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Kyoto University
A non-standard proof of
the Peano existence theorem in WKL₀

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This note is a sort of supplement to our paper [6], but can be read independently. Except for lacking in the proof of our self-embedding theorem that every countable non-standard model of WKL₀ has a proper initial part isomorphic to itself, our argument here is essentially self-contained. The goal of this note is to carry out the popular non-standard proof of the Peano existence theorem for solutions of ordinary differential equations within WKL₀.

The usual standard proof of Peano's theorem depends much on the Ascoli lemma, by which one can make a solution of initial value problem from a sequence of piecewise linear approximations. It was Simpson [3] who first proved the theorem within WKL₀ by avoiding the use of the Ascoli lemma. In regard to the program of Reverse Mathematics, he [3] has actually shown that Peano's theorem is provably equivalent to WKL₀ over RCA₀, while the Ascoli lemma is equivalent to the stronger system ACA₀. Subsequently, we [2] obtained another WKL₀ proof of Peano's theorem based on a version of Schauder's fixed point theorem. See [4], [5] for more information.

On the other hand, the non-standard proof of Peano's theorem is also known to be free from the Ascoli lemma. Thus, the non-standard proof and the WKL₀ proofs share the same feelings of constructivity (cf. Albeverio et
al. [1, p.31]). In fact, by our self-embedding theorem, a considerable portion of non-standard analysis could be developed in WKL₀.

To begin with, recall some basic definitions and the self-embedding theorem. The system RCA₀ consists of the axioms of ordered semirings, $\Sigma^0_1$ induction and $\Delta^0_1$ comprehension, and WKL₀ is obtained from RCA₀ by adding weak König's lemma: every infinite tree of sequences of 0's and 1's has an infinite path. A structure $V$ of second-order arithmetic is often expressed as a pair $(M, S)$, where $M$ is its first-order part and $S$ consists of subsets of (the underlying set of) $M$. For an initial segment $I$ of $M$, let $V^I = (I, S^I)$ where $S^I = \{X \cap I : X \in S\}$. Now, we have

**The Self-Embedding Theorem.** Let $V = (M, S)$ be a countable non-standard model of WKL₀. Then there exists a proper initial part $V^I = (I, S^I)$ of $V$ and an isomorphism $f: V \to V^I$.


Fix a countable non-standard model $V = (M, S)$ of WKL₀, in which we are going to develop analysis. By the above theorem, $V$ has an initial part isomorphic to itself. Since the initial part and $V$ are isomorphic to each other, they may exchange their roles, and so they can be regarded as $V$ and its extension, respectively. Then, let $*V = (*M, *S)$ denote an isomorphic extension of $V$, which will be used as a non-standard universe.

Following our paper [6], a real number in the closed unit interval $[0,1]$ is defined as its binary expansion. Intuitively, a binary function $\alpha$ codes the real $\sum \frac{\alpha(i)}{2^i}$. Then, each real in $V$ is an initial segment of a $*V$-finite sequence. A set $F$ of pairs of finite binary sequences is said to be (a code for) a continuous (partial) function $f$ from $[0, 1]$ to itself if the following conditions hold:
1. if \((s, t) \in F\) and \((s, t') \in F\), then \(t\) extends \(t'\) or \(t'\) extends \(t\);
2. if \((s, t) \in F\) and \(s'\) extends \(s\), then \((s', t) \in F\);
3. if \((s, t) \in F\) and \(t\) extends \(t'\), then \((s, t') \in F\).

For a sequence \(s\) with length \(lh(s)\), we set
\[
a_s = \sum_{i < lh(s)} \frac{s(i)}{2^{i+1}}, \quad b_s = a_s + \frac{1}{2^{lh(s)+1}}.
\]

Then, \((s, t) \in F\) intuitively means that the image of open interval \((a_s, b_s)\) via \(f\) is included in the closed interval \([a_t, b_t]\). Finally, we write \(f(\alpha) = \beta\) iff for each \(M\)-finite initial segment \(t\) of \(\beta\), there exists an \(M\)-finite initial segment \(s\) of \(\alpha\) such that \((s, t) \in F\).

Suppose that \(F\) is a code for a “total” continuous function in \(V\). Let \(*F\) be a set of \(*V\) such that \(F = *F \cap V\). Since “\(F\) is a code for a continuous function” is a \(\Pi^0_1\) predicate, by overspill, there is a \(p \notin M\) such that \(*F\) satisfies the above three conditions for all the binary sequences with length \(\leq p\). Fix such a \(p\). Let \(\text{Seq}(p)\) be the set of binary sequences with length \(p\). We then define the function \(*f\) on \(\text{Seq}(p)\) by
\[
*f(\bar{s}) = \text{the longest sequence } \bar{t}\text{ such that } (\bar{s}, \bar{t}) \in *F \text{ and } lh(\bar{t}) \leq p.
\]

It is clear from conditions 1 and 2 that this function is well-defined. It is also obvious that for each \(\bar{s} \in \text{Seq}(p)\), the length of \(*f(\bar{s})\) is not in \(M\), since \(f\) is total. Again by overspill, there is a \(q \notin M\) such that the length of \(*f(\bar{s})\) is \(\geq q\) for every \(\bar{s} \in \text{Seq}(p)\). So, by pruning, \(*f\) can be seen as a function from \(\text{Seq}(p)\) to \(\text{Seq}(q)\).

**Lemma 1.** Let \(f\) be a total continuous function in \(V\). And let \(*f\) be a function from \(\text{Seq}(p)\) to \(\text{Seq}(q)\) constructed as above. Then, \(f(\bar{s} \cap M) = *f(\bar{s}) \cap M\) for each \(\bar{s} \in \text{Seq}(p)\).
Proof. Let \( y = f(\bar{s} \cap M) \). Choose any \( M \)-finite initial segment \( t \) of \( y \). By the definition of \( f(\alpha) = \beta \), there exists an \( M \)-finite initial segment \( s \) of \( \bar{s} \) such that \((s, t) \in \mathcal{F}\). Hence we have \((\bar{s}, t) \in \mathcal{F} \) by condition 2 of the definition of continuous partial functions. So, \( t \) must be an initial segment of \(*f(\bar{s})\) by condition 1. Since \( t \) is chosen as an arbitrary initial segment of \( y \), \( y \) is also an initial segment of \(*f(\bar{s})\). \[
\]

**Theorem 2** (WKLo). Any continuous function \( f \) on \([0,1]\) attains a maximal value.

Proof. If \(*f\) is maximal at \( \bar{s} \in \text{Seq}(p) \), \( f \) attains a maximal value \(*f(\bar{s}) \cap M\) at \( \bar{s} \cap M \).

Next, we show the converse to Lemma 1.

**Lemma 3.** Suppose we are first given a function \(*f\): Seq(p) \( \rightarrow \) Seq(q) with \( p, q \notin M \) such that for all \( \bar{s}, \bar{t} \in \text{Seq}(p) \),

\[
(*) \quad \bar{s} \cap M = \bar{t} \cap M \Rightarrow *f(\bar{s}) \cap M = *f(\bar{t}) \cap M.
\]

Then there exists a continuous function \( f \) in \( V \) such that \( f(\bar{s} \cap M) = *f(\bar{s}) \cap M \) for all \( \bar{s} \in \text{Seq}(p) \).

Proof. We first put

\[
*F = \{(s,t) \in \bigcup_{r \leq p} \text{Seq}(r) \times \bigcup_{r \leq q} \text{Seq}(r) : \forall \bar{s} \in \text{Seq}(p) \ (s \subseteq \bar{s} \rightarrow t \subseteq *f(\bar{s}))\}.
\]

Then it is easy to see that \(*F\) satisfies the three conditions of continuous functions with respect to sequences \( s \in \bigcup_{r \leq p} \text{Seq}(r) \) and \( t \in \bigcup_{r \leq q} \text{Seq}(r) \). Hence, \( F = *F \cap M \) is a code for a continuous (partial) function in \( V \).

To show that \( F \) is total and \( f(\bar{s} \cap M) = *f(\bar{s}) \cap M \), take any real \( \alpha \in [0,1] \). Let \( \bar{s} \in \text{Seq}(p) \) be a sequence extending \( \alpha \), and \( t \) be any \( M \)-finite initial segment of \(*f(\bar{s})\). By condition (\(*\)), for any \( s \subseteq \bar{s} \) such that \( s \notin M \), we have \((s, t) \in *F\).
So, by underspill, there is an $M$-finite $s \subseteq \tilde{s}$ such that $(s, t) \in \ast F$, hence $(s, t) \in F$. This shows that $f(\alpha)$ is defined and its value is $\ast f(\tilde{s}) \cap M$. Thus, $F$ is a code for a desired continuous function $f$. \[\]

**Theorem 4** (WKL$_0$). Any continuous function $f$ on $[0, 1]$ is uniformly continuous, that is, for each $n \in M$, there exists $m \in M$ such that $\forall s \in \text{Seq}(m) \exists t \in \text{Seq}(n) \ (s, t) \in F$.

Proof. Fix any $n \in M$. As in the proofs of the above lemmas, we can easily see that for each $p \notin M$, $\forall s \in \text{Seq}(p) \exists t \in \text{Seq}(n) \ (s, t) \in F$. Hence, also by underspill, there exists $m \in M$ such that $\forall s \in \text{Seq}(m) \exists t \in \text{Seq}(n) \ (s, t) \in F$. \[\]

**Theorem 5** (WKL$_0$). Any continuous function $f$ on $[\alpha, \beta] \subseteq [0, 1]$ is Riemann integrable.

Proof. With the help of Theorem 4, the usual argument using the upper and lower sums works. \[\]

**Remark.** The Riemann integral of a continuous function $f$ on $[0, 1]$ is given by

$$
\int_0^1 f(x)dx = \lim_{n \to \infty} \sum_{s \in \text{Seq}(n)} \max_{\sigma \geq s} f(\sigma) \cdot \frac{1}{2^n} = \lim_{n \to \infty} \sum_{s \in \text{Seq}(n)} \min_{\sigma \geq s} f(\sigma) \cdot \frac{1}{2^n} = (\sum_{s \in \text{Seq}(p)} \ast f(\tilde{s}) \cdot \frac{1}{2^p}) \cap M.
$$

**Theorem 6** (WKL$_0$). Let $f(x, y)$ be a continuous function from $D = [0, 1]^2$ to $[0,1]$. Then the initial value problem

$$
\frac{dy}{dx} = f(x, y), \quad y(0) = 0
$$

has a solution $y(x)$ on the interval $[0,1]$. (The Peano Existence Theorem)

Proof. Given a continuous function $f(x, y)$, we take $\ast f, p \notin M, q \notin M$ as before so that
\[ f = ^*f \cap V, \quad ^*f: \text{Seq}(p) \times \text{Seq}(p) \to \text{Seq}(q). \]

Then define a function \( ^*y: \text{Seq}(p) \to \text{Seq}(p+q) \) by recursion as follows:

\[
^*y\left( \frac{0}{2^p} \right) = \frac{0}{2^{p+q}},
\]

\[
^*y\left( \frac{i+1}{2^p} \right) = ^*y\left( \frac{i}{2^p} \right) + \frac{1}{2^p} \cdot ^*f\left( \frac{i}{2^p}, ^*y\left( \frac{i}{2^p} \right)^p \right),
\]

where a fraction form \( \frac{i}{2^p} \) denotes the binary sequence in \( \text{Seq}(p) \) encoding the real \( \frac{i}{2^p} \), and \( ^*y\left( \frac{i}{2^p} \right)^p \) is the initial segment of \( ^*y\left( \frac{i}{2^p} \right) \) with length \( p \).

First, it is easy to see that

\[
\left| ^*y\left( \frac{i}{2^p} \right) - ^*y\left( \frac{j}{2^p} \right) \right| \leq \frac{|i-j|}{2^p},
\]

since \( |^*f(x)| \leq 1 \). So, by Lemma 3, there exists a continuous function \( y(x) \) in \( V \) such that \( y\left( \frac{i}{2^p} \cap M \right) = ^*y\left( \frac{i}{2^p} \right) \cap M \). By the definition of \( ^*y \),

\[
^*y\left( \frac{k}{2^p} \right) = \sum_{i<k}^*f\left( \frac{i}{2^p}, ^*y\left( \frac{i}{2^p} \right)^p \right) \cdot \frac{1}{2^p}.
\]

We also have

\[
\int_{0}^{\frac{k}{2^p} \cap M} f(x, y) \, dx = \left( \sum_{i<k}^*f\left( \frac{i}{2^p}, ^*y\left( \frac{i}{2^p} \right)^p \right) \cdot \frac{1}{2^p} \right) \cap M,
\]

by the remark after Theorem 5. So, letting \( \alpha = \frac{k}{2^p} \cap M \), we have

\[
y(\alpha) = \int_{0}^{\alpha} f(x, y) \, dx.
\]

Thus, \( y(x) \) is a solution of the differential equation.
References


