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Some remarks on representations of fundamental generalized inverse *-semigroups

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Abstract

W.D. Munn [4] described that every fundamental inverse semigroup can be faithfully represented by isomorphisms among principal ideals of the semilattice of idempotents of it. Also T. Imaoka [3] has given a generalization of the Preston-Vagner representation for generalized inverse *-semigroups by using a concept of a structure sandwich set of an inverse subsemigroup of the symmetric inverse semigroup on a set.

In this paper, we shall construct a fundamental regular *-semigroup $\mathcal{F}\mathcal{G}\mathcal{I}_{X(\pi)}$ on a set $X$ with a partition $\pi: X = \Sigma\{X_i : i \in I\}$, and obtain a faithful representation of a fundamental generalized inverse *-semigroup into *-semigroup $\mathcal{F}\mathcal{G}\mathcal{I}_{X(\pi)}$ on a set $X(\pi)$.

1 Introduction

A semigroup $S$ with a unary operation $*: S \to S$ is called a regular *-semigroup if it satisfies followings

(1) $(x^*)^* = x$,
(2) $(xy)^* = y^*x^*$,
(3) $xx^*x = x$.

Let $S$ be a regular *-semigroup. Then an idempotent $e$ of $S$ is called a projection if it satisfies $e^* = e$. For any subset $A$ of $S$, denote the sets of idempotents and projections of $A$ by $E(A)$ and $P(A)$, respectively. The following result is well-known and we use it frequently throughout this paper.

Result 1.1 (see [2]) Let $S$ be a regular *-semigroup. Then we have the followings:

(1) $E(S) = P(S)^2$;
(2) for any $a \in S$ and $e \in P(S), a^*ea \in P(S)$;
(3) each $\mathcal{L}$-class and each $\mathcal{R}$-class have one and only one projection.
A regular \(*\)-semigroup \(S\) is called a \emph{generalized inverse \(*\)-semigroup} if \(E(S)\) forms a normal band, that is, \(E(S)\) satisfies the identity \(xyzx = xzyx\), equivalently, \(P(S)\) satisfies the identity \(xyzw = xzyw\).

Let \(S\) be a regular \(*\)-semigroup and \(\rho\) a congruence on \(S\). Then \(\rho\) is called a \(\ast\)-congruence if \((x, y) \in \rho\) implies \((x^*, y^*) \in \rho\).

Result 1.2 (see \cite{2}) Let \(S\) be a regular \(*\)-semigroup. Then,
\[
\mu = \{(a, b) \in S \times S : aea^* = beb^* \text{ and } a^*ea = b^*eb \text{ for all } e \in P(S)\}
\]
is the maximum idempotent separating \(\ast\)-congruence on \(S\).

A regular \(*\)-semigroup \(S\) is said to be \emph{fundamental} if \(\mu\) is the equality relation on \(S\).

The notation and terminology are those of \cite{1} and \cite{2} unless otherwise stated.

2 Structure sandwich sets

Let \(\mathcal{I}_X\) be the symmetric inverse semigroup on a set \(X\). For any subset \(A\) of \(X\), \(1_A\) means the identity mapping on \(A\). Let \(A\) be an inverse subsemigroup of \(\mathcal{I}_X\) and \(\theta : A \times A \to A\) a mapping. Denote the image \((\alpha, \beta)\theta\) of an ordered pair \((\alpha, \beta)\) by \(\theta_{\alpha, \beta}\). Set \(\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in A\}\). If \(\mathcal{M}\) satisfies the following conditions:
\[
\begin{align*}
(1) & \quad \theta_{\alpha, \beta}^{-1} = \theta_{\beta^{-1}, \alpha^{-1}}, \\
(2) & \quad \theta_{\alpha, \alpha^{-1}} = 1_{r(\alpha)}, \\
(3) & \quad \theta_{1_{d(\alpha)}, \alpha} = 1_{d(\alpha)}, \\
(4) & \quad \theta_{\alpha, \beta}\theta_{\alpha} = \theta_{\alpha, \beta}\theta_{\beta}, \gamma = \theta_{\alpha, \beta}\theta_{\beta, \gamma}.
\end{align*}
\]
we call it the \emph{structure sandwich set} of \(A\) determined by \(\theta\). The following lemma and theorem are obvious by the definition.

Lemma 2.1 Let \(\mathcal{M}\) be the structure sandwich set of \(A\) determined by \(\theta\). Then,
\[
\theta_{\alpha, 1_{r(\alpha)}} = 1_{r(\alpha)}.
\]

Theorem 2.2 (see \cite{3}) Let \(A\) be an inverse subsemigroup of the symmetric inverse semigroup \(\mathcal{I}_X\) on a set \(X\), and \(\mathcal{M}\) the structure sandwich set of \(A\) determined by a mapping \(\theta : A \times A \to A\). Define a multiplication \(\circ\) and a unary operation \(\ast\) on \(A\) as follows:
\[
\alpha \circ \beta = \alpha \theta_{\alpha, \beta} \beta, \\
\alpha^* = \alpha^{-1}.
\]
Then \(A(\circ, \ast)\) becomes a regular \(*\)-semigroup with \(E(A(\circ, \ast)) = \{\alpha \in A(\circ, \ast) : \theta_{\alpha, \alpha} = \alpha^{-1}\}\) and \(P(A(\circ, \ast)) = \{1_{d(\alpha)} : \alpha \in A(\circ, \ast)\}\).

We denote \(A(\circ, \ast)\), defined above, by \(A(\mathcal{M})\).
3 Construction

Let $X$ be a set and $\pi : X = \Sigma\{X_i : i \in I\}$ a partition of $X$. In this case, we denote $X$ by $X(\pi)$. A subset $A$ of $X$ is called a $\pi$-singleton subset of $X(\pi)$ if $|A \cap X_i| \leq 1$ for all $i \in I$. A mapping $\alpha \in I_X$ is called a $\pi$-singleton bijection of $X(\pi)$ if $d(\alpha)$ and $r(\alpha)$ are $\pi$-singleton subsets of $X(\pi)$. Denote the set of all $\pi$-singleton bijections of $X(\pi)$ by $\mathcal{FGI}_{X(\pi)}$. The following lemma is clear.

Lemma 3.1 The set $\mathcal{FGI}_{X(\pi)}$, defined above, is an inverse subsemigroup of $I_X$.

For any $\alpha, \beta \in \mathcal{FGI}_{X(\pi)}$, define a mapping $\theta_{\alpha, \beta}$ as follows:

- $d(\theta_{\alpha, \beta}) = \{e \in r(\alpha) : \text{there exist } i \in I \text{ and } f \in d(\beta) \text{ such that } e, f \in X_i\}$,
- $r(\theta_{\alpha, \beta}) = \{f \in d(\beta) : \text{there exist } i \in I \text{ and } e \in r(\alpha) \text{ such that } e, f \in X_i\}$,
- $e\theta_{\alpha, \beta} = f$, where $r(\alpha) \cap X_i = \{e\}$ and $d(\beta) \cap X_i = \{f\}$.

Proposition 3.2 The set $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in \mathcal{FGI}_{X(\pi)}\}$ is the structure sandwich set of $\mathcal{FGI}_{X(\pi)}$ determined by a mapping $\theta : (\alpha, \beta) \mapsto \theta_{\alpha, \beta}$. Therefore $\mathcal{FGI}_{X(\pi)}(\mathcal{M})$ is a regular $\ast$-semigroup. Furthermore $\mathcal{FGI}_{X(\pi)}(\mathcal{M})$ becomes a fundamental generalized inverse $\ast$-semigroup.

Proof. We can prove the proposition by similar argument of Lemma 3.4 and Lemma 3.5 of [3].

4 Representation

Let $S$ be a fundamental generalized inverse $\ast$-semigroup. In this section, we denote $E(S)$ and $P(S)$ simply by $E$ and $P$, respectively. Let $E \sim \Sigma\{E_i : i \in I\}$ be the structure decomposition of the normal band $E$. Then $\pi : P = \Sigma\{P_i : i \in I\}$ is a partition of $P$, where $P_i = P(E_i)$. Then $\mathcal{FGI}_{P(\pi)}(\mathcal{M})$, constructed in the preceding section, is a fundamental generalized inverse $\ast$-semigroup.

Lemma 4.1 For any $a \in S$, $Sa \cap P = Sa^*a \cap P$ is a $\pi$-singleton subset of $P(\pi)$.

Proof. Put $a^*a = e$ and assume that $f, g \in Se \cap P_i$. Then there exist $x, y \in S$ such that $f = xe$ and $g = ye$. Hence $f = efe$ and $g = ege$. Since $E_i$ is a rectangular band and $E$ is a normal band, $f = fgf = efegefe = egefege = gfg = g$. This shows that $Sa \cap P$ is a $\pi$-singleton subset of $P(\pi)$.

Lemma 4.2 For any $a \in S$, let $\tau_a : Sa^* \cap P \to Sa \cap P$ be a mapping defined by $e\tau_a = a^*ea$.

Then, for any $a \in S$, $\tau_a \in \mathcal{FGI}_{P(\pi)}(\mathcal{M})$ and $\tau_a^{-1} = \tau_{a^*}$.
Proof. By Lemma 4.1, $d(\tau_a)$ and $\tau_a$ are $\pi$-singleton subsets of $P(\pi)$. Let $e$ and $f$ are any elements of $Sa^* \cap P$ such that $e\tau_a = f\tau_a$, that is, $a^*ea = a^*fa$. Then there exist $x, y \in S$ such that $e = xa^*$ and $f = ya^*$. Thus $e = e^*e = ax^*xa^* = aa^*ax^*xa^*aa^* = aa^*eax^* = aa^*fya^* = aa^*fya^* = ay^*ya^* = ay^*ya^* = f$. This implies that $\tau_a$ is injective. Next, let $f$ be any element of $Sa \cap P$. Then there exists $x \in S$ such that $f = xa$. Put $e = aa^*x^*xax^*$. Then $e \in Sa^* \cap P$ and $e\tau_a = a^*ea = a^*(aa^*x^*xax^*)a = a^*x^*xa = f$. This implies that $\tau_a$ is bijective. The last statement of the lemma is obvious.

Lemma 4.3 For any $a, b \in S$, $\theta_{\tau_a, \tau_b} = \tau_{a*abb^*}$.

Proof. Let $e$ be any element of $d(\theta_{\tau_a, \tau_b})$. Then $e \in Sa \cap P$ and there exist $i \in I$ and $f \in Sb^* \cap P$ such that $e, f \in P$. Hence there exist $x, y \in S$ such that $e = xa$ and $f = yb^*$. Thus $e = e f e = x ay^*ya^* = x ay^*ya^* = x ay^*ya^* = yb^*$. Then $f \in Sb^* \cap P$ and it is clear that $e$ and $f$ are contained in a same $P$. Therefore $e \in d(\theta_{\tau_a, \tau_b})$ and $e\tau_{a*abb^*} = bb^*a*a^*aa^*abb^* = bb^*a^*ax^*xax^*abb^* = f = e\tau_{a*abb^*}$.

Lemma 4.4 Define a mapping $\phi : S \to \mathcal{F}_{\pi}(\mathcal{M})$ by

$$a^* \phi = \tau_a.$$

Then $\phi$ is a $*$-monomorphism.

Proof. To prove that $\phi$ is a homomorphism it is sufficient to show that, for any $a, b \in S, d(\tau_a \circ \tau_b) = d(\tau_{ab})$. Let $a$ and $b$ be any elements of $S$. Then,

$$d(\tau_a \circ \tau_b) = d(\tau_a \tau_{a*abb^*} \tau_b) \quad \text{by Lemma 4.3}$$

$$= \{Sa \cap P \cap (Sa*abb^* \cap P \cap Sb^*)\tau_{a*abb^*}^{-1}\} \tau_a^{-1}$$

$$= \{Sa \cap P \cap (Sa*abb^* \cap P \cap Sb^*)\tau_{a*abb^*}^{-1}\} \tau_a^{-1}$$

$$= \{Sa \cap P \cap Sb^*a^*a\} \tau_a^{-1} \quad \text{(since $\tau_{a*abb^*}^{-1} = \tau_{bb^*a*a}$)}$$

$$= \{Sb^*a^*a \cap P\} \tau_a^{-1}.$$

Hence let $e$ be any element of $d(\tau_a \circ \tau_b)$. Then there exists $f \in Sb^*a^*a \cap P$ such that $e\tau_a = f\tau_a$, that is $a^*ea = f$. Since $e \in Sa^*$ and $f \in Sb^*a^*a$, there exist $x, y \in S$ such that $e = xa^*$ and $f = yb^*a^*a$. Thus $e = e^*e = ax^*xa^* = aa^*ax^*xa^*aa^* = a(\phi\tau_a)a^* = afa^* = ayb^*a^* \in Sb^*a^* \cap P = d(\tau_{ab})$.

Conversely let $e$ be any element of $d(\tau_{ab}) = Sb^*a^* \cap P$. Then there exists $x \in S$ such that $e = xb^*a^*$. Hence $e\tau_a = xb^*a^*\tau_a = a^*xb^*a^*a \in Sb^*a^*a \cap P$. Thus $e \in (Sb^*a^*a \cap P) \tau_a^{-1} \text{ and so } d(\tau_a \circ \tau_b) = d(\tau_{ab})$.

Let $a$ and $b$ be any elements of $S$ such that $a^* \phi = b \phi$, that is, $\tau_a = \tau_b$. Then it is easy to show that $a^*a = bb^*$ and $a^*a = b^*b$. Hence assume that $(a, b) \notin \mu$. Then there exists $e \in P$ such that $ae^* \neq eb^* \text{ or } a^*ea \neq b^*eb$. 

If $aea^* \neq beb^*$, we have $aea^*\tau_a = beb^*\tau_b$, since $a^*aea^*a = b^*beb^*b$. This contradicts that $\tau_a = \tau_b$ is injective. Similarly, if $a^*ea \neq b^*eb$, we have $a^*ea^*a \tau_a = b^*eb^*b \tau_b$, that is, $a^*ea = b^*eb$. This contradicts the hypothesis. Therefore, $(a, b) \in \mu$. Since $S$ is fundamental, we have that $a = b$. Thus $\phi$ is injective.

By Lemma 4.2, $\phi$ is compatible with a unary operation $*$ and so $\phi$ is a $*$-monomorphism.

Now we have the following theorem.

**Theorem 4.5** Every fundamental generalized inverse $*$-semigroup has a faithful representation into $\mathcal{F}GL_{X(\pi)}$ on a set $X(\pi)$.

**References**


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