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CR of a reduction for classical natural deduction

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In [1], we introduced a reduction-procedure for first order classical natural deduction with full logical symbols, and proved the weak normalization theorem of the reduction. The reduction defined in [1] is simple, and it is a natural extension of Prawitz's reduction for intuitionistic natural deduction [5][6]. In this note, we show the fact that Church-Rosser property (CR) holds for the reduction introduced in [1]. We give an outline of a proof of the theorem. For the details, see [2].

1 Basic definitions and notations

1.1 System

In this paper, we investigate the natural deduction system for the first order classical logic. Our system contains all logical symbols, that is; & (and), $\vee$ (or), $\supset$ (implies), $\neg$ (not), $\forall$ (for all), and $\exists$ (there exists). The inference rules are the introduction and elimination rules for each logical symbol, and the classical absurdity rule shown by the following schema.

Classical absurdity rule

$$\frac{\begin{array}{c}[\neg A]\\ \bot\end{array}}{A}\ (\bot_c)$$

Regularity of $(\bot_c)$ **.** It is assumed that any assumption formula discharged by any application of $(\bot_c)$ in a derivation is the major premiss of an application of $(\neg E)$. Notice that if a derivation which does not satisfy the regularity of $(\bot_c)$ is given, then we can easily transform it to a regular one [1]. By definition of our reduction which will be stated in the next section, it will easily be verified that; if Π' is the derivation obtained by our reduction from a derivation Π satisfying the regularity of $(\bot_c)$, then Π' is also regular.

1.2 Notational conventions

(1) Small Greek letters α, β, ... are used as syntactical variables for formula-occurrences in derivations. If α is an formula-occurrence of a formula A, $\mathit{Form}(\alpha)$ denotes the formula A. We make a distinction between inference rules and applications of inference rules in derivations. If I is an application of an inference rule in a derivation, $\mathit{Inf}(I)$ denotes the inference rule applied at I. For example, if I is an application of $(\vee E)$ in a derivation, then $\mathit{Inf}(I)$ is the inference rule $(\vee E)$. When I is an application of an inference rule in a derivation, we call I a D-inference [3] (in [10]).

(2) Let Π be a derivation. $FO(\Pi)$ denotes the set of all formula-occurrences in Π. Notations $oa(\Pi)$, $OA(\Pi)$, $end(\Pi)$, $END(\Pi)$, $li(\Pi)$, and $LI(\Pi)$ are defined by the following:

$oa(\Pi) = \{\alpha \in FO(\Pi) \mid \alpha \text{ is an open assumption of } \Pi\}$
$OA(\Pi) = \{Form(\alpha) \mid \alpha \in oa(\Pi)\}$
$end(\Pi)$ is the end formula-occurrence of Π.
$END(\Pi) = Form(end(\Pi))$
$li(\Pi)$ is the last D-inference of Π.
Namely, $li(\Pi)$ is the D-inference whose conclusion is $end(\Pi)$.
$LI(\Pi) = Inf(li(\Pi))$

$li(\Pi)$ and $LI(\Pi)$ are defined in the case that the length of Π is greater than 1, that is, there is at least one D-inference in Π. For a formula-occurrence α in Π, $sbd(\alpha)$ denotes the subderivation of Π satisfying $end(sbd(\alpha)) = \alpha$. Let I be an D-inference in Π. Notations $pm(I)$, $cl(I)$, and $dc(I)$ are defined by the following:

$pm(I) = \{\alpha \in FO(\Pi) \mid \alpha \text{ is a premiss of } I\}$
$cl(I)$ is the conclusion of I.
$dc(I) = \{\alpha \in FO(\Pi) \mid \alpha \text{ is discharged by } I\}$

Moreover, in the case that $Inf(I)$ is an elimination rule, notations $mj(I)$, $MJ(I)$, and $mn(I)$ are defined by the following:

$mj(I)$ is the major premiss of I.
$MJ(I) = Form(mj(I))$
$mn(I) = \{\alpha \in FO(\Pi) \mid \alpha \text{ is a minor premiss of } I\}$

(3) Let Π, a, and t be a derivation, a free variable, and a term respectively. If the figure obtained by substituting t for all occurrences a in Π is a derivation, we denote the derivation by $\Pi(t/a)$. Let A be a formula. The notation $\genfrac{}{}{0pt}{}{[A]}{\Pi}$ is used in the following situation, that is, $[A]$ in $\genfrac{}{}{0pt}{}{[A]}{\Pi}$ denotes a subset, say O, of $oa(\Pi)$ satisfying that $Form(\alpha) = A$ holds for all α in Π. Let Σ be a derivation satisfying $END(\Sigma) = A$. If the figure obtained by substituting Σ for all elements of the subset of $oa(\Pi)$ denoted by $[A]$ in $\genfrac{}{}{0pt}{}{[A]}{\Pi}$ is a derivation, we denote the derivation by $\begin{matrix}\Sigma\\ [A]\\ \Pi\end{matrix}$. When a derivation Π is denoted by $\dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A}$, it means that Π equals to $\dfrac{\Pi_0}{A}$, $\dfrac{\Pi_0 \quad \Pi_1}{A}$, or $\dfrac{\Pi_1 \quad \Pi_1 \quad \Pi_2}{A}$ if the cardinality of $pm(li(\Pi))$ is 1, 2, or 3 respectively. The notation $\dfrac{\Pi_0 \quad (\Pi_1)}{A}$ is used similarly.

(4) $\mathcal{Z}$, $\mathcal{N}^0$, and $\mathcal{N}^+$ denote the set of all integers, the set of all non-negative integers, and the set of all positive integers respectively. For a finite set S, $Card(S)$ denotes the cardinality of S. We use $\sqcup$ and $\bigsqcup$ to denote disjoint sums.

2 Reduction and theorems

In this section, we define our reduction and state theorems about it. The aim of the reduction is to remove maximum formulae in a derivation and to obtain a normal derivation. Maximum formulae and normal derivations are defined as follows.

2.1 Definition (Maximum formula)

Let Π be a derivation. A formula-occurrence μ in Π is a maximum formula in Π iff it satisfies the following conditions.

(1) μ is the conclusion of an application of an introduction rule, $(\vee E)$, $(\exists E)$, or $(\bot_c)$.

(2) μ is the major premiss of an application of an elimination rule.

2.2 Definition (Normal derivation)

A derivation Π is normal iff it contains no maximum formula.

2.3 Definition (Contraction)

To define our reduction, first we define the contraction of Π where Π is a derivation satisfying that $mj(li(\Pi))$ is a maximum formula. Let I be the D-inference in Π satisfying $cl(I) = mj(li(\Pi))$. The contraction of Π is defined according to $Inf(I)$. In the case that $Inf(I) \neq (\bot_c)$, the contraction is the same with Prawitz's reduction for the intuitionistic logic [5][6].

2.3.1 $\bot_c$-contraction

Let $\Pi = \dfrac{\dfrac{\overset{[\neg M]}{\Pi_0}}{M}\,I \qquad (\Pi_1 \quad \Pi_2)}{C}\,K$ where $Inf(I) = (\bot_c)$, $Inf(K)$ is an elimination rule, and $[\neg M]$ in $\overset{[\neg M]}{\Pi_0}$ denotes $dc(I)$. Since Π satisfies the regularity of $(\bot_c)$, any element of $dc(I)$ is the major premiss of an application of $(\neg E)$. Let J_1, …, J_n be all the applications of $(\neg E)$ whose major premiss is discharged by I, if they exists. Let Π_0' be the derivation obtained from Π_0 by the transformation represented by the following diagram:

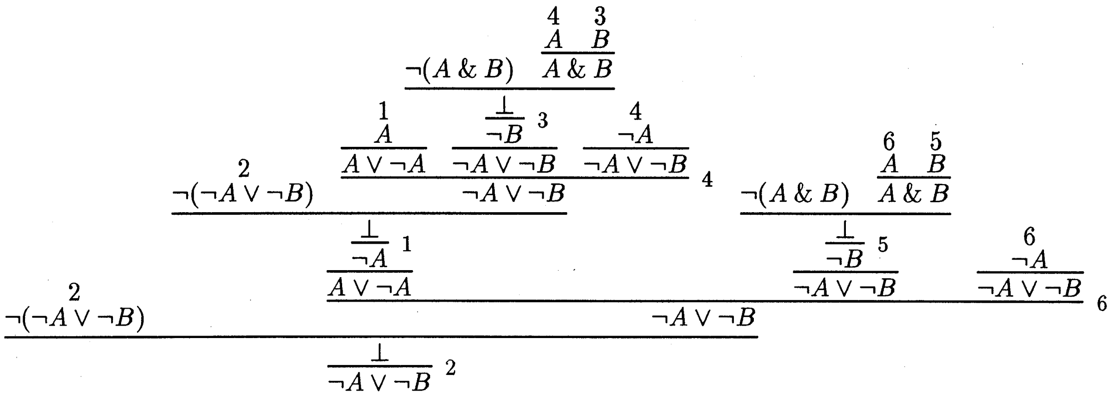
$$\dfrac{\neg M \quad M}{\bot}\,J_p \quad \longrightarrow \quad \dfrac{\neg C \qquad \dfrac{M \quad (\Pi_1 \quad \Pi_2)}{C}\,K_p'}{\bot}\,J_p'$$

where $Inf(K_p') = Inf(K)$, $Inf(J_p') = (\neg E)$, and $dc(K_p')$ is defined naturally according to $dc(K)$. These replacements are done simultaneously for all $p \in \{1, \dots, n\}$. We denote by $[\neg C]$ in $\overset{[\neg C]}{\Pi_0'}$ the set $\{mj(J_1'), \dots, mj(J_n')\}$. Then Π contracts to $\dfrac{\overset{[\neg C]}{\Pi_0'}}{C}\,I'$ where $Inf(I') = (\bot_c)$ and $dc(I')$ is $[\neg C]$ in $\overset{[\neg C]}{\Pi_0'}$.

Example of $\bot_c$-contraction

$$\dfrac{\dfrac{\dfrac{\overset{2}{\neg(A\vee\neg A)} \qquad \dfrac{\dfrac{\dfrac{\overset{2}{\neg(A\vee\neg A)} \quad \dfrac{\overset{1}{A}}{A\vee\neg A}}{\bot}}{\neg A}\,1}{A\vee\neg A}}{\bot}}{A\vee\neg A}\,2 \qquad \dfrac{\dfrac{\dfrac{\neg(A\,\&\,B) \quad \dfrac{\overset{4}{A} \;\; \overset{3}{B}}{A\,\&\,B}}{\bot}}{\neg B}\,3}{\neg A\vee\neg B} \qquad \dfrac{\overset{4}{\neg A}}{\neg A\vee\neg B}}{\neg A\vee\neg B}\,4$$

contracts to



In the figures above, the formula-occurrences indexed by a natural number are discharged by the D-inference indexed by the same number.

2.4 Immediate reducibility and Reduction sequence

We say that a derivation Π is immediately reduced to a derivation Π' iff Π' is the derivation obtained from Π by replacing a subderivation, say Γ, of Π by the derivation to which Γ contracts. A sequence $\Pi_1, \Pi_2, \ldots$ is called a reduction sequence iff for all i, Π_i is immediately reduced to Π_{i+1}.

2.5 Theorems

Now we state our theorems.

Theorem 1. (Weak normalization theorem) *For every derivation Π, we can construct a finite reduction sequence from Π to a normal derivation.*

Theorem 2. (Church-Rosser property) *If two finite reduction sequences $\Pi, \ldots, \Sigma$ and $\Pi, \ldots, \Sigma'$ are given, then we can construct two finite reduction sequences $\Sigma, \ldots, \Delta$ and $\Sigma', \ldots, \Delta$ for some derivation Δ.*

A proof of theorem 1 was given in [1]. The rest of this paper is devoted to a proof of theorem 2.

3 Segment, segment-tree, and segment-wood

3.1 Segment

We extend the definition of segment introduced by Prawitz [5] in order to treat ($\bot_c$)-contraction. Let Π be a derivation.

3.1.1 Definition($ss_\Pi(\alpha)$: segment successor of α)

A partial function on $FO(\Pi)$ denoted by ss_Π is defined as follows. Let α be a formula-occurrence in Π.

(1) If α is a minor premiss of an application, say I, of ($\vee E$) or ($\exists E$); then $ss_\Pi(\alpha)$ is the conclusion of I.

(2) If α is the minor premiss of an application of ($\neg E$) whose major premiss is discharged by an application, say I, of ($\bot_c$); then $ss_\Pi(\alpha)$ is the conclusion of I.

(3) Otherwise, $ss_\Pi(\alpha)$ is undefined.

Clearly it holds that $Form(ss_\Pi(\alpha)) = Form(\alpha)$ if $ss_\Pi(\alpha)$ is defined.

3.1.2 Definition ($sp_{\Pi}(\alpha)$: segment predecessor of α)

Let α be a formula-occurrence in Π. $sp_{\Pi}(\alpha)$ is the subset of $FO(\Pi)$ defined by

$$sp_{\Pi}(\alpha) = \{\beta \in FO(\Pi) \mid ss_{\Pi}(\beta) = \alpha\}.$$

3.1.3 Definition (segment)

A finite sequence of formula-occurrences $\alpha_1,\ldots,\alpha_n$ in Π is a segment in Π iff it satisfies the following conditions (1), (2), and (3).

(1) $sp_{\Pi}(\alpha_1) = \phi$

(2) For all $i < n$, $ss_{\Pi}(\alpha_i) = \alpha_{i+1}$

(3) $ss_{\Pi}(\alpha_n)$ is undefined.

Our definition of segment is equivalent with that introduced in [1].

3.1.4 Definition ($sd_{\Pi}(\alpha, \beta)$: segment distance from α to β)

sd_{Π} is a function from $FO(\Pi)\times FO(\Pi)$ to $\mathcal{Z}\cup\{\infty\}$ defined as follows. Let α and β be formula-occurrences in Π.

(1) If there exists a segment $\delta_1,\ldots\delta_n$ in Π satisfying $\{\alpha,\beta\} \subset \{\delta_1,\ldots,\delta_n\}$, then $sd_{\Pi}(\alpha,\beta) = y - x$ where $\alpha = \delta_x$ and $\beta = \delta_y$.

(2) Otherwise, $sd_{\Pi}(\alpha,\beta) = \infty$.

Note that sd_{Π} is well-defined. Because if two segments $\delta_1,\ldots,\delta_n$ and $\tau_1,\ldots,\tau_m$ include the same formula-occurrence, say $\delta_p = \tau_q$, then the sequences $\delta_p,\ldots,\delta_n$ and $\tau_q,\ldots,\tau_m$ are identical.

3.2 Segment-tree

To prove theorem 2, we will introduce in the next section an extended reduction (i.e. the structural reduction) which consists of $\vee E$-, $\exists E$-, or $\perp_c$-contractions applied continually for a *tree* of formula-occurrences in a derivation. Next we give the precise definition for the notion *tree* mentioned above.

3.2.1 Notation ($FO^*(\Pi)$)

We denote the set $FO(\Pi) \times \{0,1\}$ by $FO^*(\Pi)$.

3.2.2 Definition (sgt : segment-tree)

Let α be a formula-occurrence in Π, and T a subset of $FO^*(\Pi)$. The relation "*T is a segment-tree at α in Π* " holds iff one of the following conditions (a), (b), or (c) holds. It is defined by induction on the number of formula-occurrences above α.

(a) $T = \{< \alpha, 0 >\}$

(b) $sp_{\Pi}(\alpha) = \{\beta_1,\ldots,\beta_n\} \neq \phi$ where $\beta_i \neq \beta_j$ if $i \neq j$; and
$T = \{< \alpha, 0 >\} \cup \bigcup_{1\leq p\leq n} T_p$ where T_p is a segment-tree at β_p in Π for each $p \in \{1,\ldots,n\}$.

(c) α is the conclusion of an application of $(\perp_c)$; $sp_{\Pi}(\alpha) = \phi$; and
$T = \{< \alpha, 0 >, < \alpha, 1 >\}$.

We use the notation sgt for the abbreviation of segment-tree.

3.2.3 Some definitions

If T is a sgt at α in Π, then the construction of T is uniquely determined. Let T be a sgt at α in Π. We define two subsets of $FO(\Pi)$ denoted by $top(T)$ and $nf(T)$, and also define a natural number denoted by $len(T)$; by induction on the construction of T. In the following definitions of $top(T)$, $nf(T)$, and $len(T)$; (a), (b), and (c) means respectively (a), (b), and (c) in the Definition 3.2.2.

Definition ($top(T)$: tops of T)

Case (a): $top(T) = \{\alpha\}$

Case (b): $top(T) = \bigcup_{1 \le p \le n} top(T_p)$

Case (c): $top(T) = \phi$

Definition ($nf(T)$: negation-friends of T)

Case (a): $nf(T) = \phi$

Case (b): Let I be the D-inference satisfying $cl(I) = \alpha$.

$$nf(T) = \begin{cases} \bigcup_{1 \le p \le n} nf(T_p) & \text{if } Inf(I) = (\vee E) \text{ or } (\exists E) \\ dc(I) \cup \bigcup_{1 \le p \le n} nf(T_p) & \text{if } Inf(I) = (\bot_c) \end{cases}$$

Case (c): $nf(T) = \phi$

Definition ($len(T)$: length of T)

Case (a): $len(T) = 1$

Case (b): $len(T) = 1 + \max_{1 \le p \le n} len(T_p)$

Case (c): $len(T) = 2$

3.3 Segment-wood

We will introduce a notion *segment-wood*. This is used for the inductive definition of the continual reduction for a sgt at a maximum formula in a derivation.

3.3.1 Definition (connectable formula-occurrence)

A formula-occurrence α in Π is connectable in Π iff it satisfies one of the following conditions (1) or (2).

(1) $\alpha = end(\Pi)$

(2) There exists a D-inference I in Π; such that $Inf(I) = (\neg E)$, $mn(I) = \{\alpha\}$, and $mj(I) \in oa(\Pi)$.

3.3.2 Definition (sgw: segment-wood)

Let W be a subset of $FO^*(\Pi)$. W is a segment-wood in Π iff it satisfies one of the following conditions (a) or (b).

(a) $W = \phi$

(b) There exists mutually distinct formula-occurrences $\alpha_1, \ldots, \alpha_n$ in Π and subsets $T_1, \ldots, T_n$ of $FO^*(\Pi)$ such that;

(b1) for all $p, q \in \{1, \ldots, n\}$, $Form(\alpha_p) = Form(\alpha_q)$;

(b2) for all $p \in \{1, \ldots, n\}$, α_p is connectable in Π, and T_p is a sgt at α_p in Π;
and (b3) $W = \bigcup_{1 \le p \le n} T_p$.

We use the notation sgw for the abbreviation of segment-wood.

3.3.3 Definition ($cmp(W)$: component of W)

For a sgw W in Π, $cmp(W)$ is the finite set of formulae defined by

$$cmp(W) = \{Form(\alpha) \mid \text{There exists } k \in \{0,1\} \text{ such that } <\alpha, k> \in W\}$$

3.3.4 Definition ($rt(W)$: roots of W)

For a sgw W in Π, $rt(W)$ is the subset of $FO(\Pi)$ defined by

$$rt(W) = \{\alpha \in FO(\Pi) \mid <\alpha, 0> \in W \text{ and } \alpha \text{ is connectable in } \Pi\}$$

3.3.5 Definition ($W\lceil_\Gamma$)

Let W be a sgw in Π and Γ a subderivation of Π. $W\lceil_\Gamma$ is the subset of $FO^*(\Pi)$ defined by $W\lceil_\Gamma = W \cap FO^*(\Gamma)$.

3.3.6 Fact

Let W be a sgw in Π and Γ a subderivation of Π. Then, $W\lceil_\Gamma$ is a sgw in Γ and $cmp(W\lceil_\Gamma) \subset cmp(W)$.

3.3.7 Some definitions

Let W be a sgw in Π. We define three subsets of $FO(\Pi)$ denoted by $top(W)$, $on(W)$, and $nf(W)$. In the following definitions of $top(W)$, $on(W)$, and $nf(W)$; (a) and (b) means respectively (a) and (b) in the definition 3.3.2.

Definition ($top(W)$: tops of W)

Case (a): $top(W) = \phi$

Case (b): $top(W) = \bigcup_{1 \le p \le n} top(T_p)$

Definition ($on(W)$: open negation of W)

Case (a): $on(W) = \phi$

Case (b): For any $\beta \in FO(\Pi)$, $\beta \in on(W)$ is equivalent to the following condition. That is, there exists $\alpha \in rt(W) \setminus \{end(\Pi)\}$ such that $\beta = mj(I)$ where I is the D-inference satisfying $mn(I) = \{\alpha\}$.

Definition ($nf(W)$: negation-friends of W)

Case (a): $nf(W) = \phi$

Case (b): $nf(W) = on(W) \cup \bigcup_{1 \le p \le n} nf(T_p)$

4 Structural reduction

In this section, we define the structural reduction. It is applied for a sgt T at a maximum formula in a derivation where $len(T) > 1$. The structural reduction is an extension of $\vee E$-, $\exists E$-, and $\bot_c$-contractions in the following meaning. One application of $\vee E$-, $\exists E$-, or $\bot_c$-contraction removes a maximum formula μ in a derivation Π up to the elements of $sp_{\Pi}(\mu)$. The structural reduction for a sgt T at a maximum formula μ in a derivation where $len(T) > 1$ removes μ up to the elements of $top(T)$. In order to define the structural reduction, we introduce a method to *substitute* a derivation for a sgw in a derivation.

4.1 Substitution-sequence

4.1.1 Definition (substitution-sequence)

Let Π and Θ be derivations and W a sgw in Π. We call the sequence $< \Pi, W, \Theta >$ a substitution-sequence iff it satisfies the following conditions (a), (b), and (c).

(a) Any eigenvariable occurring in one of the derivations Π and Θ does not occur in the other.

(b) $LI(\Theta)$ is an elimination rule, and $mj(li(\Theta)) \in oa(\Theta)$.

(c) $cmp(W) \subset \{MJ(li(\Theta))\}$

4.1.2 Definition ($\mathcal{P}_S$, $\mathcal{E}_S^1$, $\mathcal{E}_S^2$, $\mathcal{F}_S^U$, $\mathcal{F}_S^D$)

Let S be a substitution-sequence $< \Pi, W, \Theta >$. By the following clauses from Case 0 to Case 2, we define a derivation denoted by $\mathcal{P}_S$; two subsets of $FO(\mathcal{P}_S)$ denoted by $\mathcal{E}_S^1$ and $\mathcal{E}_S^2$; and two injection from $FO(\Pi)$ to $FO(\mathcal{P}_S)$ denoted by $\mathcal{F}_S^U$ and $\mathcal{F}_S^D$; where they satisfy the following conditions (a), (b), (c), and (d). Suppose $\Theta = \dfrac{MJ(li(\Theta)) \quad (\Theta_1 \quad \Theta_2)}{END(\Theta)}$, and let $Q = Card(mn(li(\Theta)))$.

(a)

$$END(\mathcal{P}_S) = \begin{cases} END(\Theta), & \text{if } < end(\Pi), 0 > \in W, \\ END(\Pi), & \text{otherwise.} \end{cases}$$

(b) If $Q \geq 1$, then for all $\alpha \in \mathcal{E}_S^1$ it holds that $sbd(\alpha)$ is identical with Θ_1; otherwise, $\mathcal{E}_S^1 = \phi$. If $Q = 2$, then for all $\beta \in \mathcal{E}_S^2$ it holds that $sbd(\beta)$ is identical with Θ_2; otherwise, $\mathcal{E}_S^2 = \phi$.

(c) For all $\alpha \in oa(\Pi)$, $Form(\mathcal{F}_S^U(\alpha)) = \begin{cases} \neg(END(\Theta)), & \text{if } \alpha \in on(W), \\ Form(\alpha), & \text{otherwise.} \end{cases}$

(d) $oa(\mathcal{P}_S) = \{\mathcal{F}_S^U(\alpha) \mid \alpha \in oa(\Pi)\} \cup \bigcup_{l \in \{1,2\}} \bigcup_{\alpha \in \mathcal{E}_S^l} oa(sbd(\alpha))$

$\mathcal{P}_S$, $\mathcal{E}_S^1$, $\mathcal{E}_S^2$, $\mathcal{F}_S^U$, and $\mathcal{F}_S^D$ are defined by induction on the length of Π.

Case 0. If $W = \phi$:

$$\mathcal{P}_S = \Pi.$$

$$\mathcal{E}_S^1 = \mathcal{E}_S^2 = \phi.$$

$\mathcal{F}_S^U$ and $\mathcal{F}_S^D$ are the identity mapping on $FO(\Pi)$.

Case 1. If $W \neq \phi$ and the length of Π is 1:

$$\mathcal{P}_S = \Theta.$$

$$\begin{cases} \mathcal{E}_S^1 = \mathcal{E}_S^2 = \phi, & \text{if } Q = 0, \\ \mathcal{E}_S^1 = \{end(\Theta_1)\} \text{ and } \mathcal{E}_S^2 = \phi, & \text{if } Q = 1, \\ \mathcal{E}_S^1 = \{end(\Theta_1)\} \text{ and } \mathcal{E}_S^2 = \{end(\Theta_2)\}, & \text{if } Q = 2. \end{cases}$$

$$\mathcal{F}_S^U(end(\Pi)) = mj(li(\mathcal{P}_S)).$$

$$\mathcal{F}_S^D(end(\Pi)) = end(\mathcal{P}_S).$$

Case 2. If $W \neq \phi$ and the length of Π is greater than 1:

Suppose $\Pi = \dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{END(\Pi)}$. Let S_r be the substitution-sequence defined by $S_r =< \Pi_r, W\lceil_{\Pi_r}, \Theta >$ for each $r \in \{0, 1, 2\}$.

Case 2-1. If $< end(\Pi), 0 > \notin W$:

Case 2-1-1. If $end(\Pi_0) \notin on(W)$:

$$\mathcal{P}_S = \frac{\mathcal{P}_{S_0} \quad (\mathcal{P}_{S_1} \quad \mathcal{P}_{S_2})}{END(\Pi)}\, K$$

where $Inf(K) = LI(\Pi)$ and

$$dc(K) = \bigcup_{0 \leq r \leq 2} \{\mathcal{F}^U_{S_r}(\alpha) \mid \alpha \in dc(li(\Pi)) \cap FO(\Pi_r)\}.$$

$$\text{For all } l \in \{1,2\}, \mathcal{E}^l_S = \bigcup_{0 \leq r \leq 2} \mathcal{E}^l_{S_r}.$$

$$\begin{cases} \mathcal{F}^U_S(end(\Pi)) = \mathcal{F}^D_S(end(\Pi)) = end(\mathcal{P}_S). \\ \text{For all } r \in \{0,1,2\}, \text{ and for all } \alpha \in FO(\Pi_r); \\ \quad \mathcal{F}^U_S(\alpha) = \mathcal{F}^U_{S_r}(\alpha) \text{ and } \mathcal{F}^D_S(\alpha) = \mathcal{F}^D_{S_r}(\alpha). \end{cases}$$

Case 2-1-2. If $end(\Pi_0) \in on(W)$:

$$\mathcal{P}_S = \frac{\neg(END(\Theta)) \quad \mathcal{P}_{S_1}}{\bot}\, K$$

where $Inf(K) = (\neg E)$.

$$\text{For all } l \in \{1,2\}, \mathcal{E}^l_S = \mathcal{E}^l_{S_1}.$$

$$\begin{cases} \mathcal{F}^U_S(end(\Pi)) = \mathcal{F}^D_S(end(\Pi)) = end(\mathcal{P}_S). \\ \mathcal{F}^U_S(end(\Pi_0)) = \mathcal{F}^D_S(end(\Pi_0)) = mj(K). \\ \text{For all } \alpha \in FO(\Pi_1), \mathcal{F}^U_S(\alpha) = \mathcal{F}^U_{S_1}(\alpha) \text{ and } \mathcal{F}^D_S(\alpha) = \mathcal{F}^D_{S_1}(\alpha). \end{cases}$$

Case 2-2. If $< end(\Pi), 0 > \in W$:

Case 2-2-1. If $end(\Pi) \notin top(W)$:

$$\mathcal{P}_S = \frac{\mathcal{P}_{S_0} \quad (\mathcal{P}_{S_1} \quad \mathcal{P}_{S_2})}{END(\Theta)}\, K$$

where $Inf(K) = LI(\Pi)$ and

$$dc(K) = \bigcup_{0 \leq r \leq 2} \{\mathcal{F}^U_{S_r}(\alpha) \mid \alpha \in dc(li(\Pi)) \cap FO(\Pi_r)\}.$$

$$\text{For all } l \in \{1,2\}, \mathcal{E}^l_S = \bigcup_{0 \leq r \leq 2} \mathcal{E}^l_{S_r}.$$

$$\begin{cases} \mathcal{F}^U_S(end(\Pi)) = \mathcal{F}^D_S(end(\Pi)) = end(\mathcal{P}_S). \\ \text{For all } r \in \{0,1,2\}, \text{ and for all } \alpha \in FO(\Pi_r); \\ \quad \mathcal{F}^U_S(\alpha) = \mathcal{F}^U_{S_r}(\alpha) \text{ and } \mathcal{F}^D_S(\alpha) = \mathcal{F}^D_{S_r}(\alpha). \end{cases}$$

Case 2-2-2. If $end(\Pi) \in top(W)$:

$$\mathcal{P}_S = \frac{\dfrac{\mathcal{P}_{S_0} \quad (\mathcal{P}_{S_1} \quad \mathcal{P}_{S_2})}{END(\Pi)}\, K \qquad (\Theta_1 \quad \Theta_2)}{END(\Theta)}\, I$$

where $Inf(K) = LI(\Pi)$,

$$dc(K) = \bigcup_{0 \leq r \leq 2} \{\mathcal{F}^U_{S_r}(\alpha) \mid \alpha \in dc(li(\Pi)) \cap FO(\Pi_r)\},$$

$Inf(I) = LI(\Theta)$, and $dc(I)$ is identical with $dc(li(\Theta))$ as the subset of $\bigcup_{1 \leq q \leq Q} FO(\Theta_q)$.

$$\begin{cases} \mathcal{E}^1_S = \mathcal{E}^2_S = \phi, & \text{if } Q = 0, \\ \mathcal{E}^1_S = mn(I) \cup \bigcup_{0 \leq r \leq 2} \mathcal{E}^1_{S_r} \text{ and } \mathcal{E}^2_S = \phi, & \text{if } Q = 1, \\ \text{For all } l \in \{1,2\}, \mathcal{E}^l_S = \{\alpha_l\} \cup \bigcup_{0 \leq r \leq 2} \mathcal{E}^l_{S_r}, & \text{if } Q = 2, \end{cases}$$

where, in the case of $Q = 2$, α_1 and α_2 are formula-occurrences of $\mathcal{P}_S$ satisfying that $mn(I) = \{\alpha_1, \alpha_2\}$ and α_1 stands on the left hand of α_2.

$$\begin{cases} \mathcal{F}^U_S(end(\Pi)) = mj(I). \ \mathcal{F}^D_S(end(\Pi)) = end(\mathcal{P}_S). \\ \text{For all } r \in \{0,1,2\}, \text{ and for all } \alpha \in FO(\Pi_r); \\ \quad \mathcal{F}^U_S(\alpha) = \mathcal{F}^U_{S_r}(\alpha) \text{ and } \mathcal{F}^D_S(\alpha) = \mathcal{F}^D_{S_r}(\alpha). \end{cases}$$

4.2 Structural reduction

4.2.1 Definition (structural reduction)

Let Π be a derivation satisfying that $mj(li(\Pi))$ is a maximum formula in Π, and let T be a sgt at $mj(li(\Pi))$ in Π satisfying $len(T) \geq 2$. Then, the structural reduction of Π with T is the transformation of Π to the derivation $\mathcal{P}_S$ where the substitution-sequence S is defined by the following. Suppose $\Pi = \dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{END(\Pi)} K$. Let Θ be a derivation defined by $\Theta = \dfrac{END(\Pi_0) \quad (\Pi_1 \quad \Pi_2)}{END(\Pi)} K'$ where $Inf(K') = Inf(K)$, and $dc(K')$ is identical with $dc(K)$ as a subset of $FO(\Pi_1) \cup FO(\Pi_2)$. Then, the substitution-sequence S is defined by $S =< \Pi_0, T, \Theta >$. We call this substitution-sequence the *accompanying* substitution-sequence of the structural reduction of Π with T.

4.2.2 Notation

$\Pi \stackrel{SR(T)}{\longrightarrow} \Pi'$ denotes the fact that the derivation Π' is obtained by the structural reduction of Π with T.

4.2.3 Facts

We have the following facts (1) and (2) by definition.

(1) *Let α be a formula-occurrence in a derivation Π satisfying that α is the conclusion of an application of $(\vee E)$, $(\exists E)$, or $(\bot_c)$. Then, there exists exactly one sgt T at α in Π such that $len(T) = 2$.*

(2) *Let Π be a derivation satisfying that $mj(li(\Pi))$ is a maximum formula and is the conclusion of an application of $(\vee E)$, $(\exists E)$, or $(\bot_c)$. Suppose Π contracts to Π'. Then, it holds that $\Pi \stackrel{SR(T)}{\longrightarrow} \Pi'$ where T is the sgt at $mj(li(\Pi))$ in Π satisfying $len(T) = 2$.*

At the end of this section, we will state the fact that; if $\Pi \stackrel{SR(T)}{\longrightarrow} \Pi'$ holds, then there exists a reduction sequence from Π to Π' consisting of $\vee E$-, $\exists E$-, and $\bot_c$ -contractions (for subderivations).

4.2.4 Notation

For a derivation Π, we denote the set of all sgw's in Π by $SGW(\Pi)$.

4.3 Mappings

When $\Pi \stackrel{SR(T)}{\longrightarrow} \Pi'$ holds, we often need to use the *natural* mappings from $SGW(\Pi)$ to $SGW(\Pi')$ and from $oa(\Pi)$ to $oa(\Pi')$. In order to represent such mappings, we define the mappings CS_S^1, OS_S^1, CS_S^2, and OS_S^2 for a substitution-sequence S.

4.3.1 Definition (CS_S^1, OS_S^1, CS_S^2, OS_S^2)

Let S be a substitution-sequence $< \Pi, W, \Theta >$. For $U \in SGW(\Pi)$ satisfying $U \cap W = \phi$, $CS_S^1(U)$ is the subset of $FO^*(\mathcal{P}_S)$ defined by

$$CS_S^1(U) = \{< \mathcal{F}_S^D(\theta), k > | < \theta, k > \in U\}$$

For $\alpha \in oa(\Pi) \setminus on(W)$, $OS_S^1(\alpha)$ is the subset of $oa(\mathcal{P}_S)$ defined by $OS_S^1(\alpha) = \{\mathcal{F}_S^U(\alpha)\}$. For $V \in SGW(\Theta)$, $CS_S^2(V)$ is the subset of $FO^*(\mathcal{P}_S)$ defined by

$$CS_S^2(V) = \begin{cases} \bigcup_{l \in \{1,2\}} \bigcup_{\lambda \in \mathcal{E}_S^l} \{< i_\lambda(\theta), k > | < \theta, k > \in V\lceil_{\Theta_l}\} \\ \quad \cup \{< \mathcal{F}_S^D(\theta), k > | < \theta, k > \in W\}, & \text{if } < end(\Theta), 0 > \in V, \\ \bigcup_{l \in \{1,2\}} \bigcup_{\lambda \in \mathcal{E}_S^l} \{< i_\lambda(\theta), k > | < \theta, k > \in V\lceil_{\Theta_l}\}, & \text{otherwise,} \end{cases}$$

where for each $l \in \{1, 2\}$ and for each $\lambda \in \mathcal{E}_S^l$, i_λ is the canonical bijection from $FO(\Theta_l)$ ($\subset FO(\Theta)$) to $FO(sbd(\lambda))$ ($\subset FO(\mathcal{P}_S)$). For $\beta \in oa(\Theta) \setminus \{mj(li(\Theta))\}$, $OS_S^2(\beta)$ is the subset of $oa(\mathcal{P}_S)$ defined by

$$OS_S^2(\beta) = \begin{cases} \bigcup_{\lambda \in \mathcal{E}_S^1} \{i_\lambda(\beta)\}, & \text{if } \beta \in FO(\Theta_1), \\ \bigcup_{\lambda \in \mathcal{E}_S^2} \{i_\lambda(\beta)\}, & \text{if } \beta \in FO(\Theta_2), \end{cases}$$

where i_λ is defined as above.

4.3.2 Definition ($CS_{\Pi,T}$, $OS_{\Pi,T}$)

Let Π' be the derivation obtained from a derivation Π by the structural reduction of Π with T, i.e. $\Pi \stackrel{SR(T)}{\longrightarrow} \Pi'$. Suppose $\Pi = \dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{END(\Pi)}$, and let S be the accompanying substitution-sequence of the structural reduction of Π with T. For $W \in SGW(\Pi)$, $CS_{\Pi,T}(W)$ is the subset of $FO^*(\Pi')$ defined by

$$CS_{\Pi,T}(W) = CS_S^1(W\lceil_{\Pi_0}) \cup CS_S^2(W \setminus W\lceil_{\Pi_0}).$$

For $\alpha \in oa(\Pi)$, $OS_{\Pi,T}(\alpha)$ is the subset of $oa(\Pi')$ defined by

$$OS_{\Pi,T}(\alpha) = \begin{cases} OS_S^1(\alpha), & \text{if } \alpha \in FO(\Pi_0), \\ OS_S^2(\alpha), & \text{otherwise.} \end{cases}$$

4.4 Relationship between structural reductions and contractions

4.4.1 Fact

Let S be a substitution-sequence $< \Pi, W, \Theta >$. Let V_1 and V_2 be sgw's in Π satisfying $V_1 \cup V_2 = W$ and $V_1 \cap V_2 = \phi$. Let S^1 and S^2 be the substitution-sequences defined by $S^1 =< \Pi, V_1, \Theta >$ and $S^2 =< \mathcal{P}_{S^1}, CS_{S^1}^1(V_2), \Theta >$. Then, it holds that $\mathcal{P}_S = \mathcal{P}_{S^2}$.

Proof. By induction on the length of Π.

4.4.2 Definition ($supp(W)$: support of W)

Let W be a sgw in Π. $supp(W)$ is the sgw in Π defined by

$$supp(W) = \{< \alpha, 0 >\in FO^*(\Pi) \mid \alpha \in rt(W)\}$$

4.4.3 Fact

Let S be a substitution-sequence $< \Pi, W, \Theta >$. If S' is the substitution-sequence defined by $S' =< \Pi, supp(W), \Theta >$; then, it holds that there exists a reduction sequence from $\mathcal{P}_{S'}$ to $\mathcal{P}_S$ consisting of $\vee E$-, $\exists E$-, and $\bot_c$-contractions (for subderivations).

Proof. By induction on $Card(W)$. Suppose $\Theta = \dfrac{MJ(li(\Theta)) \quad (\Theta_1 \quad \Theta_2)}{END(\Theta)}$.

Case 0. If $W = \phi$: Clear.

Case 1. If $Card(rt(W)) = 1$: Without loss of generality, we can assume that $rt(W) = \{end(\Pi)\}$.

Case 1-1. If $supp(W) = W$: Clear.

Case 1-2. If $supp(W) \neq W$, $li(\Pi) = (\bot_c)$, and $< end(\Pi), 1 >\notin W$: Suppose $\Pi = \dfrac{\Pi_0}{END(\Pi)}$. Then, there exists a sgw W_0 in Π_0, such that $W = \{< end(\Pi), 0 >\} \cup W_0$. Let S_0 and S'_0 be the substitution-sequences defined by $S_0 =< \Pi_0, W_0, \Theta >$ and $S'_0 =< \Pi_0, supp(W_0), \Theta >$. Now, $\mathcal{P}_{S'}$ is of the form $\dfrac{\dfrac{\Pi_0}{END(\Pi)} \quad (\Theta_1 \quad \Theta_2)}{END(\Theta)}$. Let Π' be the derivation obtained from $\mathcal{P}_{S'}$ by $(\bot_c)$-contraction. Then, Π' is of the form $\dfrac{\mathcal{P}_{S'_0}}{END(\Theta)}$, and by induction hypothesis, there exists a reduction sequence from Π' to the derivation $\dfrac{\mathcal{P}_{S_0}}{END(\Theta)} = \mathcal{P}_S$, consisting of $\vee E$-, $\exists E$-, and $\bot_c$-contractions.

Case 1-3. If $supp(W) \neq W$, $li(\Pi) = (\bot_c)$, and $< end(\Pi), 1 >\in W$; i.e. if $W = \{< end(\Pi), 0 >, < end(\Pi), 1 >\}$: Easy.

Case 1-4. If $supp(W) \neq W$ and $li(\Pi) = (\vee E)$ or $(\exists E)$: Similarly to the case 1-2.

Case 2. If $Card(rt(W)) > 1$: Take two sgw's in Π, say V_1 and V_2, satisfying that $W = V_1 \cup V_2$, $V_1 \cap V_2 = \phi$, $V_1 \neq \phi$, and $V_2 \neq \phi$. Let X be the substitution-sequence defined by $X =< \Pi, V_1 \cup supp(V_2), \Theta >$. Let Y_1, Y_2, and Y_3 be the substitution-sequence defined by

$$Y_1 =< \Pi, supp(V_2), \Theta >, \quad Y_2 =< \mathcal{P}_{Y_1}, CS^1_{Y_1}(supp(V_1)), \Theta >,$$

and

$$Y_3 =< \mathcal{P}_{Y_1}, CS^1_{Y_1}(V_1), \Theta > .$$

Using fact 4.4.1, we have $\mathcal{P}_{S'} = \mathcal{P}_{Y_2}$ and $\mathcal{P}_X = \mathcal{P}_{Y_3}$. It holds that $Card(CS^1_{Y_1}(V_1)) = Card(V_1)$ and that $supp(CS^1_{Y_1}(V_1)) = CS^1_{Y_1}(supp(V_1))$. Hence, by induction hypothesis, there exists a reduction sequence from $\mathcal{P}_{Y_2}$ to $\mathcal{P}_{Y_3}$, i.e. from $\mathcal{P}_{S'}$ to $\mathcal{P}_X$, consisting of $\vee E$-, $\exists E$-, and $\bot_c$-contractions. Similarly, we have the existance of a reduction sequence from $\mathcal{P}_X$ to $\mathcal{P}_S$, consisting of $\vee E$-, $\exists E$-, and $\bot_c$-contractions. This leads the result.□

4.4.4 Fact

Let Π' be the derivation obtained from a derivation Π by the structural reduction of Π with T, i.e. $\Pi \stackrel{SR(T)}{\longrightarrow} \Pi'$. Then, there exists a reduction sequence from Π to Π' consisting of $\vee E$-, $\exists E$-, and $\bot_c$-contractions (for subderivations).

Proof. By fact 4.4.3.

5 1-reduction and Church-Rosser property

In this section, we define 1-reduction to prove the Church-Rosser property of our reduction. The definition of 1-reduction is an extension of that of Girard [4, pp135].

5.1 Mappings for essential reduction

5.1.1 Notation ($\Pi \stackrel{ER}{\longrightarrow} \Pi'$)

When a derivation Π' is obtained from a derivation Π by $\&_1$-, $\&_2$-, $\vee_1$-, $\vee_2$-, $\supset$-, $\neg$-, $\forall$-, or $\exists$-contraction; we denote the fact by $\Pi \stackrel{ER}{\longrightarrow} \Pi'$.

5.1.2 Definition (CE_Π, OE_Π)

Let Π and Π' be derivations satisfying $\Pi \stackrel{ER}{\longrightarrow} \Pi'$. For $W \in SGW(\Pi)$ and for $\alpha \in oa(\Pi)$, $CE_\Pi(W)$ and $OE_\Pi(\alpha)$ are the subset of $FO^*(\Pi')$ and the subset of $oa(\Pi')$ respectively, defined by the following clauses (1),...,(6).

(1) If Π' is obtained from Π by $\&_l$-contraction ($l = 1$ or 2): Suppose $\Pi = \dfrac{\dfrac{\Pi_1 \quad \Pi_2}{A_1 \,\&\, A_2}}{A_l}$. Then, $\Pi' = \Pi_l$.

Let i be the canonical bijection from $FO(\Pi_l)$ (as a subset of $FO(\Pi)$) to $FO(\Pi')$. Then, $CE_\Pi(W)$ and $OE_\Pi(\alpha)$ are defined as follows.

$$CE_\Pi(W) = \begin{cases} \{< i(\theta), k > \mid < \theta, k > \in W\lceil_{\Pi_l}\} \\ \quad \cup \{< end(\Pi'), 0 >\}, & \text{if } < end(\Pi), 0 > \in W, \\ \{< i(\theta), k > \mid < \theta, k > \in W\lceil_{\Pi_l}\}, & \text{otherwise.} \end{cases}$$

$$OE_\Pi(\alpha) = \begin{cases} \{i(\alpha)\}, & \text{if } \alpha \in FO(\Pi_l), \\ \phi, & \text{otherwise.} \end{cases}$$

(2) If Π' is obtained from Π by $\vee_l$-contraction ($l = 1$ or 2): Suppose $\Pi = \dfrac{\dfrac{\Pi_0}{A_1 \vee A_2} \quad \begin{matrix}[A_1] \\ \Pi_1\end{matrix} \quad \begin{matrix}[A_2] \\ \Pi_2\end{matrix}}{C}$.

Then, $\Pi' = \begin{matrix}\Pi_0 \\ [A_l] \\ \Pi_l\end{matrix}$. Let i be the canonical bijection from $FO(\Pi_l)$ (as a subset of $FO(\Pi)$) to $FO(\Pi_l)$ (as a subset of $FO(\Pi')$). Let Λ be the subset of $FO(\Pi')$ defined by

$$\Lambda = \{i(\theta) \mid \theta \in dc(li(\Pi)) \cap FO(\Pi_l)\}.$$

For each $\lambda \in \Lambda$, let i_λ be the canonical bijection from $FO(\Pi_0)$ (as a subset of $FO(\Pi)$) to $FO(sbd(\lambda))$. Then, $CE_\Pi(W)$ and $OE_\Pi(\alpha)$ are defined as follows.

$$CE_\Pi(W) = \begin{cases} \{<i(\theta),k>|<\theta,k>\in W\lceil_{\Pi_l}\} \cup \bigcup_{\lambda\in\Lambda}\{<i_\lambda(\theta),k>|<\theta,k>\in W\lceil_{\Pi_0}\} \\ \quad \cup\{<end(\Pi'),0>\}, & \text{if } <end(\Pi),0>\in W, \\ \{<i(\theta),k>|<\theta,k>\in W\lceil_{\Pi_l}\} \cup \bigcup_{\lambda\in\Lambda}\{<i_\lambda(\theta),k>|<\theta,k>\in W\lceil_{\Pi_0}\}, \\ & \text{otherwise.} \end{cases}$$

$$OE_\Pi(\alpha) = \begin{cases} \{i(\alpha)\}, & \text{if } \alpha \in FO(\Pi_l), \\ \phi, & \text{if } \alpha \in FO(\Pi_m) \text{ where } \{l,m\} = \{1,2\}, \\ \bigcup_{\lambda\in\Lambda}\{i_\lambda(\alpha)\}, & \text{if } \alpha \in FO(\Pi_0). \end{cases}$$

(3) If Π' is obtained from Π by $\supset$-contraction: Suppose $\Pi = \dfrac{\dfrac{\begin{array}{c}[A] \\ \Pi_0\end{array}}{A \supset B}\,I \qquad \Pi_1}{B}$. Then, $\Pi' = \begin{array}{c}\Pi_1 \\ [A] \\ \Pi_0\end{array}$.

Let i be the canonical bijection from $FO(\Pi_0)$ (as a subset of $FO(\Pi)$) to $FO(\Pi_0)$ (as a subset of $FO(\Pi')$). Let Λ be the subset of $FO(\Pi')$ defined by $\Lambda = \{i(\theta) \mid \theta \in dc(I)\}$. For each $\lambda \in \Lambda$, let i_λ be the canonical bijection from $FO(\Pi_1)$ (as a subset of $FO(\Pi)$) to $FO(sbd(\lambda))$. Then, $CE_\Pi(W)$ and $OE_\Pi(\alpha)$ is defined as follows.

$$CE_\Pi(W) = \begin{cases} \{<i(\theta),k>|<\theta,k>\in W\lceil_{\Pi_0}\} \cup \bigcup_{\lambda\in\Lambda}\{<i_\lambda(\theta),k>|<\theta,k>\in W\lceil_{\Pi_1}\} \\ \quad \cup\{<end(\Pi'),0>\}, & \text{if } <end(\Pi),0>\in W, \\ \{<i(\theta),k>|<\theta,k>\in W\lceil_{\Pi_0}\} \cup \bigcup_{\lambda\in\Lambda}\{<i_\lambda(\theta),k>|<\theta,k>\in W\lceil_{\Pi_1}\}, \\ & \text{otherwise.} \end{cases}$$

$$OE_\Pi(\alpha) = \begin{cases} \{i(\alpha)\}, & \text{if } \alpha \in FO(\Pi_0), \\ \bigcup_{\lambda\in\Lambda}\{i_\lambda(\alpha)\}, & \text{if } \alpha \in FO(\Pi_1). \end{cases}$$

(4) If Π' is obtained from Π by $\neg$-contraction: Similarly to the case (3).

(5) If Π' is obtained from Π by $\forall$-contraction: Similarly to the case (1).

(6) If Π' is obtained from Π by $\exists$-contraction: Similarly to the case (2).

5.2 1-reduction

5.2.1 Definition (1-reduction)

Let Π and Π' be derivations satisfying $END(\Pi') = END(\Pi)$ and $OA(\Pi') \subset OA(\Pi)$. The transformation of Π to Π' is called 1-reduction iff it satisfies one of the conditions (1), (2), (3), or (4) below. We denote by $\Pi \xrightarrow{1} \Pi'$ the fact that the transformation of Π to Π' is a 1-reduction. 1-reduction is defined inductively with a mapping from $SGW(\Pi)$ to $SGW(\Pi')$, denoted by $C_\Pi^{\Pi'}$, and with a mapping from $oa(\Pi)$ to the power set of $oa(\Pi')$, denoted by $O_\Pi^{\Pi'}$; where $C_\Pi^{\Pi'}$ and $O_\Pi^{\Pi'}$ satisfy the following conditions (a), (b), and (c).

(a) For all $\alpha \in oa(\Pi)$ and for all $\beta \in O_\Pi^{\Pi'}(\alpha)$, $Form(\alpha) = Form(\beta)$ holds.

(b)

$$oa(\Pi') = \bigsqcup_{\alpha \in oa(\Pi)} O_\Pi^{\Pi'}(\alpha)$$

(c) For all $W \in SGW(\Pi)$, $cmp(C_{\Pi}^{\Pi'}(W)) \subset cmp(W)$ and $on(C_{\Pi}^{\Pi'}(W)) = \bigcup_{\alpha \in on(W)} O_{\Pi}^{\Pi'}(\alpha)$ hold.

(1) Π and Π' are identical. In this case, $C_{\Pi}^{\Pi'}$ and $O_{\Pi}^{\Pi'}$ are defined as follows.
For each $W \in SGW(\Pi)$, $C_{\Pi}^{\Pi'}(W) = W$.
For each $\alpha \in oa(\Pi)$, $O_{\Pi}^{\Pi'}(\alpha) = \{\alpha\}$.

(2) Π and Π' are of the form $\dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A}\,K$ and $\dfrac{\Pi'_0 \quad (\Pi'_1 \quad \Pi'_2)}{A}\,K'$ respectively, where $\Pi_p \stackrel{1}{\longrightarrow} \Pi'_p$ (for all $p \in \{0,1,2\}$), $Inf(K') = Inf(K)$, and $dc(K') = \bigcup_{0 \le p \le 2} \bigcup_{\alpha \in dc(K) \cap FO(\Pi_p)} O_{\Pi_p}^{\Pi'_p}(\alpha)$. In this case, $C_{\Pi}^{\Pi'}$ and $O_{\Pi}^{\Pi'}$ are defined as follows.
For each $W \in SGW(\Pi)$,

$$C_{\Pi}^{\Pi'}(W) = \bigcup_{0 \le p \le 2} C_{\Pi_p}^{\Pi'_p}(W \lceil_{\Pi_p}) \cup \{< end(\Pi'), k > \mid < end(\Pi), k > \in W\} \cup E$$

where

$$E = \begin{cases} \{< end(\Pi'), 1 >\}, & \text{if } Inf(K) = (\bot_c),\ dc(K) \cap nf(W) \neq \phi, \text{ and } dc(K') = \phi, \\ \phi, & \text{otherwise.} \end{cases}$$

For each $p \in \{0,1,2\}$ and for each $\alpha \in oa(\Pi) \cap FO(\Pi_p)$, $O_{\Pi}^{\Pi'}(\alpha) = O_{\Pi_p}^{\Pi'_p}(\alpha)$.

(3) Π is of the form $\dfrac{\dfrac{\Pi_0 \quad (\Pi_1)}{M}\,I \qquad (\Pi_2 \quad \Pi_3)}{A}\,K$ where $Inf(I)$ is an introduction rule and $Inf(K)$ is an elimination rule; and

$$\dfrac{\dfrac{\Pi'_0 \quad (\Pi'_1)}{M}\,I' \qquad (\Pi'_2 \quad \Pi'_3)}{A}\,K' \quad \stackrel{ER}{\longrightarrow} \quad \Pi'$$

where $\Pi_p \stackrel{1}{\longrightarrow} \Pi'_p$ (for all $p \in \{0,\ldots,3\}$), $Inf(I') = Inf(I)$, $dc(I') = \bigcup_{\alpha \in dc(I)} O_{\Pi_0}^{\Pi'_0}(\alpha)$, $Inf(K') = Inf(K)$, and $dc(K') = \bigcup_{2 \le p \le 3} \bigcup_{\alpha \in dc(K) \cap FO(\Pi_p)} O_{\Pi_p}^{\Pi'_p}(\alpha)$. In this case, $C_{\Pi}^{\Pi'}$ and $O_{\Pi}^{\Pi'}$ are defined as follows. Let Δ be the derivation $\dfrac{\dfrac{\Pi'_0 \quad (\Pi'_1)}{M}\,I' \qquad (\Pi'_2 \quad \Pi'_3)}{A}\,K'$. For each $W \in SGW(\Pi)$, $C_{\Pi}^{\Pi'}(W) = CE_{\Delta}(W')$ where

$$W' = \begin{cases} \bigcup_{0 \le p \le 3} C_{\Pi_p}^{\Pi'_p}(W \lceil_{\Pi_p}) \cup \{< end(\Delta), 0 >\}, & \text{if } < end(\Pi), 0 > \in W, \\ \bigcup_{0 \le p \le 3} C_{\Pi_p}^{\Pi'_p}(W \lceil_{\Pi_p}), & \text{otherwise.} \end{cases}$$

For each $p \in \{0,\ldots,3\}$ and for each $\alpha \in oa(\Pi) \cap FO(\Pi_p)$, $O_{\Pi}^{\Pi'}(\alpha) = \bigcup_{\theta \in O_{\Pi_p}^{\Pi'_p}(\alpha)} OE_{\Delta}(\theta)$.

(4) Π is of the form $\dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A}\,K$ where $Inf(K)$ is an elimination rule and $LI(\Pi_0)$ is $(\vee E)$, $(\exists E)$, or $(\bot_c)$; and

$$\dfrac{\Pi'_0 \quad (\Pi'_1 \quad \Pi'_2)}{A}\,K' \quad \stackrel{SR(C_{\Pi_0}^{\Pi'_0}(T))}{\longrightarrow} \quad \Pi'$$

where $\Pi_p \stackrel{1}{\longrightarrow} \Pi'_p$ (for all $p \in \{0,1,2\}$), $Inf(K') = Inf(K)$, $dc(K') = \bigcup_{1 \le p \le 2} \bigcup_{\alpha \in dc(K) \cap FO(\Pi_p)} O_{\Pi_p}^{\Pi'_p}(\alpha)$, and T is a sgt at $end(\Pi_0)$ in Π_0 satisfying $len(T) > 1$ and $len(C_{\Pi_0}^{\Pi'_0}(T)) > 1$. In this case, $C_{\Pi}^{\Pi'}$ and

$O_{\Pi}^{\Pi'}$ are defined as follows. Let Δ be the derivation $\dfrac{\Pi_0' \quad (\Pi_1' \quad \Pi_2')}{A} K'$ and T' the sgt $C_{\Pi_0}^{\Pi_0'}(T)$ at $end(\Pi_0')$ in Π_0'. For each $W \in SGW(\Pi)$, $C_{\Pi}^{\Pi'}(W) = CS_{\Delta,T'}(W')$ where

$$W' = \begin{cases} \bigcup_{0 \le p \le 2} C_{\Pi_p}^{\Pi_p'}(W\lceil_{\Pi_p}) \cup \{< end(\Delta), 0 >\}, & \text{if } < end(\Pi), 0 > \in W, \\ \bigcup_{0 \le p \le 2} C_{\Pi_p}^{\Pi_p'}(W\lceil_{\Pi_p}), & \text{otherwise.} \end{cases}$$

For each $p \in \{0,1,2\}$ and for each $\alpha \in oa(\Pi) \cap FO(\Pi_p)$, $O_{\Pi}^{\Pi'}(\alpha) = \bigcup_{\theta \in O_{\Pi_p}^{\Pi_p'}(\alpha)} OS_{\Delta,T'}(\theta)$.

5.2.2 Notice

When derivations Π and Π' satisfying $\Pi \xrightarrow{1} \Pi'$ are given; it is assumed that the construction of $\Pi \xrightarrow{1} \Pi'$ is also given, and so, the number of the clauses in definition 5.2.1 used in the construction of $\Pi \xrightarrow{1} \Pi'$ is uniquely determined.

5.2.3 Notation ($|\Pi \xrightarrow{1} \Pi'|$, $LC(\Pi \xrightarrow{1} \Pi')$)

Let Π and Π' be derivations satisfying $\Pi \xrightarrow{1} \Pi'$. We denote by $|\Pi \xrightarrow{1} \Pi'|$ the number of the clauses in definition 5.2.1 used in the construction of $\Pi \xrightarrow{1} \Pi'$. Also we denote by $LC(\Pi \xrightarrow{1} \Pi')$ the last clause in definition 5.2.1 used in the construction of $\Pi \xrightarrow{1} \Pi'$.

5.2.4 Fact

If a derivation Π is immediately reduced to a derivation Π', then it holds that $\Pi \xrightarrow{1} \Pi'$.

Proof. By fact (2) of 4.2.3.

5.2.5 Fact

If a derivation Π is 1-reduced to a derivation Π', i.e. $\Pi \xrightarrow{1} \Pi'$, then there exists a reduction sequence from Π to Π'.

Proof. By fact 4.4.4.

5.2.6 Notation

Let Π, Π', and Π'' be derivations. For a mapping f from $SGW(\Pi)$ to $SGW(\Pi')$ and a mapping g from $SGW(\Pi')$ to $SGW(\Pi'')$, $g \circ f$ denotes the mapping from $SGW(\Pi)$ to $SGW(\Pi'')$ defined by $g \circ f(W) = g(f(W))$. Also, for a mapping F from $oa(\Pi)$ to the power set of $oa(\Pi')$ and a mapping G from $oa(\Pi')$ to the power set of $oa(\Pi'')$, $G \circ F$ denotes the mapping from $oa(\Pi)$ to the power set of $oa(\Pi'')$ defined by $G \circ F(\alpha) = \bigcup_{\theta \in F(\alpha)} G(\theta)$. We use these notations also in the case of partial mappings.

5.2.7 Main Lemma.

If $\Pi \xrightarrow{1} \Pi'$ and $\Pi \xrightarrow{1} \Pi''$ hold, then there exists a derivation Π''' such that $\Pi' \xrightarrow{1} \Pi'''$, $\Pi'' \xrightarrow{1} \Pi'''$, $C_{\Pi'}^{\Pi'''} \circ C_{\Pi}^{\Pi'} = C_{\Pi''}^{\Pi'''} \circ C_{\Pi}^{\Pi''}$, and $O_{\Pi'}^{\Pi'''} \circ O_{\Pi}^{\Pi'} = O_{\Pi''}^{\Pi'''} \circ O_{\Pi}^{\Pi''}$.

Main Lemma will be proved in the next section. Theorem 2, i.e. the Church-Rosser property of our reduction, can be easily proved using fact 5.2.4, fact 5.2.5, and Main Lemma. Here, we state theorem 2 again.

Theorem 2. (Church-Rosser property) *If two finite reduction sequences Π, ..., Σ and Π, ..., Σ' are given, then we can construct two finite reduction sequences Σ, ..., Δ and Σ', ..., Δ for some derivation Δ.*

6 Proof of Main Lemma

6.1 Lemmata

It now remains for us to establish the proof of Main Lemma. The essential parts of the proof are obtained from Lemma A (6.1.2) and Lemma B (6.1.3).

6.1.1 Notation ($W \prec V$)

Let W and V be sgw's in a derivation. We denote by $W \prec V$ the fact that $W \subset V$ and $rt(W) = rt(V)$ hold.

6.1.2 Lemma A

If $\Pi \xrightarrow{1} \Pi'$, $LC(\Pi \xrightarrow{1} \Pi')$ is (2), $\Pi \xrightarrow{ER} \Sigma$, and $\Pi' \xrightarrow{ER} \Sigma'$ hold; then, $\Sigma \xrightarrow{1} \Sigma'$, $C_{\Sigma}^{\Sigma'} \circ CE_{\Pi} = CE_{\Pi'} \circ C_{\Pi}^{\Pi'}$, and $O_{\Sigma}^{\Sigma'} \circ OE_{\Pi} = OE_{\Pi'} \circ O_{\Pi}^{\Pi'}$ hold.

6.1.3 Lemma B

Let S be a substitution-sequence $< \Pi, W, \Theta >$, and let V be a sgw in Π satisfying $W \prec V$. If $\Pi \xrightarrow{1} \Pi'$ and $\Theta \xrightarrow{1} \Theta'$ hold, and let S' be the substitution-sequence defined by $S' =< \Pi', V', \Theta' >$ where $V' = C_{\Pi}^{\Pi'}(V)$; then, the following facts (a),...,(e) hold.

(a) $\mathcal{P}_S \xrightarrow{1} \mathcal{P}_{S'}$

(b) For all $U \in SGW(\Pi)$ satisfying $U \cap V = \phi$, it holds that $C_{\mathcal{P}_S}^{\mathcal{P}_{S'}} \circ CS_S^1(U) = CS_{S'}^1 \circ C_{\Pi}^{\Pi'}(U)$.

(c) For all $\alpha \in oa(\Pi) \setminus on(V)$, it holds that $O_{\mathcal{P}_S}^{\mathcal{P}_{S'}} \circ OS_S^1(\alpha) = OS_{S'}^1 \circ O_{\Pi}^{\Pi'}(\alpha)$.

(d) $C_{\mathcal{P}_S}^{\mathcal{P}_{S'}} \circ CS_S^2 = CS_{S'}^2 \circ C_{\Theta}^{\Theta'}$

(e) For all $\alpha \in oa(\Theta) \setminus \{mj(li(\Theta))\}$, it holds that $O_{\mathcal{P}_S}^{\mathcal{P}_{S'}} \circ OS_S^2(\alpha) = OS_{S'}^2 \circ O_{\Theta}^{\Theta'}(\alpha)$.

6.1.4 Remark

Lemma A and Lemma B are proved using some facts stated in the following. We state these facts in an abbreviated form. Namely, the commutativity of mappings on sgw's and on open assumptions (e.g. (b), (c), (d), and (e) in Lemma B) is not represented in these statement. But all these facts stated in the following are hold with such commutativity.

6.2 Some facts

6.2.1 Fact

If $\Pi \xrightarrow{1} \Pi'$ holds and let a and t be a free variable and a term respectively satisfying $\Pi(t/a)$ becomes a derivation; then $\Pi'(t/a)$ is a derivation, and $\Pi(t/a) \xrightarrow{1} \Pi'(t/a)$ holds.

Proof. By induction on $|\Pi \xrightarrow{1} \Pi'|$.□

6.2.2 Fact

Let Σ and $\genfrac{}{}{0pt}{}{[A]}{\Pi}$ be derivations satisfying $END(\Sigma) = A$. Let P be the subset of $oa(\Pi)$ denoted by $[A]$ in $\genfrac{}{}{0pt}{}{[A]}{\Pi}$. Suppose that $\Sigma \stackrel{1}{\longrightarrow} \Sigma'$ and $\genfrac{}{}{0pt}{}{[A]}{\Pi} \stackrel{1}{\longrightarrow} \genfrac{}{}{0pt}{}{[A]}{\Pi'}$ hold where $[A]$ in $\genfrac{}{}{0pt}{}{[A]}{\Pi'}$ denotes the subset of $oa(\Pi')$, say P', defined by $P' = \bigcup_{\alpha \in P} O_{\Pi}^{\Pi'}(\alpha)$. Then, we have $\begin{matrix}\Sigma\\ [A]\\ \Pi\end{matrix} \stackrel{1}{\longrightarrow} \begin{matrix}\Sigma'\\ [A]\\ \Pi'\end{matrix}$.

Proof. By induction on $|\Pi \stackrel{1}{\longrightarrow} \Pi'|$. In the case that $LC(\Pi \stackrel{1}{\longrightarrow} \Pi')$ is (4), we use the following fact. That is, if S and X are substitution-sequence defined by $S =< \genfrac{}{}{0pt}{}{[B]}{\Gamma}, W, \genfrac{}{}{0pt}{}{[B]}{\Theta} >$ and $X =< \begin{matrix}\Delta\\ [B]\\ \Gamma\end{matrix}, W, \begin{matrix}\Delta\\ [B]\\ \Theta\end{matrix} >$, then it holds that $\mathcal{P}_X = \begin{matrix}\Sigma\\ [B]\\ \mathcal{P}_S\end{matrix}$ where we define $[B]$ in $\genfrac{}{}{0pt}{}{[B]}{\mathcal{P}_S}$ using OS_S^1 and OS_S^2.□

6.2.3 Fact

Let S be a substitution-sequence $< \Pi, W, \Theta >$, and let V be a sgw in Π satisfying $W \prec V$. If $\Theta \stackrel{1}{\longrightarrow} \Theta'$ holds, and let S' be a substitution-sequence defined by $S' =< \Pi, V, \Theta' >$; then, $\mathcal{P}_S \stackrel{1}{\longrightarrow} \mathcal{P}_{S'}$ holds.

Proof. By induction on the length of Π. We prove this fact in the case that $end(\Pi) \in top(W)$ and $end(\Pi) \notin top(V)$ hold, since other cases are straight-forward. Now we assume that. Suppose Π and Θ are of the form $\dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A} I$ and $\dfrac{A \quad (\Theta_1 \quad \Theta_2)}{B} K$ respectively. Then, $\mathcal{P}_S$ and $\mathcal{P}_{S'}$ are of the form $\dfrac{\dfrac{\mathcal{P}_{S_0} \quad (\mathcal{P}_{S_1} \quad \mathcal{P}_{S_2})}{A} \quad (\Theta_1 \quad \Theta_2)}{B}$ and $\dfrac{\mathcal{P}_{S'_0} \quad (\mathcal{P}_{S'_1} \quad \mathcal{P}_{S'_2})}{B}$ respectively, where $S_p =< \Pi_p, W\lceil_{\Pi_p}, \Theta >$ and $S'_p =< \Pi_p, V\lceil_{\Pi_p}, \Theta' >$ for each $p \in \{0,1,2\}$. Let V_0 and V_1 be sgw's in Π satisfying that $V = V_0 \cup V_1$, $V_0 \cap V_1 = \phi$, and $rt(V_0) = \{end(\Pi)\}$. Define substitution-sequences X and X_p for each $p \in \{0,1,2\}$ by $X =< \Pi, V_1, \Theta' >$ and $X_p =< \Pi_p, V_1\lceil_{\Pi_p}, \Theta' >$ for each $p \in \{0,1,2\}$. Denote $sbd(mj(li(\mathcal{P}_S)))$ by Δ_0. From the condition $end(\Pi) \in top(W)$ and the definition of V_0 and V_1, we have $W\lceil_{\Pi_p} \prec V_1\lceil_{\Pi_p}$ for each $p \in \{0,1,2\}$. Hence, by induction hypothesis, we have $\mathcal{P}_{S_p} \stackrel{1}{\longrightarrow} \mathcal{P}_{X_p}$ for each $p \in \{0,1,2\}$. Therefore, we have $\Delta_0 \stackrel{1}{\longrightarrow} \mathcal{P}_X$ using the clause (2) for $LC(\Delta_0 \stackrel{1}{\longrightarrow} \mathcal{P}_X)$, since $\mathcal{P}_X$ is of the form $\dfrac{\mathcal{P}_{X_0} \quad (\mathcal{P}_{X_1} \quad \mathcal{P}_{X_2})}{A}$. Let T be the sgt at $end(\Delta_0)$ in Δ_0 defined by

$$T = \{< end(\Delta_0), k > | < end(\Pi), k > \in V_0\} \cup \bigcup_{0 \le p \le 2} CS_{S_p}^1(V_0\lceil_{\Pi_p}),$$

and let T' be the sgt at $end(\mathcal{P}_X)$ in $\mathcal{P}_X$ defined by $T' = C_{\Delta_0}^{\mathcal{P}_X}(T)$. Define a substitution-sequence Y by $Y =< \mathcal{P}_X, T', \Theta' >$. By induction hypothesis (about commutativity of mappings) for Π_p, we have $T' = CS_X^1(V_0)$. Hence, by fact 4.4.1, $\mathcal{P}_{S'} = \mathcal{P}_Y$ holds. On the other hand, we have $\mathcal{P}_S \stackrel{1}{\longrightarrow} \mathcal{P}_Y$ because $\dfrac{\mathcal{P}_X \quad (\Theta'_1 \quad \Theta'_2)}{B} \stackrel{SR(T')}{\longrightarrow} \mathcal{P}_X$ holds where we suppose $\Theta' = \dfrac{A \quad (\Theta'_1 \quad \Theta'_2)}{B}$. Therefore, $\mathcal{P}_S \stackrel{1}{\longrightarrow} \mathcal{P}_{S'}$ holds.□

6.2.4 Fact

Let Π and Σ be derivations satisfying $\Pi \stackrel{ER}{\longrightarrow} \Sigma$. Let S be a substitution-sequence $< \Pi, W, \Theta >$, and X the substitution-sequence defined by $X =< \Sigma, CE_{\Pi}(W), \Theta >$. Then, $\mathcal{P}_S \stackrel{ER}{\longrightarrow} \mathcal{P}_X$ holds.

Proof. By definition of CE_{Π}.□

6.2.5 Fact

Let S, X, and Y be substitution-sequences $< \Pi, W, \Theta >$, $< \Pi, V_1, \Delta >$, and $< \Theta, V_2, \Delta >$ respectively; satisfying $W \cap V_1 = \phi$. Let $\widetilde{S}$ and $\widetilde{X}$ be the substitution-sequences defined by $\widetilde{S} =< \mathcal{P}_X, CS^1_X(W), \mathcal{P}_Y >$ and $\widetilde{X} =< \mathcal{P}_S, CS^1_S(V_1) \cup CS^2_S(V_2), \Delta >$. Then, $\mathcal{P}_{\widetilde{S}} = \mathcal{P}_{\widetilde{X}}$ holds.

Proof. By induction on the length of Π.□

6.3 Proof of lemmata

Now we prove Lemma A, Lemma B, and Main Lemma.

6.3.1 Proof of Lemma A

Since $\Pi \stackrel{ER}{\longrightarrow} \Sigma$, Π is of the form $\dfrac{\dfrac{\Pi_0 \quad (\Pi_1)}{M}\,I \qquad (\Pi_2 \quad \Pi_3)}{A}\,K$ where $Inf(I)$ is an introduction rule and $Inf(K)$ is an elimination rule. Then, Π' is of the form $\dfrac{\dfrac{\Pi'_0 \quad (\Pi'_1)}{M} \qquad (\Pi'_2 \quad \Pi'_3)}{A}$ where $\Pi_p \stackrel{1}{\longrightarrow} \Pi'_p$ for each $p \in \{0, \ldots, 3\}$, because $LC(\Pi \stackrel{1}{\longrightarrow} \Pi')$ is (2) and $Inf(I)$ is an introduction rule. Then, using fact 6.2.1 and fact 6.2.2, we have the result.□

6.3.2 Proof of Lemma B

By induction on $|\Pi \stackrel{1}{\longrightarrow} \Pi'|$.

Case 1. $LC(\Pi \stackrel{1}{\longrightarrow} \Pi')$ is (1): Use fact 6.2.3.

Case 2. $LC(\Pi \stackrel{1}{\longrightarrow} \Pi')$ is (2): Similarly with the proof of fact 6.2.3.

Case 3. $LC(\Pi \stackrel{1}{\longrightarrow} \Pi')$ is (3): Use fact 6.2.4.

Case 4. $LC(\Pi \stackrel{1}{\longrightarrow} \Pi')$ is (4): Use fact 6.2.5.□

6.3.3 Proof of Main Lemma.

By induction on $|\Pi \stackrel{1}{\longrightarrow} \Pi'| + |\Pi \stackrel{1}{\longrightarrow} \Pi''|$.

Case 1. $LC(\Pi \stackrel{1}{\longrightarrow} \Pi')$ is (1): Take Π'' for Π'''.

Case 1'. $LC(\Pi \stackrel{1}{\longrightarrow} \Pi'')$ is (1): Similarly to the case 1.

Case 2. $LC(\Pi \stackrel{1}{\longrightarrow} \Pi')$ and $LC(\Pi \stackrel{1}{\longrightarrow} \Pi'')$ are (2): Suppose Π, Π', and Π'' are of the form $\dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A}$, $\dfrac{\Pi'_0 \quad (\Pi'_1 \quad \Pi'_2)}{A}$, and $\dfrac{\Pi''_0 \quad (\Pi''_1 \quad \Pi''_2)}{A}$ respectively, where for each $p \in \{0,1,2\}$, $\Pi_p \stackrel{1}{\longrightarrow} \Pi'_p$ and $\Pi_p \stackrel{1}{\longrightarrow} \Pi''_p$ hold. Then by induction hypothesis, for each $p \in \{0,1,2\}$ there exists a derivation Π'''_p such that $\Pi'_p \stackrel{1}{\longrightarrow} \Pi'''_p$, $\Pi''_p \stackrel{1}{\longrightarrow} \Pi'''_p$, $C^{\Pi'''_p}_{\Pi'_p} \circ C^{\Pi'_p}_{\Pi_p} = C^{\Pi'''_p}_{\Pi''_p} \circ C^{\Pi''_p}_{\Pi_p}$, and $O^{\Pi'''_p}_{\Pi'_p} \circ O^{\Pi'_p}_{\Pi_p} = O^{\Pi'''_p}_{\Pi''_p} \circ O^{\Pi''_p}_{\Pi_p}$ hold. Let Π''' be the derivation of the form $\dfrac{\Pi'''_0 \quad (\Pi'''_1 \quad \Pi'''_2)}{A}$. Then, the result holds for this Π'''.

Case 3. $LC(\Pi \xrightarrow{1} \Pi')$ and $LC(\Pi \xrightarrow{1} \Pi'')$ are (3): Suppose Π is of the form $\dfrac{\dfrac{\Pi_0 \quad (\Pi_1)}{M} \qquad (\Pi_2 \quad \Pi_3)}{A}$, and suppose Π' and Π'' satisfy that

$$\dfrac{\dfrac{\Pi'_0 \quad (\Pi'_1)}{M} \qquad (\Pi'_2 \quad \Pi'_3)}{A} \xrightarrow{ER} \Pi' \text{ and } \dfrac{\dfrac{\Pi''_0 \quad (\Pi''_1)}{M} \qquad (\Pi''_2 \quad \Pi''_3)}{A} \xrightarrow{ER} \Pi'',$$

where for each $p \in \{0, \dots, 3\}$, $\Pi_p \xrightarrow{1} \Pi'_p$ and $\Pi_p \xrightarrow{1} \Pi''_p$ hold. Then by induction hypothesis, for each $p \in \{0, \dots, 3\}$ there exists a derivation Π'''_p such that $\Pi'_p \xrightarrow{1} \Pi'''_p$, $\Pi''_p \xrightarrow{1} \Pi'''_p$, $C^{\Pi'''_p}_{\Pi'_p} \circ C^{\Pi'_p}_{\Pi_p} = C^{\Pi'''_p}_{\Pi''_p} \circ C^{\Pi''_p}_{\Pi_p}$, and $O^{\Pi'''_p}_{\Pi'_p} \circ O^{\Pi'_p}_{\Pi_p} = O^{\Pi'''_p}_{\Pi''_p} \circ O^{\Pi''_p}_{\Pi_p}$ hold. Let Π''' be the derivation satisfying

$$\dfrac{\dfrac{\Pi'''_0 \quad (\Pi'''_1)}{M} \qquad (\Pi'''_2 \quad \Pi'''_3)}{A} \quad \xrightarrow{ER} \quad \Pi'''$$

Then, by Lemma A (6.1.2), the result holds for this Π'''.

Case 4. One of the $LC(\Pi \xrightarrow{1} \Pi')$ and $LC(\Pi \xrightarrow{1} \Pi'')$ is (2) and the other is (3): Similarly to the case 3.

Case 5. $LC(\Pi \xrightarrow{1} \Pi')$ and $LC(\Pi \xrightarrow{1} \Pi'')$ are (4): Suppose Π is of the form $\dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A}$ and suppose Π' and Π'' satisfy that

$$\dfrac{\Pi'_0 \quad (\Pi'_1 \quad \Pi'_2)}{A} \xrightarrow{SR(C^{\Pi'_0}_{\Pi_0}(T_1))} \Pi'$$

and

$$\dfrac{\Pi''_0 \quad (\Pi''_1 \quad \Pi''_2)}{A} \xrightarrow{SR(C^{\Pi''_0}_{\Pi_0}(T_2))} \Pi''$$

; where for each $p \in \{0, 1, 2\}$, $\Pi_p \xrightarrow{1} \Pi'_p$ and $\Pi_p \xrightarrow{1} \Pi''_p$ hold, and where T_1 and T_2 are sgt's at $end(\Pi_0)$ in Π_0 satisfying $len(T_1) > 1$, $len(C^{\Pi'_0}_{\Pi_0}(T_1)) > 1$, $len(T_2) > 1$, and $len(C^{\Pi''_0}_{\Pi_0}(T_2)) > 1$. Then, by induction hypothesis, for all $p \in \{0, 1, 2\}$ there exists a derivation Π'''_p such that $\Pi'_p \xrightarrow{1} \Pi'''_p$, $\Pi''_p \xrightarrow{1} \Pi'''_p$, $C^{\Pi'''_p}_{\Pi'_p} \circ C^{\Pi'_p}_{\Pi_p} = C^{\Pi'''_p}_{\Pi''_p} \circ C^{\Pi''_p}_{\Pi_p}$, and $O^{\Pi'''_p}_{\Pi'_p} \circ O^{\Pi'_p}_{\Pi_p} = O^{\Pi'''_p}_{\Pi''_p} \circ O^{\Pi''_p}_{\Pi_p}$ hold. Let T be the sgt at $end(\Pi_0)$ in Π_0 defined by $T = T_1 \cup T_2$, and let T''' be the sgt at $end(\Pi'''_0)$ in Π'''_0 defined by

$$T''' = C^{\Pi'''_0}_{\Pi'_0} \circ C^{\Pi'_0}_{\Pi_0}(T) = C^{\Pi'''_0}_{\Pi''_0} \circ C^{\Pi''_0}_{\Pi_0}(T)$$

Let Θ''' be the derivation of the form $\dfrac{END(\Pi'''_0) \quad (\Pi'''_1 \quad \Pi'''_2)}{A}$, and S the substitution-sequence defined by $S =< \Pi'''_0, T''', \Theta''' >$. Let Π''' be the derivation $\mathcal{P}_S$. Then by Lemma B (6.1.3), the result holds for this Π'''.

Case 6. One of the $LC(\Pi \xrightarrow{1} \Pi')$ and $LC(\Pi \xrightarrow{1} \Pi'')$ is (2) and the other is (4): Similarly to the case 5.□

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