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Some remarks on representations of fundamental generalized inverse $*$-semigroups

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Abstract

W.D. Munn [4] described that every fundamental inverse semigroup can be faithfully represented by isomorphisms among principal ideals of the semilattice of idempotents of it. Also T. Imaoka [3] has given a generalization of the Preston-Vagner representation for generalized inverse $*$-semigroups by using a concept of a structure sandwich set of an inverse subsemigroup of the symmetric inverse semigroup on a set.

In this paper, we shall construct a fundamental regular $*$-semigroup $\mathcal{F}GI_{X}(\pi)$ on a set $X$ with a partition $\pi : X = \Sigma\{X_{i} : i \in I\}$, and obtain a faithful representation of a fundamental generalized inverse $*$-semigroup into $*$-semigroup $\mathcal{F}GI_{X(\pi)}$ on a set $X(\pi)$.

1 Introduction

A semigroup $S$ with a unary operation $* : S \rightarrow S$ is called a regular $*$-semigroup if it satisfies followings

1. $(x^*)^* = x$
2. $(xy)^* = y^*x^*$
3. $xx^*x = x$

Let $S$ be a regular $*$-semigroup. Then an idempotent $e$ of $S$ is called a projection if it satisfies $e^* = e$. For any subset $A$ of $S$, denote the sets of idempotents and projections of $A$ by $E(A)$ and $P(A)$, respectively. The following result is well-known and we use it frequently throughout this paper.

Result 1.1 (see [2]) Let $S$ be a regular $*$-semigroup. Then we have the followings:

1. $E(S) = P(S)^2$;
2. for any $a \in S$ and $e \in P(S), a^*ea \in P(S)$;
3. each $L$-class and each $R$-class have one and only one projection.
A regular \*$\*$-semigroup $S$ is called a generalized inverse \*$\*$-semigroup if $E(S)$ forms a normal band, that is, $E(S)$ satisfies the identity $xyz = xzyx$, equivalently, $P(S)$ satisfies the identity $xyzw = xzyw$.

Let $S$ be a regular \*$\*$-semigroup and $\rho$ a congruence on $S$. Then $\rho$ is called a \*$\*$-congruence if $(x, y) \in \rho$ implies $(x^*, y^*) \in \rho$.

Result 1.2 (see [2]) Let $S$ be a regular \*$\*$-semigroup. Then, 

$$
\mu = \{(a, b) \in S \times S : aea^* = beb^* \text{ and } a^*ea = b^*eb \text{ for all } e \in P(S)\}
$$

is the maximum idempotent separating \*$\*$-congruence on $S$.

A regular \*$\*$-semigroup $S$ is said to be fundamental if $\mu$ is the equality relation on $S$.

The notation and terminology are those of [1] and [2] unless otherwise stated.

2 Structure sandwich sets

Let $I_X$ be the symmetric inverse semigroup on a set $X$. For any subset $A$ of $X$, $1_A$ means the identity mapping on $A$. let $\mathcal{A}$ be an inverse subsemigroup of $I_X$ and $\theta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ a mapping. Denote the image $(\alpha, \beta)\theta$ of an ordered pair $(\alpha, \beta)$ by $\theta_{\alpha, \beta}$. Set $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in A\}$. If $\mathcal{M}$ satisfies the following conditions:

(1) $\theta_{\alpha, \beta}^{-1} = \theta_{\beta^{-1}, \alpha^{-1}}$,
(2) $\theta_{\alpha, \alpha^{-1}} = 1_{r(\alpha)}$,
(3) $\theta_{1_{d(\alpha)}, \alpha} = 1_{d(\alpha)}$,
(4) $\theta_{\alpha, \beta}\theta_{\alpha^* \alpha, \beta^* \gamma} = \theta_{\alpha, \beta}\theta_{\alpha, \beta^* \gamma}$,

we call it the structure sandwich set of $A$ determined by $\theta$. The following lemma and theorem are obvious by the definition.

Lemma 2.1 Let $\mathcal{M}$ be the structure sandwich set of $A$ determined by $\theta$. Then, 

$$
\theta_{\alpha, 1_{r(\alpha)}} = 1_{r(\alpha)}.
$$

Theorem 2.2 (see [3]) Let $\mathcal{A}$ be an inverse subsemigroup of the symmetric inverse semigroup $I_X$ on a set $X$, and $\mathcal{M}$ the structure sandwich set of $\mathcal{A}$ determined by a mapping $\theta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. Define a multiplication $\circ$ and a unary operation $\ast$ on $\mathcal{A}$ as follows:

$$
\alpha \circ \beta = \alpha \theta_{\alpha, \beta} \beta, \quad \alpha^* = \alpha^{-1}.
$$

Then $\mathcal{A}(\circ, \ast)$ becomes a regular \*$\*$-semigroup with $E(\mathcal{A}(\circ, \ast)) = \{\alpha \in \mathcal{A}(\circ, \ast) : \theta_{\alpha, \alpha} = \alpha^{-1}\}$ and $P(\mathcal{A}(\circ, \ast)) = \{1_{d(\alpha)} : \alpha \in \mathcal{A}(\circ, \ast)\}$.

We denote $\mathcal{A}(\circ, \ast)$, defined above, by $\mathcal{A}(\mathcal{M})$. 
3 Construction

Let $X$ be a set and $\pi : X = \Sigma\{X_i : i \in I\}$ a partition of $X$. In this case, we denote $X$ by $X(\pi)$. A subset $A$ of $X$ is called a $\pi$-singleton subset of $X(\pi)$ if $|A \cap X_i| \leq 1$ for all $i \in I$. A mapping $\alpha \in I_X$ is called a $\pi$-singleton bijection of $X(\pi)$ if $d(\alpha)$ and $r(\alpha)$ are $\pi$-singleton subsets of $X(\pi)$. Denote the set of all $\pi$-singleton bijections of $X(\pi)$ by $\mathcal{F}\mathcal{G}\mathcal{I}_{X(\pi)}$. The following lemma is clear.

Lemma 3.1 The set $\mathcal{F}\mathcal{G}\mathcal{I}_{X(\pi)}$, defined above, is an inverse subsemigroup of $I_X$.

For any $\alpha, \beta \in \mathcal{F}\mathcal{G}\mathcal{I}_{X(\pi)}$, define a mapping $\theta_{\alpha, \beta}$ as follows:

$$d(\theta_{\alpha, \beta}) = \{e \in r(\alpha) : \text{there exist } i \in I \text{ and } f \in d(\beta) \text{ such that } e, f \in X_i\},$$

$$r(\theta_{\alpha, \beta}) = \{f \in d(\beta) : \text{there exist } i \in I \text{ and } e \in r(\alpha) \text{ such that } e, f \in X_i\}.$$

$$e\theta_{\alpha, \beta} = f, \text{ where } r(\alpha) \cap X_i = \{e\} \text{ and } d(\beta) \cap X_i = \{f\}.$$

Proposition 3.2 The set $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in \mathcal{F}\mathcal{G}\mathcal{I}_{X(\pi)}\}$ is the structure sandwich set of $\mathcal{F}\mathcal{G}\mathcal{I}_{X(\pi)}$ determined by a mapping $\theta : (\alpha, \beta) \mapsto \theta_{\alpha, \beta}$. Therefore $\mathcal{F}\mathcal{G}\mathcal{I}_{X(\pi)}(\mathcal{M})$ is a regular $\ast$-semigroup. Furthermore $\mathcal{F}\mathcal{G}\mathcal{I}_{X(\pi)}(\mathcal{M})$ becomes a fundamental generalized inverse $\ast$-semigroup.

Proof. We can prove the proposition by similar argument of Lemma 3.4 and Lemma 3.5 of [3].

4 Representation

Let $S$ be a fundamental generalized inverse $\ast$-semigroup. In this section, we denote $E(S)$ and $P(S)$ simply by $E$ and $P$, respectively. Let $E \sim \Sigma\{E_i : i \in I\}$ be the structure decomposition of the normal band $E$. Then $\pi : P = \Sigma\{P_i : i \in I\}$ is a partition of $P$, where $P_i = P(E_i)$. Then $\mathcal{F}\mathcal{G}\mathcal{I}_{P(\pi)}(\mathcal{M})$, constructed in the preceding section, is a fundamental generalized inverse $\ast$-semigroup.

Lemma 4.1 For any $a \in S$, $S_a \cap P(= S_a^* a \cap P)$ is a $\pi$-singleton subset of $P(\pi)$.

Proof. Put $a^* a = e$ and assume that $f, g \in S_e \cap P_i$. Then there exist $x, y \in S$ such that $f = xe$ and $g = ye$. Hence $f = efe$ and $g = ege$. Since $E_i$ is a rectangular band and $E$ is a normal band, $f = fgf = efegefe = egefege = gfg = g$. This shows that $S_a \cap P$ is a $\pi$-singleton subset of $P(\pi)$.

Lemma 4.2 For any $a \in S$, let $\tau_a : S_a^* \cap P \rightarrow S_a \cap P$ be a mapping defined by

$$e\tau_a = a^* ea.$$

Then, for any $a \in S$, $\tau_a \in \mathcal{F}\mathcal{G}\mathcal{I}_{P(\pi)}(\mathcal{M})$ and $\tau_a^{-1} = \tau_a^*$. 


Proof. By Lemma 4.1, $d(\tau_a)$ and $r(\tau_a)$ are $\pi$-singleton subsets of $P(\pi)$. Let $e$ and $f$ are any elements of $Sa^* \cap P$ such that $e\tau_a = f\tau_a$, that is, $a^*ea = a^*fa$. Then there exist $x, y \in S$ such that $e = xa^*$ and $f = ya^*$. Thus $e = e^*e = ax^*xa^* = aa^*ax^*xa^*aa^* = aa^*eaa^* = aa^*fqa^* = aa^*fqa^*$ implies $ay^*ya^* = a^*y^*ya^* = f^*f = f$. This implies that $\tau_a$ is injective. Next, let $f$ be any element of $S a \cap P$. Then there exists $x \in S$ such that $f = xa$. Put $e = aa^*x^*xa^*$. Then $e \in Sa^* \cap P$ and $e\tau_a = a^*(aa^*x^*xa^*)a = a^*x^*xa = f$. This implies that $\tau_a$ is bijective. The last statement of the lemma is obvious.

Lemma 4.3 For any $a, b \in S$, $\theta_{\tau_a, \tau_b} = \tau_{a^*abb^*}$.

Proof. Let $e$ be any element of $d(\theta_{\tau_a, \tau_b})$. Then $e \in Sa \cap P$ and there exist $i \in I$ and $f \in S b^* \cap P$ such that $e, f \in P$. Hence there exist $x, y \in S$ such that $e = xa$ and $f = yb^*$. Thus $e = efe = xayb^*xa = xayb^*xabb^*a^*a \in S b^*a^*a \cap P = d(\tau_{aabb^*})$.

Conversely let $e$ be any element of $d(\tau_{aabb^*}) = S b^*a^*a \cap P$. Then there exists $x \in S$ such that $e = xbb^*a^*a$. Hence $e \in Sa \cap P$. Put $f = b^*a^*ax^*xa^*abb^*$. Then $f \in S b^* \cap P$ and it is clear that $e$ and $f$ are contained in a same $P_i$. Therefore $e \in d(\theta_{\tau_a, \tau_b})$ and $e\tau_{aabb^*} = b^*a^*aeab^*abb^* = b^*a^*ax^*xa^*abb^* = f = e\tau_{\tau_a, \tau_b}$.

Lemma 4.4 Define a mapping $\phi : S \to F G I P(\pi)(M)$ by

$$a\phi = \tau_a.$$

Then $\phi$ is a $*$-monomorphism.

Proof. To prove that $\phi$ is a homomorphism it is sufficient to show that, for any $a, b \in S$, $d(\tau_a \circ \tau_b) = d(\tau_{ab})$. Let $a$ and $b$ be any elements of $S$. Then,

$$d(\tau_a \circ \tau_b) = d(\tau_a \tau_{a^*abb^*} \tau_b) \quad \text{(by Lemma 4.3)}$$

$$= \{Sa \cap P \cap (Sa^*abb^* \cap P \cap Sb^*)\tau_{a^*abb^*}^{-1}\} \tau_a^{-1}$$

$$= \{Sa \cap P \cap (Sa^*abb^* \cap P)\tau_{a^*abb^*}^{-1}\} \tau_a^{-1}$$

$$= (Sa \cap P \cap S b^*a^*a)\tau_a^{-1} \quad \text{(since } \tau_{a^*abb^*}^{-1} = \tau_{b^*a^*a})$$

$$= (S b^*a^*a \cap P)\tau_a^{-1}.$$

Hence let $e$ be any element of $d(\tau_a \circ \tau_b)$. Then there exists $f \in S b^*a^*a \cap P$ such that $e\tau_a = f\tau_a$, that is $a^*ea = f$. Since $e \in Sa^*$ and $f \in S b^*a^*a$, there exist $x, y \in S$ such that $e = xa^*$ and $f = yb^*a^*a$. Thus $e = e^*e = ax^*xa^* = aa^*(ax^*xa^*)a^* = a\phi(a^*e)$ and $a^*fa = ayb^*a^* \in S b^*a^*a \cap P = d(\tau_{ab})$.

Conversely let $e$ be any element of $d(\tau_{ab}) = S b^*a^*a \cap P$. Then there exists $x \in S$ such that $e = xbb^*a^*$. Hence $e\tau_a = xbb^*a^*\tau_a = a^*xb^*a^*a \in S b^*a^*a \cap P$. Thus $e \in (S b^*a^*a \cap P)\tau_a^{-1}$ and so $d(\tau_a \circ \tau_b) = d(\tau_{ab})$.

Let $a$ and $b$ be any elements of $S$ such that $a\phi = b\phi$, that is, $\tau_a = \tau_b$. Then it is easy to show that $a^*e = b\phi$, that $a^*e = b\phi$. Hence assume that $(a, b) \not\in \mu$. Then there exists $e \in P$ such that $a e a^* \neq b^*eb$ or $a^*e = b^*eb$. 
If $aea^* \neq beb^*$, we have $aea^* \tau_a = beb^* \tau_b$, since $a^*aea^*a = b^*beb^*b$. This contradicts that $\tau_a = \tau_b$ is injective. Similarly, if $a^*ea \neq b^*eb$, we have $aa^*ea^*a = bb^*eb^*b$, that is, $a^*ea = b^*eb$. This contradicts the hypothesis. Therefore, $(a, b) \in \mu$. Since $S$ is fundamental, we have that $a = b$. Thus $\phi$ is injective.

By Lemma 4.2, $\phi$ is compatible with a unary operation $*$ and so $\phi$ is a $*$-monomorphism.

Now we have the following theorem.

**Theorem 4.5** Every fundamental generalized inverse $*$-semigroup has a faithful representation into $\mathcal{FGL}_X(\pi)$ on a set $X(\pi)$.

**References**


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