On representations of locally inverse $\ast$-semigroups

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On representations of locally inverse *-semigroups

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Abstract

The purpose of this paper is to obtain an analogous representation of the Preston-Vagner Representation for locally inverse *-semigroups which is a generalization of [7].

Firstly, by introducing a concept of a $\pi$-set (which is slightly different from the one in [7]), we shall construct the $\pi$-symmetric locally inverse *-semigroup $\mathcal{L}\mathcal{I}_{X(\pi)}(\mathcal{M})$ on a $\pi$-set $X(\pi';\omega;\{\sigma_{e,f}\})$, and show that $\mathcal{L}\mathcal{I}_{X(\pi)}(\mathcal{M})$ is a locally inverse *-semigroup and that any locally inverse *-semigroup can be embedded up to *-isomorphism in $\mathcal{L}\mathcal{I}_{X(\pi)}(\mathcal{M})$ on a $\pi$-set $X(\pi';\omega;\{\sigma_{e,f}\})$. Moreover, we shall show that the wreath product (in the sense of Cowan[1]) of locally inverse *-semigroups is also a locally inverse *-semigroup.

1 Introduction

A semigroup $S$ with a unary operation $*: S \to S$ is called a regular *-semigroup if it satisfies

(i) $(x^*)^* = x,$
(ii) $(xy)^* = y^*x^*,$
(iii) $xx^*x = x.$

Let $S$ be a regular *-semigroup. An idempotent $e$ in $S$ is called a projection if it satisfies $e^* = e$. For any subset $A$ of $S$, denote the sets of idempotents and projections of $A$ by $E(A)$ and $P(A)$, respectively. The following result is well-known, and we use it frequently throughout this paper.

\[This\ is\ the\ abstract\ and\ the\ details\ will\ be\ published\ elsewhere.\ The\ results\ of\ \S\ 2\ and\ 4\ were\ obtained\ after\ the\ conference.\]
Result 1.1 (see [4]). Let $S$ be a regular $\ast$-semigroup. Then we have the followings:

1. $E(S) = P(S)^2$, more precisely, for any $e \in E(S)$, there exist $f, g \in P(S)$ such that $fR\text{e}Lg$ and $e = fg$;
2. for any $a \in S$ and $e \in P(S)$, $a^{\ast}e a \in P(S)$;
3. each $L$-class and each $R$-class have one and only one projection.

A regular $\ast$-semigroup $S$ is called a locally inverse $\ast$-semigroup if, for any $e \in E(S)$, $eSe$ is an inverse subsemigroup of $S$.

Lemma 1.2 A regular $\ast$-semigroup $S$ is a locally inverse $\ast$-semigroup if and only if, for each $e \in P(S)$, $eSe$ is an inverse subsemigroup of $S$.

A regular $\ast$-semigroup $S$ is called a generalized inverse $\ast$-semigroup if $E(S)$ forms a normal band, that is, $E(S)$ satisfies the identity $xyzx = zyx$. It is obvious that a generalized inverse $\ast$-semigroup is a locally inverse $\ast$-semigroup.

Remark. It is clear that a regular $\ast$-semigroup $S$ is a generalized inverse $\ast$-semigroup if and only if it satisfies the following condition:

for any $e, f, g, h \in P(S)$, $efgh = egfh$ (in $S$).

However, we remark that even if a locally inverse $\ast$-semigroup $S$ satisfies the condition

for any $e, f, g \in P(S)$, $efge = egfe$ (in $S$),

it is not always a generalized inverse $\ast$-semigroup. The second remark of [6] is its counterexample.

Let $X$ be a set. If $X = \bigcup\{X_i : i \in I\}$ is a partition of $X$, denote it by $X = \sum\{X_i : i \in I\}$. For a mapping $\alpha : A \rightarrow B$, denote the domain and the range of $\alpha$ by $d(\alpha)$ and $r(\alpha)$, respectively. For a subset $C$ of $A$, $\alpha|_C$ means the restriction of $\alpha$ to $C$.

Let $I_X$ be the symmetric inverse semigroup on a set $X$. For a subset $A$ of $X$, $1_A$ means the identity mapping on $A$. Let $A$ be an inverse subsemigroup of $I_X$ and $\theta : A \times A \rightarrow A$ a mapping. Denote the image $(\alpha, \beta)\theta$ of an ordered pair $(\alpha, \beta)$ by $\theta_{\alpha, \beta}$. Set $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in A\}$. If $\mathcal{M}$ satisfies the following conditions:

(C1) $\theta_{\alpha, \beta}^{-1} = \theta_{\beta^{-1}, \alpha^{-1}}$
(C2) $\theta_{\alpha, \alpha^{-1}} = 1_{r(\alpha)}$
(C3) $\theta_{1_{d(\alpha)}, \alpha} = 1_{d(\alpha)}$
(C4) $\theta_{\alpha, \beta}\theta_{\alpha, \beta, \gamma} = \theta_{\alpha, \beta, \gamma}\theta_{\beta, \gamma}$

we call it the structure sandwich set of $A$ determined by $\theta$. 
Result 1.3 (see [7]) Let $A$ be an inverse subsemigroup of the symmetric inverse semigroup $I_X$ on a set $X$, and $M$ the structure sandwich set of $A$ determined by a mapping $\theta : A \times A \rightarrow A$. Define a multiplication $\circ$ and a unary operation $*$ on $A$ as follows:

\[ \alpha \circ \beta = \alpha \theta_{\alpha,\beta} \beta, \]
\[ \alpha^* = \alpha^{-1} \]

Then $A(\circ, *)$ becomes a regular $*$-semigroup.

Hereafter, we call such a semigroup $A(\circ, *)$ a regular $*$-semigroup of partial one-to-one mappings determined by the structure sandwich set $M$, and denote it by $A(M)$. The notation and terminology are those of [3] and [4], unless otherwise stated.

In § 2, we shall firstly consider a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ which is a set $X$ with a partition $\pi' : X = \sum \{X_e : e \in \Lambda\}$, a reflexive and symmetric relation $\omega$ on $\Lambda$ and a set of mappings $\{\sigma_{e,f} : (e, f) \in \omega\}$, where $\sigma_{e,f}$ is a bijection of $X_e$ onto $X_f$. We remark that a $\pi$-set, defined in this paper, is slightly different from the one in [7], which is called a strong $\pi$-set in this paper. The set $LI_{X(\pi')}$, say, of all partial one-to-one $\pi$-mappings on $X(\pi'; \omega; \{\sigma_{e,f}\})$ is an inverse subsemigroup of $I_X$. By using $\{\sigma_{e,f} : (e, f) \in \omega\}$, we shall construct a structure sandwich set $M$, and show that $LI_{X(\pi')} (M)$ is a locally inverse $*$-semigroup. We call such a semigroup $LI_{X(\pi')} (M)$ the $\pi$-symmetric locally inverse $*$-semigroup on a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ with the structure sandwich set $M$.

In § 3, we shall show that any locally inverse $*$-semigroup is embedded up to $*$-isomorphism in the $\pi$-symmetric locally inverse $*$-semigroup $LI_{X(\pi')} (M)$ on a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$.

As a generalization of [2], Cowan [1] gave us the definition of the wreath product $SwrT(X)$ of inverse semigroups $S$ and $T(X)$, where $T(X)$ is an inverse subsemigroup of $I_X$. And he showed that the wreath product $SwrT(X)$ is also an inverse semigroup. In § 4, we shall show that the wreath product of locally inverse $*$-semigroups $S$ and $T(X)$ ($\subseteq LI_{X(\pi')} (M)$) is a locally inverse $*$-semigroup. Moreover, we shall obtain that the wreath product of generalized inverse $*$-semigroups is also a generalized inverse $*$-semigroup.

2 $\pi$-Symmetric locally inverse $*$-semigroups

Let $X$ be a non-empty set. If there exist a partition $X = \sum \{X_e : e \in \Lambda\}$ and a reflexive and symmetric relation $\omega$ on $\Lambda$ such that

(i) for each $(e, f) \in \omega$, there exists a bijection $\sigma_{e,f} : X_e \rightarrow X_f$,

(ii) for all $e \in \Lambda$, $\sigma_{e,e} = 1_{X_e}$,

(iii) for any $(e, f) \in \omega$, $\sigma_{f,e} = \sigma_{e,f}^{-1}$,
then $X$ is called a $\pi$-set with a partition $\pi': X = \sum \{ X_e : e \in \Lambda \}$, a relation $\omega$ and a set of mappings $\{ \sigma_{e,f} : (e, f) \in \omega \}$, and denote it by $X(\pi'; \omega; \{ \sigma_{e,f} \})$, or simply by $X(\pi')$. If a $\pi$-set $X(\pi'; \omega; \{ \sigma_{e,f} \})$ satisfies the following two conditions

(iv) $\omega$ is transitive, that is, it is an equivalence relation,

(v) for $(e, f), (f, g) \in \omega$, $\sigma_{e,f} \sigma_{f,g} = \sigma_{e,g}$,

it is called a strong $\pi$-set.

Let $X(\pi'; \omega; \{ \sigma_{e,f} \})$ be a $\pi$-set. A subset $A$ of $X$ is called a $\pi$-single subset of $X$ if for each $e \in \Lambda$, there exists at most one element $f \in \Lambda$ such that $X_f \cap A \neq \emptyset$ and $(e, f) \in \omega$. We consider the empty set as a $\pi$-single subset. Denote the family of all $\pi$-single subsets of $X(\pi'; \omega; \{ \sigma_{e,f} \})$ by $T$.

A mapping $\alpha$ in the symmetric inverse semigroup $I_X$ on $X$ is called a partial one-to-one $\pi$-mapping of $X(\pi'; \omega; \{ \sigma_{e,f} \})$ if $d(\alpha)$ and $r(\alpha)$ are $\pi$-single subsets. Let $LI_X(\pi')$ be the set of all partial one-to-one $\pi$-mappings of $X(\pi'; \omega; \{ \sigma_{e,f} \})$, that is, $LI_X(\pi') = \{ \alpha \in I_X : d(\alpha), r(\alpha) \in T \}$. The following lemma is clear.

**Lemma 2.1** The set $LI_X(\pi')$, defined above, is an inverse subsemigroup of $I_X$.

For $A, B \in T$, define a mapping $\theta_{A,B}$ as follows:

$$d(\theta_{A,B}) = \{ x \in A : \text{there exist } e, f \in \Lambda \text{ such that } (e, f) \in \omega, x \in X_e \text{ and } x \sigma_{e,f} \in B \},$$

$$r(\theta_{A,B}) = \{ y \in B : \text{there exist } e, f \in \Lambda \text{ such that } (e, f) \in \omega, y \in X_f \text{ and } y \sigma_{f,e} \in A \},$$

$$x \theta_{A,B} = x \sigma_{e,f} \quad (x \in d(\theta_{A,B}) \cap X_e, (e, f) \in \omega).$$

For any $\alpha, \beta \in LI_X(\pi')$, define $\theta_{\alpha,\beta} = \theta_{r(\alpha),d(\beta)}$. Since a subset of $\pi$-single subset is also a $\pi$-single subset, we have that $\theta_{\alpha,\beta} \in LI_X(\pi')$ for all $\alpha, \beta \in LI_X(\pi')$. Let $M = \{ \theta_{\alpha,\beta} : \alpha, \beta \in LI_X(\pi') \}$.

**Lemma 2.2** The set $M$, defined above, is the structure sandwich set of $LI_X(\pi')$ determined by a mapping $\theta : LI_X(\pi') \times LI_X(\pi') \rightarrow LI_X(\pi') (\alpha, \beta) \mapsto \theta_{\alpha,\beta}$. Therefore, $LI_X(\pi')(M)$ is a regular $*$-semigroup.

We call the set $M$, defined above, the structure sandwich set determined by a $\pi$-set $X(\pi'; \omega; \{ \sigma_{e,f} \})$.

It is clear that each projection of $LI_X(\pi')(M)$ is the identity mapping $1_A$ on a $\pi$-single subset $A$. Let $1_A$ be any projection and let $\alpha, \beta$ be any projections of $1_A \circ LI_X(\pi') \circ 1_A$. There exist $B, C \in T$ such that $\alpha = 1_A \circ 1_B \circ 1_A$ and $\beta = 1_A \circ 1_C \circ 1_A$. Then $\alpha = \theta_{A,B} \theta_{A,B}^{-1} = 1d(\theta_{A,B})$ and $\beta = 1d(\theta_{A,C})$. Since $d(\theta_{A,B}) \subseteq A$ and $d(\theta_{A,C}) \subseteq A$, $\theta_{A,B} \theta_{A,C} \subseteq \theta_{A,A} = 1_A$. 


Similarly $\theta_{1}^{*}_{A,C} \cdot \theta_{A,B}^{*} \subseteq 1_{A}$. Then $\alpha \circ \beta = \beta \circ \alpha$. Therefore, $\mathcal{L}I_{X(\pi)}(M)$ is a locally inverse $*$-semigroup. We call it the $\pi$-symmetric locally inverse $*$-semigroup with the structure sandwich set $M$. Now, we have the following theorem.

**Theorem 2.3** Let $X$ be a $\pi$-set with a partition $\pi' : X = \sum\{X_{e} : e \in \Lambda\}$, a relation $\omega$ on $\Lambda$ and a set of mappings $\{\sigma_{e,f} : (e, f) \in \omega\}$, and let $M$ be the structure sandwich set determined by $X(\pi'; \omega; \{\sigma_{e,f}\})$. Then $\mathcal{L}I_{X(\pi')}$ is an inverse subsemigroup of $I_{X}$. Moreover, $\mathcal{L}I_{X(\pi')}(M)$ is a locally inverse $*$-semigroup.

Let $X(\pi'; \omega; \{\sigma_{e,f}\})$ be a strong $\pi$-set, where $\pi' = \sum\{X_{e} : e \in \Lambda\}$. Since $\omega$ is an equivalence relation on $\Lambda$, there exists the partition $\Lambda = \sum\{\Lambda_{i} : i \in I\}$ induced by $\omega$. For each $i \in I$, denote the subset $\cup\{X_{e} : e \in \Lambda_{i}\}$ by $X_{i}$.

**Lemma 2.4** A subset $A$ of $X$ is a $\pi$-single subset if and only if it satisfies the condition that for any $i \in I$, $\Lambda \cap X_{i} \neq \emptyset$ implies $A \cap X_{i} \subseteq X_{e}$ for some $e \in \Lambda_{i}$.

Let $M$ be the structure sandwich set determined by $X(\pi'; \omega; \{\sigma_{e,f}\})$. By Theorem 3.6 of [7], $\mathcal{L}I_{X(\pi')}(M)$ is a generalized inverse $*$-semigroup. We call such a semigroup the $\pi$-symmetric generalized inverse $*$-semigroup and denote it by $\mathcal{G}I_{X(\pi')}(M)$ instead of $\mathcal{L}I_{X(\pi')}(M)$.

**Corollary 2.5** (Theorem 3.6 [7]) Let $X$ be a strong $\pi$-set with a partition $\pi' : X = \sum\{X_{e} : e \in \Lambda\}$, an equivalence relation $\omega$ on $\Lambda$ and a set of mappings $\{\sigma_{e,f} : (e, f) \in \omega\}$, and let $M$ be the structure sandwich set determined by $X(\pi'; \omega)$. Then $\mathcal{G}I_{X(\pi')}(M)$ is a generalized inverse $*$-semigroup.

## 3 Representations

Let $S$ be a locally inverse $*$-semigroup and $I_{S}$ the symmetric inverse semigroup on $S$. In this section, denote $E(S)$ and $P(S)$ simply by $E$ and $P$, respectively. Since each $L$-class has one and only one projection, $\pi' : S = \sum\{L_{e} : e \in P\}$ is a partition of $S$, where $L_{e}$ denotes the $L$-class containing $e$. Let $\omega = \{(e, f) \in P \times P : eRgLf$ for some $g \in E\}$. It is clear that $\omega$ is a reflexive and symmetric relation on $P$. For $(e, f) \in \omega$, define $\sigma_{e,f} : L_{e} \rightarrow L_{f}$ by $x\sigma_{e,f} = xf$. It follows from Green's Lemma that $S(\pi'; \omega; \{\sigma_{e,f}\})$ is a $\pi$-set. Let $T$ be the set of all $\pi$-single subsets of $S(\pi'; \omega; \{\sigma_{e,f}\})$ and $M$ the structure sandwich set determined by $S(\pi'; \omega; \{\sigma_{e,f}\})$. By Theorem 2.5, $\mathcal{L}I_{S(\pi')}(M)$ is a locally inverse $*$-semigroup.

For any $a \in S$, let $\rho_{a} : Sa^{*} \rightarrow Sa$ be a mapping defined by $x\rho_{a} = xa$. 

$x\rho_{a} = xa$. 
It is trivial that $\rho_a$ and $\rho_{a^*}$ are mutually inverse mappings of $Sa^*$ and $Sa$ onto each other, and hence $\rho_{a} \in \mathcal{I}_S$. A subset of $S$ is said to be $L$-full if it is a union of some $L$-classes of $S$.

Lemma 3.1 (1) For any $a \in S$, $\rho_{a} \in \mathcal{LI}_S(X)$.

(2) For any $a, b \in S$, $\theta_{\rho_{a}, \rho_{b}} = \rho_{a^*ab^*}$. Therefore, $\rho_{a} \circ \rho_{b} = \rho_{a}\rho_{a^*ab^*}\rho_{b}$.

By the lemma above and Theorem 2.2 of [5], we can easily see the following lemma.

Lemma 3.2 Define a mapping $\phi: S \to \mathcal{LI}_S(X)$ by

$$a\phi = \rho_{a}.$$ 

Then $\phi$ is a $*$-monomorphism.

Now, we have the main theorem.

Theorem 3.3 A locally inverse $*$-semigroup can be embedded up to $*$-isomorphism in the $\pi$-symmetric locally inverse $*$-semigroup $\mathcal{LI}_X(X(\pi))$ on a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ with the structure sandwich set $\mathcal{M}$ determined by $X(\pi'; \omega; \{\sigma_{e,f}\})$.

If $S$ is a generalized inverse $*$-semigroup, then a $\pi$-set $S(\pi'; \omega; \{\sigma_{e,f}\})$, constructed above, is a strong $\pi$-set. For, let $(e, f), (f, g) \in \omega$. Then there exist $h, k \in E(S)$ such that $eRhLf$ and $fRkLg$. Since $S$ is a generalized inverse $*$-semigroup, $efg = eg \in E(S)$ and $eRegLg$. In this case, it follows from [7] that $\sigma_{e,f}\sigma_{f,g} = \sigma_{e,g}$, and hence $S(\pi'; \omega; \{\sigma_{e,f}\})$ is a strong $\pi$-set. Then we have the following corollary.

Corollary 3.4 (Theorem 4.8 [7]). A generalized inverse $*$-semigroup can be embedded up to $*$-isomorphism in $\mathcal{GI}_X(X(\pi'))$ on a strong $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$.

4 Wreath products

Let $S$ and $T$ be locally inverse $*$-semigroups. By Theorem 3.3, $T$ can be embedded in the $\pi$-symmetric locally inverse $*$-semigroup $\mathcal{LI}_X(X(\pi))$ on a $\pi$-set $X(\pi'; \omega; \{\sigma_{e,f}\})$ with the structure sandwich set $\mathcal{M}$ determined by $X(\pi'; \omega; \{\sigma_{e,f}\})$. In this case, we can consider $T$ as a locally inverse $*$-subsemigroup of $\mathcal{LI}_X(X(\pi))$, and so denote it by $T(X)$.

By $X^S$, denote the set of all mappings from the family $T$ of $\pi$-single subsets of $X(\pi'; \omega; \{\sigma_{e,f}\})$ into $S$, and define a multiplication on $X^S$ by

$$d(\psi_1\psi_2) = d(\psi_1) \cap d(\psi_2),$$
$$x(\psi_1\psi_2) = (x\psi_1)(x\psi_2).$$
For any $\alpha \in \mathcal{L}_{X(\pi')}$ and $\psi \in XS$, let us define $^\alpha \psi (\in XS)$ by

$$^\alpha \psi = \alpha \theta_{\alpha, \psi} \psi,$$

where $\theta_{\alpha, \psi} = \theta_{r(\alpha), d(\psi)} \in M$.

Let $\psi \in XS$ and $\alpha \in \mathcal{L}_{X(\pi')}$ such that $d(\psi) = d(\alpha)$. Define a mapping $\psi^*_\alpha (\in XS)$ by

$$d(\psi^*_\alpha) = d(\alpha^{-1})1,$$
$$x^{\psi^*_\alpha} = (x^{\alpha^{-1}} \theta_{\alpha^{-1}, \psi} 1 \psi 1)^*$$

Since $r(\alpha^{-1}) = d(\alpha) = d(\psi)$, $\theta_{\alpha^{-1}, \psi} = 1_{d(\alpha)}$ and hence $x^{\psi^*_\alpha} = (x^{\alpha^{-1}} \psi)^*$ for all $x \in d(\psi^*_\alpha)$.

Now, we define the (right) wreath product $S \mathrm{w} \mathrm{r} T(X)$ of $S$ and $T(X)$ as follows:

$$S \mathrm{w} \mathrm{r} T(X) = \{(\psi, \alpha) \in XS \times T(X) : d(\psi) = d(\alpha)\},$$
$$(\psi, \alpha)(\varphi, \beta) = (\psi^\alpha \varphi, \alpha \circ \beta),$$
$$(\psi, \alpha) = (\psi^*_\alpha, \alpha^{-1}).$$

Let $(\psi, \alpha), (\varphi, \beta) \in S \mathrm{w} \mathrm{r} T(X)$. Then

$$x \in d(\psi^\alpha \varphi) \iff x \in d(\psi) \text{ and } x \in d(\alpha^{-1})1.$$  
$$(\psi, \alpha)(\varphi, \beta) = (\psi^\alpha \varphi, \alpha \circ \beta),$$  
$$(\psi, \alpha) = (\psi^*_\alpha, \alpha^{-1}).$$

Then $(\psi, \alpha)(\varphi, \beta) = (\psi^\alpha \varphi) \in S \mathrm{w} \mathrm{r} T(X)$, and hence $S \mathrm{w} \mathrm{r} T(X)$ is closed under the multiplication. It immediately follows from the definition of $\psi^*_\alpha$ that $S \mathrm{w} \mathrm{r} T(X)$ is closed under the unary operation $\ast$.

**Theorem 4.1** Let $S$ and $T(X)$ be locally inverse $\ast$-semigroups. Then $S \mathrm{w} \mathrm{r} T(X)$ is a locally inverse $\ast$-semigroup. Moreover, we have

$$P(S \mathrm{w} \mathrm{r} T(X)) = \{(\psi, 1_A) \in S \mathrm{w} \mathrm{r} T(X) : A \in T \text{ and } r(\psi) \subseteq P(S)\},$$
$$E(S \mathrm{w} \mathrm{r} T(X)) = \{(\psi, \alpha) \in S \mathrm{w} \mathrm{r} T(X) : \alpha \in E(T(X)) \text{ and } r(\psi) \subseteq E(S)\}.$$

Next, we shall consider wreath products of generalized inverse $\ast$-semigroups. Let $S$ and $T(X) \subseteq G_{X(\pi')}(\mathcal{M})$ be generalized inverse $\ast$-semigroups.

**Lemma 4.2** Let $A, B, C$ be a $\pi$-single subsets of a strong $\pi$-set $X(\pi'\omega;\{\sigma_{e,f}\})$, and let $\psi \in XS$ such that $d(\psi) = C$. Then, for any $x \in d(1_A \circ 1_B \circ 1_C)$, $x^{1_A \circ 1_B \psi} = x^{1_A \psi}$.

By using the lemma above, we have the following theorem.

**Theorem 4.3** Let $S$ and $T(X) \subseteq G_{X(\pi')}(\mathcal{M})$ be generalized inverse $\ast$-semigroups, then $S \mathrm{w} \mathrm{r} T(X)$ is a generalized inverse $\ast$-semigroup.
References


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