Some remarks on representations of fundamental generalized inverse $\ast$-semigroups (Semigroups, Formal Languages and Combinatorics on Words)

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Some remarks on representations of fundamental generalized inverse \(*\)-semigroups

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Abstract

W.D.Munn [4] described that every fundamental inverse semigroup can be faithfully represented by isomorphisms among principal ideals of the semilattice of idempotents of it. Also T.Imaoka [3] has given a generalization of the Preston-Vagner representation for generalized inverse \(*\)-semigroups by using a concept of a structure sandwich set of an inverse subsemigroup of the symmetric inverse semigroup on a set.

In this paper, we shall construct a fundamental regular \(*\)-semigroup \(\mathcal{F}\mathcal{G}\mathcal{I}_X(\pi)\) on a set \(X\) with a partion \(\pi : X = \Sigma\{X_i : i \in I\}\), and obtain a faithful representation of a fundamental generalized inverse \(*\)-semigroup into \(*\)-semigroup \(\mathcal{F}\mathcal{G}\mathcal{I}_X(\pi)\) on a set \(X(\pi)\).

1 Introduction

A semigroup \(S\) with a unary operation \(* : S \to S\) is called a regular \(*\)-semigroup if it satisfies followings

\[
(1) \quad (x^*)^* = x,
(2) \quad (xy)^* = y^*x^*,
(3) \quad xx^*x = x.
\]

Let \(S\) be a regular \(*\)-semigroup. Then an idempotent \(e\) of \(S\) is called a projection if it satisfies \(e^* = e\). For any subset \(A\) of \(S\), denote the sets of idempotents and projections of \(A\) by \(E(A)\) and \(P(A)\), respectively. The following result is well-known and we use it frequently throughout this paper.

Result 1.1 (see [2]) Let \(S\) be a regular \(*\)-semigroup. Then we have the followings:

(1) \(E(S) = P(S)^2;\)
(2) for any \(a \in S\) and \(e \in P(S)\), \(a^*ea \in P(S);\)
(3) each \(L\)-class and each \(R\)-class have one and only one projection.
A regular \(*\)-semigroup \(S\) is called a generalized inverse \(*\)-semigroup if \(E(S)\) forms a normal band, that is, \(E(S)\) satisfies the identity \(xyz=xzy\), equivalently, \(P(S)\) satisfies the identity \(xyzw=xzyw\).

Let \(S\) be a regular \(*\)-semigroup and \(\rho\) a congruence on \(S\). Then \(\rho\) is called a \(*\)-congruence if \((x, y) \in \rho\) implies \((x^*, y^*) \in \rho\).

**Result 1.2** (see [2]) Let \(S\) be a regular \(*\)-semigroup. Then,

\[
\mu = \{(a, b) \in S \times S : aea^* = beb^* \text{ and } a^*ea = b^*eb \text{ for all } e \in P(S)\}
\]

is the maximum idempotent separating \(*\)-congruence on \(S\).

A regular \(*\)-semigroup \(S\) is said to be fundamental if \(\mu\) is the equality relation on \(S\).

The notation and terminology are those of [1] and [2] unless otherwise stated.

## 2 Structure sandwich sets

Let \(\mathcal{I}_X\) be the symmetric inverse semigroup on a set \(X\). For any subset \(A\) of \(X\), \(1_A\) means the identity mapping on \(A\). Let \(\mathcal{A}\) be an inverse subsemigroup of \(\mathcal{I}_X\) and \(\theta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}\) a mapping. Denote the image \((\alpha, \beta)\theta\) of an ordered pair \((\alpha, \beta)\) by \(\theta_{\alpha, \beta}\). Set \(\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in \mathcal{A}\}\). If \(\mathcal{M}\) satisfies the following conditions:

\[
\begin{align*}
(1) & \quad \theta_{a, \beta^{-1}}^{-1} = \theta_{\beta^{-1}, \alpha^{-1}}, \\
(2) & \quad \theta_{\alpha, a^{-1}} = 1_{r(\alpha)}, \\
(3) & \quad \theta_{1_{d(\alpha)}, \alpha} = 1_{d(\alpha)}, \\
(4) & \quad \theta_{\alpha, \beta} \theta_{a, \beta, \gamma} = \theta_{a, \theta_{\beta, \gamma}, \gamma} \theta_{\beta, \gamma},
\end{align*}
\]

we call it the structure sandwich set of \(\mathcal{A}\) determined by \(\theta\). The following lemma and theorem are obvious by the definition.

**Lemma 2.1** Let \(\mathcal{M}\) be the structure sandwich set of \(\mathcal{A}\) determined by \(\theta\). Then,

\[
\theta_{\alpha, 1_{r(\alpha)}} = 1_{r(\alpha)}.
\]

**Theorem 2.2** (see [3]) Let \(\mathcal{A}\) be an inverse subsemigroup of the symmetric inverse semigroup \(\mathcal{I}_X\) on a set \(X\), and \(\mathcal{M}\) the structure sandwich set of \(\mathcal{A}\) determined by a mapping \(\theta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}\). Define a multiplication \(\circ\) and a unary operation \(\ast\) on \(\mathcal{A}\) as follows:

\[
\begin{align*}
\alpha \circ \beta &= a_{\theta_{\alpha, \beta}}, \\
\alpha^\ast &= \alpha^{-1}.
\end{align*}
\]

Then \(\mathcal{A}(\circ, \ast)\) becomes a regular \(*\)-semigroup with \(E(\mathcal{A}(\circ, \ast)) = \{\alpha \in \mathcal{A}(\circ, \ast) : \theta_{\alpha, \alpha} = \alpha^{-1}\}\) and \(P(\mathcal{A}(\circ, \ast)) = \{1_{d(\alpha)} : \alpha \in \mathcal{A}(\circ, \ast)\}\).

We denote \(\mathcal{A}(\circ, \ast)\), defined above, by \(\mathcal{A}(\mathcal{M})\).
3 Construction

Let $X$ be a set and $\pi : X = \Sigma\{X_i : i \in I\}$ a partition of $X$. In this case, we denote $X$ by $X(\pi)$. A subset $A$ of $X$ is called a $\pi$-singleton subset of $X(\pi)$ if $|A \cap X_i| \leq 1$ for all $i \in I$. A mapping $\alpha \in I_X$ is called a $\pi$-singleton bijection of $X(\pi)$ if $d(\alpha)$ and $r(\alpha)$ are $\pi$-singleton subsets of $X(\pi)$. Denote the set of all $\pi$-singleton bijections of $X(\pi)$ by $\mathcal{F} \mathcal{G} \mathcal{I}_{X(\pi)}$. The following lemma is clear.

Lemma 3.1 The set $\mathcal{F} \mathcal{G} \mathcal{I}_{X(\pi)}$, defined above, is an inverse subsemigroup of $I_X$.

For any $\alpha, \beta \in \mathcal{F} \mathcal{G} \mathcal{I}_{X(\pi)}$, define a mapping $\theta_{\alpha, \beta}$ as follows:

\[
\begin{align*}
\theta_{\alpha, \beta} : & \text{ where } \alpha \cap \beta \subseteq \mathcal{F} \mathcal{G} \mathcal{I}_{X(\pi)} \\
& e\theta_{\alpha, \beta} = f, \text{ where } r(\alpha) \cap X_i = \{e\} \text{ and } d(\beta) \cap X_i = \{f\}.
\end{align*}
\]

Proposition 3.2 The set $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in \mathcal{F} \mathcal{G} \mathcal{I}_{X(\pi)}\}$ is the structure sandwich set of $\mathcal{F} \mathcal{G} \mathcal{I}_{X(\pi)}$ determined by a mapping $\theta : (\alpha, \beta) \mapsto \theta_{\alpha, \beta}$. Therefore $\mathcal{F} \mathcal{G} \mathcal{I}_{X(\pi)}(\mathcal{M})$ is a regular $\ast$-semigroup. Furthermore, $\mathcal{F} \mathcal{G} \mathcal{I}_{X(\pi)}(\mathcal{M})$ becomes a fundamental generalized inverse $\ast$-semigroup.

Proof. We can prove the proposition by similar argument of Lemma 3.4 and Lemma 3.5 of [3].

4 Representation

Let $S$ be a fundamental generalized inverse $\ast$-semigroup. In this section, we denote $E(S)$ and $P(S)$ simply by $E$ and $P$, respectively. Let $E \sim \Sigma\{E_i : i \in I\}$ be the structure decomposition of the normal band $E$. Then $\pi : P = \Sigma\{P_i : i \in I\}$ is a partition of $P$, where $P_i = P(E_i)$. Then $\mathcal{F} \mathcal{G} \mathcal{I}_{P(\pi)}(\mathcal{M})$, constructed in the preceding section, is a fundamental generalized inverse $\ast$-semigroup.

Lemma 4.1 For any $a \in S$, $Sa \cap P(=Sa^*a \cap P)$ is a $\pi$-singleton subset of $P(\pi)$.

Proof. Put $a^*a = e$ and assume that $f, g \in S \cap P_i$. Then there exist $x, y \in S$ such that $f = xe$ and $g = ye$. Hence $f = efe$ and $g = ege$. Since $E_i$ is a rectangular band and $E$ is a normal band, $f = fgf = efegefe = egefege = gfg = g$. This shows that $Sa \cap P$ is a $\pi$-singleton subset of $P(\pi)$.

Lemma 4.2 For any $a \in S$, let $\tau_a : Sa^* \cap P \rightarrow Sa \cap P$ be a mapping defined by

\[
\tau_a = a^*ea.
\]

Then, for any $a \in S$, $\tau_a \in \mathcal{F} \mathcal{G} \mathcal{I}_{P(\pi)}(\mathcal{M})$ and $\tau_a^{-1} = \tau_{a^*}$.
Proof. By Lemma 4.1, $d(\tau_a)$ and $\tau(\tau_a)$ are $\pi$-singleton subsets of $P(\pi)$. Let $e$ and $f$ are any elements of $S$ such that $e\tau_a = f\tau_a$, that is, $a*e_a = a*f_a$. Then there exist $x, y \in S$ such that $e = xa^*$ and $f = ya^*$. Thus $e = e_xe = ax*xa^* = aa*ax*xa^*aa^* = aa*eaa^* = aa*f*faa^* = aa*^f*faa^* = aa*ay*ya^*aa^* = ay*ya^* = f^*f = f$. This implies that $\tau_a$ is injective. Next, let $f$ be any element of $S$ such that $f = xa$. Put $e = aa^*x*xaa^*$. Then $e \in Sa^* \cap P$ and $e\tau_a = a*ea = a*(aa^*x*xaa^*)a = a*x*xa = f$. This implies that $\tau_a$ is bijective. The last statement of the lemma is obvious.

Lemma 4.3 For any $a, b \in S$, $\theta_{\tau_a, \tau_b} = \tau_a^{*abb^*}$. 

Proof. Let $e$ be any element of $d(\theta_{\tau_a, \tau_b})$. Then $e \in Sa \cap P$ and there exist $i \in I$ and $j \in Sb^* \cap P$ such that $e, f \in P$. Hence there exist $x, y \in S$ such that $e = xa$ and $f = yb^*$. Thus $e = e_xe = xayb^*xa = xayb^*abb^*a^*a \in Sbb^*a^*a \cap P = d(\tau_{aabb^*})$.

Conversely let $e$ be any element of $d(\theta_{\tau_a, \tau_b}) = Sbb^*a^*a \cap P$. Then there exists $e \in S$ such that $e = xbb^*a^*a$. Hence $e \in Sa \cap P$. Put $f = bb^*a^*ax*xaabb^*$. Then $f \in Sb^* \cap P$ and it is clear that $e$ and $f$ are contained in a same $P$. Therefore $e \in d(\theta_{\tau_a, \tau_b})$ and $e\tau_{aabb^*} = bb^*a^*aa^*abb^* = bb^*a^*ax*xaabb^* = f = e\theta_{\tau_a, \tau_b}$.

Lemma 4.4 Define a mapping $\phi : S \rightarrow \mathcal{FGL}(p)(\mathcal{M})$ by

$$a\phi = \tau_a.$$ 

Then $\phi$ is a $*$-monomorphism.

Proof. To prove that $\phi$ is a homomorphism it is sufficient to show that, for any $a, b \in S$, $d(\tau_a \circ \tau_b) = d(\tau_{ab})$. Let $a$ and $b$ be any elements of $S$. Then,

$$d(\tau_a \circ \tau_b) = d(\tau_a \tau_{a*abb^*} \tau_b) \quad \text{(by Lemma 4.3)}$$

$$= \{Sa \cap P \cap (Sa*abb^* \cap P \cap Sb^*)\tau_{a*abb^*}^{-1}\} \tau_a^{-1}$$

$$= \{Sa \cap P \cap (Sa*abb^* \cap P)\tau_{a*abb^*}^{-1}\} \tau_a^{-1}$$

$$= (Sa \cap P \cap Sbb^*a^*a)\tau_a^{-1} \quad \text{(since } \tau_a^{-1} = \tau_{a*abb^*})$$

$$= (Sbb^*a^*a \cap P)\tau_a^{-1}.$$ 

Hence let $e$ be any element of $d(\tau_a \circ \tau_b)$. Then there exists $f \in Sbb^*a^*a \cap P$ such that $e\tau_a = f$, that is $a*ea = f$. Since $e \in Sa^*$ and $f \in Sbb^*a^*a$, there exist $x, y \in S$ such that $e = xa^*$ and $f = ybb^*a^*a$. Thus $e = e_xe = ax*xa^* = aa*(ax*xa^*)a = a(a*ea)a^* = afa^* = aybb^*a^* \in Sbb^*a^*a \cap P = d(\tau_{ab})$.

Conversely let $e$ be any elements of $d(\tau_{ab}) = Sbb^*a^*a \cap P$. Then there exists $x \in S$ such that $e = xb^*a^*$. Hence $e\tau_a = xb^*a^*\tau_a = a*xb^*a^*a \in Sbb^*a^*a \cap P$. Thus $e \in (Sbb^*a^*a \cap P)\tau_a^{-1}$ and so $d(\tau_a \circ \tau_b) = d(\tau_{ab})$.

Let $a$ and $b$ be any elements of $S$ such that $a\phi = b\phi$, that is, $\tau_a = \tau_b$. Then it is easy to show that $aa^* = bb^*$ and $a*^a = b^*b$. Hence assume that $(a, b) \notin \mu$. Then there exists $e \in P$ such that $aia^* \neq bia^*$ and $a*e_a \neq b*e_b$. 
If $aea^* \neq beb^*$, we have $aea^* \tau_a = beb^* \tau_b$, since $a^*aea^*a = b^*beb^*b$. This contradicts that $\tau_a = \tau_b$ is injective. Similarly, if $a^*ea \neq b^*eb$, we have $aa^*ea^* \tau_a = bb^*eb^* \tau_b$, that is, $a^*ea = b^*eb$. This contradicts the hypothesis. Therefore, $(a, b) \in \mu$. Since $S$ is fundamental, we have that $a = b$. Thus $\phi$ is injective.

By Lemma 4.2, $\phi$ is compatible with a unary operation $*$ and so $\phi$ is a $*$-monomorphism.

Now we have the following theorem.

**Theorem 4.5** Every fundamental generalized inverse $*$-semigroup has a faithful representation into $\mathcal{FGI}_{X(\pi)}$ on a set $X(\pi)$.

**References**


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