THE FREEBOUNDARY IN A MINIMIZATION PROBLEM

KWON CHO AND HI JUN CHOE

1. INTRODUCTION

In this paper we study a minimization problem

$$\min I^\lambda(u) = \min \int_\Omega \frac{|\nabla u|^p}{p} + \frac{\lambda}{\gamma+1} u^{\gamma+1} \, dx, \quad p \geq 2, \quad \gamma \in [0, p-1)$$

with respect to $K = W_0^{1,p} + u_0$, where $\lambda$ is a positive constant. Here we consider the case boundary data $u_0$ is constant, say, $u_0 = 1$. The motivation of this problem comes from reaction diffusion models. We refer various references in [6] and [8] for practical motivations.

From variational principle we note that the minimizer satisfies the Euler-Lagrange equation

$$\text{div} \left( |\nabla u|^{p-2} \nabla u \right) = \lambda u^\gamma \quad \text{in } \Omega.$$ 

In fact the existence and uniqueness follows from convexity of the functional $I^\lambda$ on $W_0^{1,p} + u_0$. An interesting fact is that if $\gamma < p - 1$, then there appears deadcore $N_\lambda(u) = \{ x \in \Omega : u(x) = 0 \}$. Here we call $F(u) = \partial\{ u > 0 \}$ the free boundary.

We shall study the nature of free boundary and deadcore. Our main result is that if $\partial\Omega$ has positive mean curvature, then the smooth portion of free boundary has also positive mean curvature. Hence in two dimensional case if $\Omega$ is convex, then the deadcore is also convex. Friedman and Phillips[8] considered the case when $p = 2$. Moreover the convexity of the graph of the solutions to various minimization problems were considered by many authors([4], [10]).

We also study the asymptotic behaviour of free boundary with respect to $\lambda$. Indeed for two dimensional case van Duijn and Peletier[7] studied the behaviour of free boundary for discontinuous boundary data.
We assume $\partial\Omega$ is smooth and use the following symbol, $B_R(x_0) = \{x : |x - x_0| < R\}$.

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2. **Asymptotic behavior of deadcore as $\lambda \to \infty$**

In this section we study the asymptotic behavior of $u_\lambda$ as $\lambda$ goes to $\infty$. First we prove that $u_\lambda$ decreases at each point as $\lambda \to \infty$. This follows from standard comparison method.

**Lemma 2.1.** Let $0 < \lambda_2 < \lambda_1$, then $u_{\lambda_2} < u_{\lambda_1}$ on $\{x \in \Omega : u_{\lambda_2}(x) > 0\}$.

**Proof.** We regularize $I^\lambda$ by
\[
\int_{\Omega} \frac{1}{p} \left( \epsilon + |\nabla u|^2 \right)^\frac{p}{2} + \frac{\lambda}{\gamma + 1} u^{\gamma+1} \, dx, \quad u = 1 \text{ on } \partial\Omega
\]
and let $u_\lambda^\epsilon$ be the minimizer. Then $u_\lambda^\epsilon \in C^{2,\alpha}(\Omega)$ for all $0 < \alpha < 1$. If $w(x) = u_{\lambda_1}^\epsilon(x) - u_{\lambda_2}^\epsilon(x)$ attains a positive maximum at $x_0 \in \Omega$, then
\[
0 \geq \text{div} \left( \left( \epsilon + |\nabla u_{\lambda_2}^\epsilon|^2 \right)^\frac{p-2}{2} \nabla u_{\lambda_1}^\epsilon - \left( \epsilon + |\nabla u_{\lambda_2}^\epsilon|^2 \right)^\frac{p-2}{2} \nabla u_{\lambda_2}^\epsilon \right)
= \lambda_1 \left( u_{\lambda_1}^\epsilon(x_0) \right)^\gamma - \lambda_2 \left( u_{\lambda_2}^\epsilon(x_0) \right)^\gamma
\geq (\lambda_1 - \lambda_2) u_{\lambda_2}^\epsilon(x_0) > 0.
\]
Note that $\nabla u_{\lambda_1}^\epsilon(x_0) = \nabla u_{\lambda_2}^\epsilon(x_0)$. Hence we get $a_{ij} w_{ij} > 0$ for
\[
a_{ij} = \left( \epsilon + |\nabla u_{\lambda_2}^\epsilon|^2 \right)^\frac{p-2}{2} \left( \delta_{ij} + \frac{u_{\lambda_2}^\epsilon,x_i u_{\lambda_2}^\epsilon,x_j}{\epsilon + |\nabla u_{\lambda_2}^\epsilon|^2} \right)
\]
and this contradicts to the assumption $w$ attains maximum at $x_0$. \qed

Consequently we have
\[
N_{\lambda_1} \subseteq \text{int } N_{\lambda_2} \quad \text{if } 0 < \lambda_1 < \lambda_2.
\]

The following theorem is our main result in this section and the case when $p = 2$ was considered by Friedman and Phillips[8].

We define $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$. 


Theorem 2.2. There exist positive constants $a$, $c$ and $\lambda_0$ depending only on $n$, $p$ and $\gamma$ such that

$$\Omega_{a/\psi(X+c/(\psi)^2} \subset N_\lambda \subset \Omega_{a/\psi(X-c/(\psi)^2}$$

for all $\lambda > \lambda_0$.

Proof. We let $w_\lambda(x) = u_\lambda \left( \frac{x}{\sqrt{\lambda}} \right)$, then

$$\text{div}(|\nabla w_\lambda|^{p-2}\nabla w_\lambda) = w^\gamma.$$ 

Hence from elliptic estimate

$$|\nabla w| \leq C$$

since $|w| = |u| \leq 1$. Hence we get $|\nabla u| \leq c\sqrt{\lambda}$ and

$$N_\lambda \subset \Omega_{c/\psi(X}.$$ 

On the other hand if we set $v(x) = A|x - x_0|^\frac{p}{p-\gamma}$, then

$$\text{div}(|\nabla v|^{p-2}\nabla v) = C_0^p A^{p-1-\gamma} v^\gamma,$$

where $C_0 = \frac{(p^{-1}(\gamma + 1)(p-1))^{\frac{1}{2}}}{p - 1 - \gamma}$. We take $A$ satisfying $Ad^{\frac{p}{p-1-\gamma}} = 1$, where $d = \text{dist}(x_0, \partial \Omega)$, then $v \geq 1$ on $\partial \Omega$. If $C_0^p A^{p-1-\gamma} = \lambda$, that is, $d = \frac{C_0}{\sqrt{\lambda}}$, then $v \geq u$ and $v(x_0) = u(x_0) = 0$. This implies

(1) $$\Omega_{c/\psi(X} \subset N_\lambda \subset \Omega_{c/\psi(X}.$$ 

Now we refine the previous estimates. Let $y \in \partial \Omega$ and $B_R \subset \Omega$ such that $y \in \partial B_R$. Let $U$ be the radial minimizer if $I^\lambda$, then $u^\lambda \leq U$ and $U'(r) \geq 0$. $U$ satisfies

$$(p-1)|U'|^{p-2}U'' + \frac{n-1}{r}|U'|^{p-2}U' = \lambda U^\gamma$$

and

$$Z(s) = U \left( R - \frac{\gamma_0}{\sqrt{\lambda}} + \frac{s}{\sqrt{\lambda}} \right) \quad (\gamma_0 \text{ is to be determined})$$

satisfies

(2) $$\frac{p-1}{\rho \sqrt{\lambda} + s} |Z'|^{p-2}Z' = Z^\gamma,$$
where \( \rho = R - \frac{\gamma_0}{\sqrt{\lambda}} \).

From (1) \( \gamma_0 \leq C \) independent \( \lambda \). Multiplying both side of (2) by \( Z'(s) \), we get

\[
\frac{p-1}{p} (|Z'|^p)' + \frac{n-1}{\rho \sqrt[4]{\lambda} + s} |Z'|^p = Z^\gamma Z'.
\]

Hence we obtain

\[
(|Z'|^p)' + \frac{(n-1)p}{p-1} \frac{1}{\rho \sqrt[4]{\lambda} + s} |Z'|^p = \frac{p}{(p-1)(\gamma + 1)} (Z^{\gamma+1})'
\]

and

\[
(|Z'|^p)' + \frac{C}{\sqrt[4]{\lambda}} |Z'|^p \geq \frac{p}{(p-1)(\gamma + 1)} (Z^{\gamma+1})'
\]

for some \( C \). From this we obtain

\[
\left( e^{C_s/\sqrt[4]{\lambda}} |Z'|^p \right)' \geq \frac{p}{(p-1)\gamma + 1} e^{C_s/\sqrt[4]{\lambda}} (Z^{\gamma+1})'
\]

and

\[
|Z'|^p(s) \geq e^{-C_s/\sqrt[4]{\lambda}} \frac{p}{(p-1)(\gamma + 1)} \int_0^s e^{C_t/\sqrt[4]{\lambda}} (Z^{\gamma+1})' dt
\]

\[
= \frac{p}{(p-1)(\gamma + 1)} Z^{\gamma+1}(s) - \frac{p}{(p-1)(\gamma + 1)} \frac{C}{\sqrt[4]{\lambda}} \int_0^s e^{-C(s-t)/\sqrt[4]{\lambda}} Z^{\gamma+1} dt.
\]

Recalling that \( Z'(t) \geq 0 \) we get

\[
|Z'|^p(s) \geq \frac{p}{(p-1)(\gamma + 1)} \left( 1 - \frac{C}{\sqrt[4]{\lambda}} \right) Z^{\gamma+1}(s).
\]

On the other hand

\[
\begin{cases}
\eta'(s) = \left( \frac{p}{(p-1)\gamma + 1} \eta^{\gamma+1}(s) \right)^{\frac{1}{2}} \\
\eta(0) = \left( \frac{p}{(p-1)\gamma + 1} \eta^{\gamma+1}(s) \right)^{\frac{1}{2}}
\end{cases}
\]

has a unique solution as long as \( \eta > 0 \). It determines a unique number \( a > 0 \) such that

\[
\eta(-a) = 0.
\]
Letting $\zeta(s) = \eta(-a + s)$ we have

$$
\begin{aligned}
\zeta'(s) &= \left( \frac{p}{(p-1)(\gamma+1)} \zeta^{\gamma+1}(s) \right)^{\frac{1}{p}} \quad \text{for } 0 < s < a \\
\zeta(s) &= 0 \quad \text{for } 0 < s < a \\
\zeta(0) &= 0 \\
\zeta(a) &= 1.
\end{aligned}
$$

The function

$$
\tilde{\zeta}(s) = \zeta \left( s \left( 1 - \frac{C}{\sqrt[\lambda]{p}} \right) \right)
$$

satisfies

$$
(\tilde{\zeta}(s))' = \left( 1 - \frac{C}{\sqrt[\lambda]{p}} \right)^{\frac{1}{p}} \left( \frac{p}{(p-1)(\gamma+1)} \tilde{\zeta}^{\gamma+1} \right)^{\frac{1}{p}}.
$$

By comparison we also have

$$
Z(s) \geq \tilde{\zeta}(s) = \zeta \left( s \left( 1 - \frac{C}{\sqrt[\lambda]{p}} \right) \right).
$$

Since $U(R) = 1$ implies $Z(\gamma_0) = 1$, we conclude that

$$
\gamma_0 \left( 1 - \frac{C}{\sqrt[\lambda]{p}} \right)^{\frac{1}{p}} \leq a.
$$

Recalling that

$$u_\lambda \leq U,$$

we deduce that

$$u_\lambda(|x - x_0|) \leq Z(|x - x_0|) \leq U \left( R - \gamma_0 \frac{\sqrt[\lambda]{p}}{\sqrt[\lambda]{\lambda}} + \frac{|x - x_0|}{\sqrt[\lambda]{\lambda}} \right)$$

and $U(x_0) = 0$ implies

$$N_\lambda \supset \Omega_{R-\gamma_0/\sqrt[\lambda]{\lambda}} \supset B_{R-a/\sqrt[\lambda]{\lambda}}.$$

This completes the first part of the theorem.

To prove the second part we let $v$ be the radial solution of

$$
\begin{aligned}
\text{div}(|\nabla v|^{p-2} \nabla v) &= \lambda v^\gamma \quad \text{in } B_{R_1} \setminus B_{R_0} \\
v &= 1 \quad \text{on } \partial B_{R_0} \\
v &= 0 \quad \text{on } \partial B_{R_1},
\end{aligned}
$$

respectively.
where $B_{R_{1}} \supset \Omega$ and $B_{R_{0}} \cap \Omega = \{y\}$ for some $y$. Then from comparison $v \leq u_{\lambda}$ and $v'(r) \leq 0$. Then considering $\Omega(s) = V\left(R + \frac{\gamma}{\sqrt{\lambda}} - \frac{s}{\sqrt{\lambda}}\right)$ as in the proof of the first part, we prove the second part. \[\square\]

3. Convexity of deadcore

The following maximum principle for polynomial growth case is relatively well known (see Chapter 7 in [13]).

**Lemma 3.1.** Let $\Omega$ be a bounded regular ($\partial \Omega \in C^{2}$) open set. Then if $\partial \Omega$ has nonnegative mean curvature, then for every $x \in \bar{\Omega}$

$$|\nabla u(x)|^{p} \leq \frac{p}{p-1} \frac{\lambda}{\gamma + 1} \left(u^{\gamma+1}(x) - m^{\gamma+1}\right),$$

where $m = \min_{x \in \Omega} u(x)$.

**Corollary 3.2.** Let $\Omega$ be convex domain in $\mathbb{R}^{n}$ and let $x_{m}$ be the point at which the minimum $u(x_{m}) = m \geq 0$ occurs. Then

$$\text{dist}(x_{m}, \partial \Omega) \geq \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{m}^{1} \left(\frac{\lambda}{\gamma + 1} \left(s^{\gamma+1} - m^{\gamma+1}\right)\right)^{-\frac{1}{p}} ds$$

In particular the null set $N$ is empty if

$$\rho < \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{0}^{1} \left(\frac{\lambda}{\gamma + 1}\right)^{-\frac{1}{p}} s^{-\frac{\gamma+1}{p}}.$$  

**Proof.** Let $x_{1} \in \partial \Omega$ and let $r$ be the arc length on straight segment joining $x_{m}$ to $x_{1}$. Let $x_{2}$ be a point in this segment such that $u(x_{2}) = m$ and $u(x) > m$ for all $x$ between $x_{2}$ and $x_{1}$.

Then

$$\frac{du}{dr} \leq |\nabla u| \leq \left(\frac{p}{p-1}\right)^{\frac{1}{p}} \left(\int_{m}^{u} f(t) dt\right)^{\frac{1}{p}}.$$

So

$$\frac{dr}{du} \geq \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \frac{1}{\left(\int_{m}^{u} f(t) dt\right)^{\frac{1}{p}}}.$$
and integrating from $x_2$ to $x_1$,

$$\text{dist}(x_m, x_1) \geq \text{dist}(x_2, x_1) \geq \left(\frac{p - 1}{p} \frac{\gamma + 1}{\lambda}\right)^\frac{1}{p} \int_m (s^{\gamma + 1} - m^{\gamma + 1})\frac{ds}{(s^{\gamma + 1} + m^{\gamma + 1})^{\frac{1}{p}}}.$$

\[\square\]

We let

$$\psi(u) = \left(\frac{p - 1}{p} \frac{\gamma + 1}{\lambda}\right)\frac{p}{p - \gamma + 1} u^{\frac{p - \gamma}{p}} - \lambda,$$

then from the Hausdorff measure estimate of free boundary [5] we have

$$\text{div}(|\nabla \psi|^{p-2} \nabla \psi) = d\Lambda + I_{\{u > 0\}} C \psi^{-1}(1 - |\nabla \psi|^p),$$

where $d\Lambda = d\mathcal{H}^{n-1} F_{\text{reg}}(u) + \theta(x) d\mathcal{H}^{n-1} F_{\text{sing}}(u)$ and $C$ depends only on $n$, $p$, $\gamma$, $\theta$ bounded. Here $I$ is the usual characteristic function. Moreover

$$\psi^{-1}(1 - |\nabla \psi|^p) \in L^1_{\text{loc}}.$$

From Green's formula we note that if $D$ is a subdomain of $\Omega$ with piecewise smooth boundary $\partial D$ and with $\mathcal{H}^{n-1}(F(u) \cap \partial D) = 0$, then

$$\int_D \text{div}(|\nabla \psi|^{p-2} \nabla \psi) \, dx = \int_{\partial D \cap \{u > 0\}} |\nabla \psi|^{p-2} \nabla \psi \cdot \nu \, d\mathcal{H}^{n-1}.$$

Hence from the above observation if $D$ has piecewise smooth boundary and $\mathcal{H}^{n-1}(F(u) \cap \partial D) = 0$, then

$$\int_{D \cap F_{\text{reg}}} d\mathcal{H}^{n-1} + \int_{D \cap F_{\text{sing}}} \theta d\mathcal{H}^{n-1} = -\int_{D \cap \{u > 0\}} \text{div}(|\nabla \psi|^{p-2} \nabla \psi) \, dx$$

$$+ \int_{\partial D \cap \{u > 0\}} |\nabla \psi|^{p-2} \nabla \psi \cdot \nu \, d\mathcal{H}^{n-1}.$$

Therefore with the argument by Friedman and Phillips (see Theorem 4.3 in [8]) we prove the following Corollary.

**Corollary 3.3.** Every $C^2$ portion of $F(u)$ has nonnegative mean curvature.
REFERENCES

5. H.J. Choe, Hausdorff measure of free boundary in a minimization problem, preprint

DEPARTMENT OF MATHEMATICS, POHANG UNIVERSITY OF SCIENCE AND TECHNOLOGY, POHANG, KYUNGBUK, REPUBLIC OF KOREA, 790-600