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<th>A non-standard proof of the Peano existence theorem in WKL$_0$ (Mathematical Incompleteness in Arithmetic)</th>
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<tr>
<td>Author(s)</td>
<td>Tanaka, Kazuyuki</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1995), 912: 57-63</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1995-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/59562">http://hdl.handle.net/2433/59562</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
A non-standard proof of
the Peano existence theorem in $\text{WKL}_0$

Kazuyuki Tanaka
Tohoku University
Mathematical Institute, Kawauchi
Sendai 980-77, Japan
tanaka@math.tohoku.ac.jp

This note is a sort of supplement to our paper [6], but can be read independently. Except for lacking in the proof of our self-embedding theorem that every countable non-standard model of $\text{WKL}_0$ has a proper initial part isomorphic to itself, our argument here is essentially self-contained. The goal of this note is to carry out the popular non-standard proof of the Peano existence theorem for solutions of ordinary differential equations within $\text{WKL}_0$.

The usual standard proof of Peano's theorem depends much on the Ascoli lemma, by which one can make a solution of initial value problem from a sequence of piecewise linear approximations. It was Simpson [3] who first proved the theorem within $\text{WKL}_0$ by avoiding the use of the Ascoli lemma. In regard to the program of Reverse Mathematics, he [3] has actually shown that Peano's theorem is provably equivalent to $\text{WKL}_0$ over $\text{RCA}_0$, while the Ascoli lemma is equivalent to the stronger system $\text{ACA}_0$. Subsequently, we [2] obtained another $\text{WKL}_0$ proof of Peano's theorem based on a version of Schauder's fixed point theorem. See [4], [5] for more information.

On the other hand, the non-standard proof of Peano's theorem is also known to be free from the Ascoli lemma. Thus, the non-standard proof and the $\text{WKL}_0$ proofs share the same feelings of constructivity (cf. Albeverio et
al. [1, p.31]). In fact, by our self-embedding theorem, a considerable portion of non-standard analysis could be developed in WKL₀.

To begin with, recall some basic definitions and the self-embedding theorem. The system RCA₀ consists of the axioms of ordered semirings, Σ¹ induction and Δ¹ comprehension, and WKL₀ is obtained from RCA₀ by adding weak König's lemma: every infinite tree of sequences of 0's and 1's has an infinite path. A structure V of second-order arithmetic is often expressed as a pair (M, S), where M is its first-order part and S consists of subsets of (the underlying set of) M. For an initial segment I of M, let $V^I = (I, S^I)$ where $S^I = \{X \cap I : X \in S\}$. Now, we have

**The Self-Embedding Theorem.** Let $V = (M, S)$ be a countable non-standard model of WKL₀. Then there exists a proper initial part $V^I = (I, S^I)$ of V and an isomorphism $f: V \rightarrow V^I$.


Fix a countable non-standard model $V = (M, S)$ of WKL₀, in which we are going to develop analysis. By the above theorem, V has an initial part isomorphic to itself. Since the initial part and V are isomorphic to each other, they may exchange their roles, and so they can be regarded as V and its extension, respectively. Then, let $*V = (*M, *S)$ denote an isomorphic extension of V, which will be used as a non-standard universe.

Following our paper [6], a **real number** in the closed unit interval [0,1] is defined as its binary expansion. Intuitively, a binary function $\alpha$ codes the real $\sum \frac{\alpha(i)}{2^{i+1}}$. Then, each real in V is an initial segment of a *V-finite sequence. A set F of pairs of finite binary sequences is said to be (a code for) a **continuous (partial) function** f from [0, 1] to itself if the following conditions hold:
1. if \((s, t) \in F\) and \((s, t') \in F\), then \(t\) extends \(t'\) or \(t'\) extends \(t\);
2. if \((s, t) \in F\) and \(s'\) extends \(s\), then \((s', t) \in F\);
3. if \((s, t) \in F\) and \(t\) extends \(t'\), then \((s, t') \in F\).

For a sequence \(s\) with length \(lh(s)\), we set
\[
as = \sum_{i < lh(s)} \frac{s(i)}{2^{i+1}}, \quad b_s = a_s + \frac{1}{2^{lh(s)+1}}.
\]
Then, \((s, t) \in F\) intuitively means that the image of open interval \((a_s, b_s)\) via \(f\) is included in the closed interval \([a_t, b_t]\). Finally, we write \(f(\alpha) = \beta\) iff for each \(M\)-finite initial segment \(t\) of \(\beta\), there exists an \(M\)-finite initial segment \(s\) of \(\alpha\) such that \((s, t) \in F\).

Suppose that \(F\) is a code for a "total" continuous function in \(V\). Let \(*F\) be a set of \(*V\) such that \(F = *F \cap V\). Since "\(F\) is a code for a continuous function" is a \(\Pi^0_1\) predicate, by overspill, there is a \(p \notin M\) such that \(*F\) satisfies the above three conditions for all the binary sequences with length \(\leq p\). Fix such a \(p\). Let \(\text{Seq}(p)\) be the set of binary sequences with length \(p\). We then define the function \(*f\) on \(\text{Seq}(p)\) by
\[*f(\tilde{s}) = \text{the longest sequence } t \text{ such that } (\tilde{s}, t) \in *F \text{ and } lh(t) \leq p.*\]

It is clear from conditions 1 and 2 that this function is well-defined. It is also obvious that for each \(\tilde{s} \in \text{Seq}(p)\), the length of \(*f(\tilde{s})\) is not in \(M\), since \(f\) is total. Again by overspill, there is a \(q \notin M\) such that the length of \(*f(\tilde{s})\) is \(\geq q\) for every \(\tilde{s} \in \text{Seq}(p)\). So, by pruning, \(*f\) can be seen as a function from \(\text{Seq}(p)\) to \(\text{Seq}(q)\).

**Lemma 1.** Let \(f\) be a total continuous function in \(V\). And let \(*f\) be a function from \(\text{Seq}(p)\) to \(\text{Seq}(q)\) constructed as above. Then, \(f(\tilde{s} \cap M) = *f(\tilde{s}) \cap M\) for each \(\tilde{s} \in \text{Seq}(p)\).
Proof. Let $y = f(\tilde{s} \cap M)$. Choose any M-finite initial segment $t$ of $y$. By the definition of $f(\alpha) = \beta$, there exists an M-finite initial segment $s$ of $\tilde{s}$ such that $(s, t) \in F$. Hence we have $(\tilde{s}, t) \in \ast F$ by condition 2 of the definition of continuous partial functions. So, $t$ must be an initial segment of $\ast f(\tilde{s})$ by condition 1. Since $t$ is chosen as an arbitrary initial segment of $y$, $y$ is also an initial segment of $\ast f(\tilde{s})$. 

\[ \square \]

**Theorem 2 (WKL₀).** Any continuous function $f$ on $[0, 1]$ attains a maximal value.

Proof. If $\ast f$ is maximal at $\tilde{s} \in \text{Seq}(p)$, $f$ attains a maximal value $\ast f(\tilde{s}) \cap M$ at $\tilde{s} \cap M$.

Next, we show the converse to Lemma 1.

**Lemma 3.** Suppose we are first given a function $\ast f$: $\text{Seq}(p) \rightarrow \text{Seq}(q)$ with $p, q \not\in M$ such that for all $\tilde{s}, t \in \text{Seq}(p)$,

\begin{equation}
\tag{*} \tilde{s} \cap M = t \cap M \Rightarrow \ast f(\tilde{s}) \cap M = \ast f(t) \cap M.
\end{equation}

Then there exists a continuous function $f$ in $V$ such that $f(\tilde{s} \cap M) = \ast f(\tilde{s}) \cap M$ for all $\tilde{s} \in \text{Seq}(p)$.

Proof. We first put

\[ \ast F = \{ (s, t) \in \bigcup_{r \leq p} \text{Seq}(r) \times \bigcup_{r \leq q} \text{Seq}(r) : \forall \tilde{s} \in \text{Seq}(p) \ (s \subseteq \tilde{s} \rightarrow t \subseteq \ast f(\tilde{s})) \}. \]

Then it is easy to see that $\ast F$ satisfies the three conditions of continuous functions with respect to sequences $s \in \bigcup_{r \leq p} \text{Seq}(r)$ and $t \in \bigcup_{r \leq q} \text{Seq}(r)$. Hence, $F = \ast F \cap M$ is a code for a continuous (partial) function in $V$.

To show that $F$ is total and $f(\tilde{s} \cap M) = \ast f(\tilde{s}) \cap M$, take any real $\alpha \in [0, 1]$. Let $\tilde{s} \in \text{Seq}(p)$ be a sequence extending $\alpha$, and $t$ be any M-finite initial segment of $\ast f(\tilde{s})$. By condition (*), for any $s \subseteq \tilde{s}$ such that $s \not\subseteq M$, we have $(s, t) \in \ast F$. 

\[ \square \]
So, by underspill, there is an $\mathrm{M}$-finite $s \subseteq \tilde{s}$ such that $(s, t) \in \ast \mathrm{F}$, hence $(s, t) \in \mathrm{F}$. This shows that $f(\alpha)$ is defined and its value is $\ast f(\tilde{s}) \cap \mathrm{M}$. Thus, $\mathrm{F}$ is a code for a desired continuous function $f$. 

**Theorem 4** (WKL$_0$). Any continuous function $f$ on $[0, 1]$ is uniformly continuous, that is, for each $n \in \mathrm{M}$, there exists $m \in \mathrm{M}$ such that $\forall s \in \mathrm{Seq}(m) \exists t \in \mathrm{Seq}(n) (s, t) \in \mathrm{F}$. 

Proof. Fix any $n \in \mathrm{M}$. As in the proofs of the above lemmas, we can easily see that for each $p \not\in \mathrm{M}$, $\forall s \in \mathrm{Seq}(p) \exists t \in \mathrm{Seq}(n) (s, t) \in \mathrm{F}$. Hence, also by underspill, there exists $m \in \mathrm{M}$ such that $\forall s \in \mathrm{Seq}(m) \exists t \in \mathrm{Seq}(n) (s, t) \in \mathrm{F}$. 

**Theorem 5** (WKL$_0$). Any continuous function $f$ on $[\alpha, \beta] \subseteq [0, 1]$ is Riemann integrable. 

Proof. With the help of Theorem 4, the usual argument using the upper and lower sums works. 

**Remark.** The Riemann integral of a continuous function $f$ on $[0, 1]$ is given by 

$$
\int_{0}^{1} f(x) dx = \lim_{n \to \infty} \sum_{s \in \mathrm{Seq}(n)} \max_{\alpha \geq s} f(\alpha) \cdot \frac{1}{2^n} = \lim_{n \to \infty} \sum_{s \in \mathrm{Seq}(n)} \min_{\alpha \leq s} f(\alpha) \cdot \frac{1}{2^n} \\
= (\sum_{p \in \mathrm{Seq}(p)} \ast f(\tilde{s}) \cdot \frac{1}{2^n}) \cap \mathrm{M}.
$$

**Theorem 6** (WKL$_0$). Let $f(x, y)$ be a continuous function from $\mathrm{D} = [0, 1]^2$ to $[0,1]$. Then the initial value problem 

$$
\frac{dy}{dx} = f(x, y), \quad y(0) = 0
$$

has a solution $y(x)$ on the interval $[0,1]$. (The Peano Existence Theorem) 

Proof. Given a continuous function $f(x, y)$, we take $\ast f$, $p \not\in \mathrm{M}$, $q \not\in \mathrm{M}$ as before so that
\[ f = *f \cap V, \quad *f: \text{Seq}(p) \times \text{Seq}(p) \to \text{Seq}(q). \]

Then define a function \(*y: \text{Seq}(p) \to \text{Seq}(p+q)\) by recursion as follows:

\[
* y\left(\frac{0}{2^p}\right) = \frac{0}{2^{p+q}}, \\
* y\left(\frac{i+1}{2^p}\right) = * y\left(\frac{i}{2^p}\right) + \frac{1}{2^p} \cdot f\left(\frac{i}{2^p}, * y\left(\frac{i}{2^p}\right)^\lceil p\right),
\]

where a fraction form \(\frac{i}{2^p}\) denotes the binary sequence in \(\text{Seq}(p)\) encoding the real \(\frac{i}{2^p}\), and \(* y\left(\frac{i}{2^p}\right)^\lceil p\) is the initial segment of \(* y\left(\frac{i}{2^p}\right)\) with length \(p\).

First, it is easy to see that

\[
|* y\left(\frac{i}{2^p}\right) - * y\left(\frac{j}{2^p}\right)| \leq \frac{|i-j|}{2^p},
\]

since \(|*f(x)| \leq 1\). So, by Lemma 3, there exists a continuous function \(y(x)\) in \(V\) such that \(y\left(\frac{i}{2^p} \cap M\right) = * y\left(\frac{i}{2^p}\right) \cap M\). By the definition of \(*y\),

\[
* y\left(\frac{k}{2^p}\right) = \sum_{i<k} * f\left(\frac{i}{2^p}, * y\left(\frac{i}{2^p}\right)^\lceil p\right) \cdot \frac{1}{2^p}.
\]

We also have

\[
\int_0^{k/2^p \cap M} f(x,y) \, dx = \left(\sum_{i<k} * f\left(\frac{i}{2^p}, * y\left(\frac{i}{2^p}\right)^\lceil p\right) \cdot \frac{1}{2^p}\right) \cap M,
\]

by the remark after Theorem 5. So, letting \(\alpha = \frac{k}{2^p} \cap M\), we have

\[
y(\alpha) = \int_0^\alpha f(x,y) \, dx.
\]

Thus, \(y(x)\) is a solution of the differential equation.
References


