambda\} \), a reflexive and symmetric relation \( \omega \) on \( \Lambda \) and a set of mappings \( \{\sigma_{e,f} : (e, f) \in \omega\} \), where \( \sigma_{e,f} \) is a bijection of \( X_e \) onto \( X_f \). We remark that a \( \pi \)-set, defined in this paper, is slightly different from the one in [7], which is called a strong \( \pi \)-set in this paper. The set \( \mathcal{L} \mathcal{I}_X(\pi') \), say, of all partial one-to-one \( \pi \)-mappings on \( X(\pi' ; \omega ; \{\sigma_{e,f}\}) \) is an inverse subsemigroup of \( \mathcal{I}_X \). By using \( \{\sigma_{e,f} : (e, f) \in \omega\} \), we shall construct a structure sandwich set \( \mathcal{M} \), and show that \( \mathcal{L} \mathcal{I}_X(\pi')(\mathcal{M}) \) is a locally inverse \( * \)-semigroup. We call such a semigroup \( \mathcal{L} \mathcal{I}_X(\pi')(\mathcal{M}) \) the \( \pi \)-symmetric locally inverse \( * \)-semigroup on a \( \pi \)-set \( X(\pi' ; \omega ; \{\sigma_{e,f}\}) \) with the structure sandwich set \( \mathcal{M} \).

In §3, we shall show that any locally inverse \( * \)-semigroup is embedded up to \( * \)-isomorphism in the \( \pi \)-symmetric locally inverse \( * \)-semigroup \( \mathcal{L} \mathcal{I}_X(\pi')(\mathcal{M}) \) on a \( \pi \)-set \( X(\pi' ; \omega ; \{\sigma_{e,f}\}) \).

As a generalization of [2], Cowan [1] gave us the definition of the wreath product \( \text{Swr} T(X) \) of inverse semigroups \( S \) and \( T(X) \), where \( T(X) \) is an inverse subsemigroup of \( \mathcal{I}_X \). And he showed that the wreath product \( \text{Swr} T(X) \) is also an inverse semigroup. In §4, we shall show that the wreath product of locally inverse \( * \)-semigroups \( S \) and \( T(X) \) (\( \subseteq \mathcal{L} \mathcal{I}_X(\pi') \)) is a locally inverse \( * \)-semigroup. Moreover, we shall obtain that the wreath product of generalized inverse \( * \)-semigroups is also a generalized inverse \( * \)-semigroup.

2 \( \pi \)-Symmetric locally inverse \( * \)-semigroups

Let \( X \) be a non-empty set. If there exist a partition \( X = \sum \{X_e : e \in \Lambda\} \) and a reflexive and symmetric relation \( \omega \) on \( \Lambda \) such that

(i) for each \( (e, f) \in \omega \), there exists a bijection \( \sigma_{e,f} : X_e \rightarrow X_f \),

(ii) for all \( e \in \Lambda \), \( \sigma_{e,e} = 1_{X_e} \),

(iii) for any \( (e, f) \in \omega \), \( \sigma_{f,e} = \sigma_{e,f}^{-1} \),
then $X$ is called a $\pi$-set with a partition $\pi' : X = \sum \{ X_e : e \in \Lambda \}$, a relation $\omega$ and a set of mappings $\{ \sigma_{e,f} : (e, f) \in \omega \}$, and denote it by $X(\pi'; \omega; \{ \sigma_{e,f} \})$, or simply by $X(\pi')$. If a $\pi$-set $X(\pi'; \omega; \{ \sigma_{e,f} \})$ satisfies the following two conditions

(iv) $\omega$ is transitive, that is, it is an equivalence relation,
(v) for $(e, f), (f, g) \in \omega$, $\sigma_{e,f}\sigma_{f,g} = \sigma_{e,g}$,

it is called a strong $\pi$-set.

Let $X(\pi'; \omega; \{ \sigma_{e,f} \})$ be a $\pi$-set. A subset $A$ of $X$ is called a $\pi$-single subset of $X$ if for each $e \in \Lambda$, there exists at most one element $f \in \Lambda$ such that $X_f \cap A \neq \emptyset$ and $(e, f) \in \omega$. We consider the empty set as a $\pi$-single subset. Denote the family of all $\pi$-single subsets of $X(\pi'; \omega; \{ \sigma_{e,f} \})$ by $T$.

A mapping $\alpha$ in the symmetric inverse semigroup $I_X$ on $X$ is called a partial one-to-one $\pi$-mapping of $X(\pi'; \omega; \{ \sigma_{e,f} \})$ if $d(\alpha)$ and $r(\alpha)$ are $\pi$-single subsets. Let $LI_{X(\pi')}$ be the set of all partial one-to-one $\pi$-mappings of $X(\pi'; \omega; \{ \sigma_{e,f} \})$, that is, $LI_{X(\pi')} = \{ \alpha \in I_X : d(\alpha), r(\alpha) \in T \}$. The following lemma is clear.

**Lemma 2.1** The set $LI_{X(\pi')}$, defined above, is an inverse subsemigroup of $I_X$.

For $A, B \in T$, define a mapping $\theta_{A,B}$ as follows:

$$d(\theta_{A,B}) = \{ x \in A : \text{there exist } e, f \in \Lambda \text{ such that } (e, f) \in \omega, x \in X_e \text{ and } x\sigma_{e,f} \in B \},$$

$$(3.1) \quad r(\theta_{A,B}) = \{ y \in B : \text{there exist } e, f \in \Lambda \text{ such that } (e, f) \in \omega, y \in X_f \text{ and } y\sigma_{f,e} \in A \},$$

$$x\theta_{A,B} = x\sigma_{e,f} \quad (x \in d(\theta_{A,B}) \cap X_e, (e, f) \in \omega).$$

For any $\alpha, \beta \in LI_{X(\pi')}$, define $\theta_{\alpha,\beta} = \theta_{r(\alpha),d(\beta)}$. Since a subset of $\pi$-single subset is also a $\pi$-single subset, we have that $\theta_{\alpha,\beta} \in LI_{X(\pi')}$ for all $\alpha, \beta \in LI_{X(\pi')}$. Let $M = \{ \theta_{\alpha,\beta} : \alpha, \beta \in LI_{X(\pi')} \}$.

**Lemma 2.2** The set $M$, defined above, is the structure sandwich set of $LI_{X(\pi')}$ determined by a mapping $\theta : LI_{X(\pi')} \times LI_{X(\pi')} \to LI_{X(\pi')}$ ($(\alpha, \beta) \mapsto \theta_{\alpha,\beta}$). Therefore, $LI_{X(\pi')}(M)$ is a regular $\ast$-semigroup.

We call the set $M$, defined above, the structure sandwich set determined by a $\pi$-set $X(\pi'; \omega; \{ \sigma_{e,f} \})$.

It is clear that each projection of $LI_{X(\pi')}(M)$ is the identity mapping $1_A$ on a $\pi$-single subset $A$. Let $1_A$ be any projection and let $\alpha, \beta$ be any projections of $1_A \circ LI_{X(\pi')} \circ 1_A$. There exist $B, C \in T$ such that $\alpha = 1_A \circ 1_B \circ 1_A$ and $\beta = 1_A \circ 1_C \circ 1_A$. Then $\alpha = \theta_{A,B}\theta_{A,B}^{-1} = 1_{d(\theta_{A,B})} \text{ and } \beta = 1_{d(\theta_{A,C})}$. Since $d(\theta_{A,B}) \subseteq A$ and $d(\theta_{A,C}) \subseteq A$, $\theta_{A,B}\theta_{A,B}^{-1} = 1_A \text{ and } 1_{d(\theta_{A,C})} \subseteq 1_A$. 
Similarly $\theta_{1}t_{A}t_{C}:t_{A}t_{B}\subseteq 1_{A}$. Then $\alpha \circ \beta = \beta \circ \alpha$. Therefore, $\mathcal{L}\mathcal{I}_{X(\pi)}(\mathcal{M})$ is a locally inverse $\ast$-semigroup. We call it the $\pi$-symmetric locally inverse $\ast$-semigroup on $X(\pi';\omega;\{\sigma_{e,f}\})$ with the structure sandwich set $\mathcal{M}$. Now, we have the following theorem.

**Theorem 2.3** Let $X$ be a $\pi$-set with a partition $\pi' : X = \sum\{X_{e} : e \in \Lambda\}$, a relation $\omega$ on $\Lambda$ and a set of mappings $\{\sigma_{e,f} : (e, f) \in \omega\}$, and let $\mathcal{M}$ be the structure sandwich set determined by $X(\pi';\omega;\{\sigma_{e,f}\})$. Then $\mathcal{L}\mathcal{I}_{X(\pi')}(\mathcal{M})$ is an inverse subsemigroup of $\mathcal{I}_{X}$. Moreover, $\mathcal{L}\mathcal{I}_{X(\pi')}(\mathcal{M})$ is a locally inverse $\ast$-semigroup.

Let $X(\pi';\omega;\{\sigma_{e,f}\})$ be a strong $\pi$-set, where $\pi' = \sum\{X_{e} : e \in \Lambda\}$. Since $\omega$ is an equivalence relation on $\Lambda$, there exists the partition $\Lambda = \sum\{\Lambda_{i} : i \in I\}$ induced by $\omega$. For each $i \in I$, denote the subset $\bigcup\{X_{e} : e \in \Lambda_{i}\}$ by $X_{i}$.

**Lemma 2.4** A subset $A$ of $X$ is a $\pi$-single subset if and only if it satisfies the condition that for any $i \in I$, $A \cap X_{i} \neq \emptyset$ implies $A \cap X_{i} \subseteq X_{e}$ for some $e \in \Lambda_{i}$.

Let $\mathcal{M}$ be the structure sandwich set determined by $X(\pi';\omega;\{\sigma_{e,f}\})$. By Theorem 3.6 of [7], $\mathcal{L}\mathcal{I}_{X(\pi')}(\mathcal{M})$ is a generalized inverse $\ast$-semigroup. We call such a semigroup the $\pi$-symmetric generalized inverse $\ast$-semigroup and denote it by $\mathcal{G}\mathcal{I}_{X(\pi')}(\mathcal{M})$ instead of $\mathcal{L}\mathcal{I}_{X(\pi')}(\mathcal{M})$.

**Corollary 2.5** (Theorem 3.6 [7]) Let $X$ be a strong $\pi$-set with a partition $\pi' : X = \sum\{X_{e} : e \in \Lambda\}$, an equivalence relation $\omega$ on $\Lambda$ and a set of mappings $\{\sigma_{e,f} : (e, f) \in \omega\}$, and let $\mathcal{M}$ be the structure sandwich set determined by $X(\pi';\omega)$. Then $\mathcal{G}\mathcal{I}_{X(\pi')}(\mathcal{M})$ is a generalized inverse $\ast$-semigroup.

### 3 Representations

Let $S$ be a locally inverse $\ast$-semigroup and $\mathcal{I}_{S}$ the symmetric inverse semigroup on $S$. In this section, denote $E(S)$ and $P(S)$ simply by $E$ and $P$, respectively. Since each $L$-class has one and only one projection, $\pi' : S = \sum\{L_{e} : e \in P\}$ is a partition of $S$, where $L_{e}$ denotes the $L$-class containing $e$. Let $\omega = \{(e, f) \in P \times P : eRgLf \text{ for some } g \in E\}$. It is clear that $\omega$ is a reflexive and symmetric relation on $P$. For $(e, f) \in \omega$, define $\sigma_{e,f} : L_{e} \rightarrow L_{f}$ by $x\sigma_{e,f} = xf$. It follows from Green's Lemma that $S(\pi';\omega;\{\sigma_{e,f}\})$ is a $\pi$-set. Let $T$ be the set of all $\pi$-single subsets of $S(\pi';\omega;\{\sigma_{e,f}\})$ and $\mathcal{M}$ the structure sandwich set determined by $S(\pi';\omega;\{\sigma_{e,f}\})$. By Theorem 2.5, $\mathcal{L}\mathcal{I}_{S(\pi')}(\mathcal{M})$ is a locally inverse $\ast$-semigroup.

For any $a \in S$, let $\rho_{a} : Sa \rightarrow Sa$ be a mapping defined by $x\rho_{a} = xa$. 
It is trivial that $\rho_a$ and $\rho_{a^*}$ are mutually inverse mappings of $S\alpha^*$ and $S\alpha$ onto each other, and hence $\rho_{a} \in I_S$. A subset of $S$ is said to be $L$-full if it is a union of some $L$-classes of $S$.

**Lemma 3.1** (1) For any $a \in S$, $\rho_a \in LI_{S(\pi')}$.

(2) For any $a, b \in S$, $\theta_{\rho_a, \rho_b} = \rho_{a^*} \rho_{a} \rho_{a^*} \rho_{b}$. Therefore, $\rho_a \circ \rho_b = \rho_a \rho_{a^*} \rho_{a^*} \rho_{b}$.

By the lemma above and Theorem 2.2 of [5], we can easily see the following lemma.

**Lemma 3.2** Define a mapping $\phi : S \rightarrow LI_{S(\pi')} (M)$ by

$$a\phi = \rho_a.$$

Then $\phi$ is an $*$-monomorphism.

Now, we have the main theorem.

**Theorem 3.3** A locally inverse $*$-semigroup can be embedded up to $*$-isomorphism in the $\pi$-symmetric locally inverse $*$-semigroup $LI_{X(\pi')}(M)$ on a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ with the structure sandwich set $M$ determined by $X(\pi'; \omega; \{\sigma_{e,f}\})$.

If $S$ is a generalized inverse $*$-semigroup, then a $\pi$-set $S(\pi'; \omega; \{\sigma_{e,f}\})$, constructed above, is a strong $\pi$-set. For, let $(e, f), (f, g) \in \omega$. Then there exist $h, k \in E(S)$ such that $e R h L f$ and $f R k L g$. Since $S$ is a generalized inverse $*$-semigroup, $efg = eg \in E(S)$ and $e R e g L g$. In this case, it follows from [7] that $\sigma_{e,f} \sigma_{f,g} = \sigma_{e,g}$, and hence $S(\pi'; \omega; \{\sigma_{e,f}\})$ is a strong $\pi$-set. Then we have the following corollary.

**Corollary 3.4** (Theorem 4.8 [7]). A generalized inverse $*$-semigroup can be embedded up to $*$-isomorphism in $GI_{X(\pi')}$ on a strong $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$.

### 4 Wreath products

Let $S$ and $T$ be locally inverse $*$-semigroups. By Theorem 3.3, $T$ can be embedded in the $\pi$-symmetric locally inverse $*$-semigroup $LI_{X(\pi')}(M)$ on a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ with the structure sandwich set $M$ determined by $X(\pi'; \omega; \{\sigma_{e,f}\})$. In this case, we can consider $T$ as a locally inverse $*$-subsemigroup of $LI_{X(\pi')}(M)$, and so denote it by $T(X)$.

By $XS$, denote the set of all mappings from the family $T$ of $\pi$-single subsets of $X(\pi'; \omega; \{\sigma_{e,f}\})$ into $S$, and define a multiplication on $XS$ by

$$d(\psi_1 \psi_2) = d(\psi_1) \cap d(\psi_2),$$

$$x(\psi_1 \psi_2) = (x\psi_1)(x\psi_2).$$
For any $\alpha \in \mathcal{L}_{X(\pi')}$ and $\psi \in XS$, let us define $^a\psi (\in XS)$ by

$$^a\psi = \alpha \theta_{\alpha,\psi} \psi,$$

where $\theta_{\alpha,\psi} = \theta_{r(\alpha), d(\psi)} \in \mathcal{M}$.

Let $\psi \in XS$ and $\alpha \in L\mathcal{I}_{X(\pi')}$ such that $d(\psi) = d(\alpha)$. Define a mapping $\psi^*_\alpha (\in XS)$ by

$$d(\psi^*_\alpha) = d(\alpha^{-1}1),$$

$$x\psi^*_\alpha = (x\alpha^{-1}\theta_{\alpha^{-1},\psi}1\psi 1)^*.$$

Since $r(\alpha^{-1}) = d(\alpha) = d(\psi)$, $\theta_{\alpha^{-1},\psi} = 1_{d(\alpha)}$ and hence $x\psi^*_\alpha = (x\alpha^{-1}\psi)^*$ for all $x \in d(\psi^*_\alpha)$.

Now, we define the (right) wreath product $SwrT(X)$ of $S$ and $T(X)$ as follows:

$$SwrT(X) = \{(\psi, \alpha) \in XS \times T(X) : d(\psi) = d(\alpha)\},$$

$$(\psi, \alpha)(\varphi, \beta) = (\psi^\alpha \varphi, \alpha \circ \beta),$$

$$(\psi, \alpha) = (\psi^*_\alpha, \alpha^{-1}).$$

Let $(\psi, \alpha), (\varphi, \beta) \in SwrT(X)$. Then

$$x \in d(\psi^\alpha \varphi) \iff x \in d(\psi) \text{ and } x \in d(\alpha^{-1}1),$$

$$x \in d(\alpha), x\alpha \in d(\theta_{\alpha,\psi}) \text{ and } x\alpha\theta_{\alpha,\psi} \in d(\varphi),$$

$$x \in d(\alpha), x\alpha \in d(\theta_{\alpha,\beta}) \text{ and } x\alpha\theta_{\alpha,\beta} \in d(\beta),$$

$$x \in d(\alpha\theta_{\alpha,\beta} \beta) = d(\alpha \circ \beta).$$

Then $(\psi, \alpha)(\varphi, \beta) = (\psi^\alpha \varphi) \in SwrT(X)$, and hence SwrT(X) is closed under the multiplication. It immediately follows from the definition of $\psi^*_\alpha$ that SwrT(X) is closed under the unary operation $*$. 

**Theorem 4.1** Let $S$ and $T(X)$ be locally inverse $*$-semigroups. Then SwrT(X) is a locally inverse $*$-semigroup. Moreover, we have

$$P(SwrT(X)) = \{(\psi, 1_A) \in SwrT(X) : A \in T \text{ and } r(\psi) \subseteq P(S)\},$$

$$E(SwrT(X)) = \{(\psi, \alpha) \in SwrT(X) : \alpha \in E(T(X)) \text{ and } r(\psi) \subseteq E(S)\}.$$

Next, we shall consider wreath products of generalized inverse $*$-semigroups. Let $S$ and $T(X) (\subseteq GI_{X(\pi')}(\mathcal{M}))$ be generalized inverse $*$-semigroups.

**Lemma 4.2** Let $A, B, C$ be a $\pi$-single subsets of a strong $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$, and let $\psi \in XS$ such that $d(\psi) = C$. Then, for any $x \in d(1_A \circ 1B \circ 1_C)$, $x^{\sigma_{e,f}1\psi} = x^{1\psi}$. 

By using the lemma above, we have the following theorem.

**Theorem 4.3** Let $S$ and $T(X) (\subseteq GI_{X(\pi)}(\mathcal{M}))$ be generalized inverse $*$-semigroups, then SwrT(X) is a generalized inverse $*$-semigroup.
References


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