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Convex entropy function and symmetrization of the relativistic Euler equation

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1 Introduction

The motion of a relativistic perfect fluid in the Minkowski space-time is governed by

\[
\begin{align*}
\frac{\partial}{\partial t} & \left( \frac{\rho c^2 + p}{c^2 - v^2} - p \right) + \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left( \frac{\rho c^2 + p}{c^2 - v^2} v_k \right) = 0, \\
\frac{\partial}{\partial t} & \left( \frac{\rho c^2 + p}{c^2 - v^2} v_i \right) + \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left( \frac{\rho c^2 + p}{c^2 - v^2} v_i v_k + p \delta_{ik} \right) = 0, \quad i = 1, 2, 3.
\end{align*}
\]

Here $c$ denotes the speed of light, $p$ the pressure, $(v_1, v_2, v_3)$ the velocity of the fluid particle, $\rho$ the mass-energy density of the fluid (as measured in units of mass in a reference frame moving with the fluid particle) and $v^2 = v_1^2 + v_2^2 + v_3^2$. The fluid is assumed to be barotropic, which means that the equation (1.1) is to be supplemented with the equation of state

\[
p(\rho) = p(\rho),
\]

where $p(\rho)$ is a given function of $\rho$ only.

For the case of one space dimension, Smoller and Temple [7] constructed global weak solutions to (1.1) for the isentropic case $p(\rho) = a^2 \rho$ with $0 < a < c$, and Chen [1] for the case $p(\rho) = a^2 \rho^\gamma$ with $a > 0$ and $\gamma > 1$.

In our previous paper [6], the existence of local smooth solutions was proved for three space dimensions, with $p(\rho) = a^2 \rho$, $0 < a < c$. Our objective here is to extend this results to the general equation of state (1.2), under the sole assumption that

\[
\begin{align*}
p(\rho) & \in C^\infty(\rho_*, \rho^*), \\
p(\rho) > 0, \quad 0 < p'(\rho) < c^2 \quad & \text{for} \quad \rho \in (\rho_*, \rho^*),
\end{align*}
\]
where $\rho_\ast$ and $\rho^\ast$ are some constants such that $0 \leq \rho_\ast < \rho^\ast \leq \infty$. Note that if $p(\rho) = c^2 \rho^\gamma$, then $\rho_\ast = 0$ while $\rho^\ast = \infty$ if $\gamma = 1$ and $\rho^\ast = \{c^2/((\gamma c)^2)\}^{1/(\gamma - 1)}$ if $\gamma > 1$.

We consider the initial value problem to (1.1) with the initial condition

\begin{equation}
(1.4) \begin{cases}
\rho|_{t=0} = \rho_0(x), \\
v_i|_{t=0} = v_{0i}(x), \quad i = 1, 2, 3.
\end{cases}
\end{equation}

The main result of this paper is,

**Theorem 1.1.** Assume (1.3) for $p(\rho)$. Suppose that the initial data $\rho_0$ and $(v_{01}, v_{02}, v_{03})$ belong to the locally uniform Sobolev space $H^s_{al} = H^s_{al}(\mathbb{R}^3), s \geq 3, (\mathbb{3})$ and that there exist a positive constant $\delta$ sufficiently small so that

$$\rho_\ast + \delta \leq \rho(x) \leq \rho^\ast - \delta,$$

$$v_0^2(x) = v_{01}^2(x) + v_{02}^2(x) + v_{03}^2(x) \leq (1 - \delta)c^2,$$

hold for all $x \in \mathbb{R}^3$. Then, the Cauchy problem (1.1), (1.2) and (1.4) has a unique solution satisfying

\begin{equation}
(1.5) \quad (\rho, v_1, v_2, v_3) \in L^\infty(0, T; H^s_{al}) \cap C([0, T]; H^s_{loc}) \cap C([0, T]; H^{s-1}_{loc}),
\end{equation}

with $\rho_\ast < \rho(x, t) < \rho^\ast$ and $v^2(x, t) < c^2$, and moreover,

\begin{equation}
(1.6) \quad (\rho, v_1, v_2, v_3) \in C([0, T]; H^{s+\epsilon}_{al}) \cap C([0, T]; H^{s-1+\epsilon}_{al}),
\end{equation}

for any $\epsilon > 0$. Here $T > 0$ depends only on $\delta$ and the $H^s_{al}$-norm of the initial data.

As in [6], we shall prove the theorem by symmetrizing (1.1) and applying the Friedlichs-Lax-Kato theory [3], [5] of symmetric hyperbolic systems. According to Godunov [2], a suitable symmetrizer can be constructed if a strictly convex entropy function exists. In §3, it is shown that such an entropy function exists for (1.1), and in §2, the symmetrizer it induces is discussed. Finally in §4, the non-relativistic limit of the solutions to (1.1) as $c \rightarrow \infty$ is shown to be a solution of the non-relativistic Euler equation with the same equation of state (1.2).
2 Symmetrization

Theorem 1.1 can be proved if there is a change of variables

\begin{equation}
(2.1) \quad z = (\rho, v_1, v_2, v_3)^T \rightarrow u = (u_0, u_1, u_2, u_3)^T,
\end{equation}

which reduces the system (1.1) to a system of the form

\begin{equation}
(2.2) \quad A^0(u) \frac{\partial u}{\partial t} + \sum_{\alpha=1}^{3} A^\alpha(u) \frac{\partial u}{\partial x_\alpha} = 0,
\end{equation}

whose coefficient matrices $A^\alpha(u), \alpha = 0, 1, 2, 3,$ satisfy the condition

\begin{equation}
(2.3) \quad (i) \quad \text{they are all real symmetric and smooth in } u, \text{ and}
\end{equation}

\begin{equation}
(ii) \quad A^0(u) \text{ is positive definite.}
\end{equation}

The system (2.2) satisfying (2.3) is called a symmetric hyperbolic system, see [3], [5]. We claim that for (1.1), one of such changes of variables is given by

\begin{equation}
(2.4) \quad \begin{cases}
 u_0 &= -\frac{c^3 K e^{\phi(\rho)}}{(\rho c^2 + p)(c^2 - v^2)^{1/2}} + c^2, \\
 u_j &= \frac{c K e^{\phi(\rho)}}{(\rho c^2 + p)(c^2 - v^2)^{1/2}} v_j, \quad j = 1, 2, 3,
\end{cases}
\end{equation}

where

\begin{equation}
(2.5) \quad \phi(\rho) = \int_\rho^\rho \frac{c^2}{\rho c^2 + p(\rho)} d\rho, \quad K = c^2 \bar{\rho} + p(\bar{\rho}),
\end{equation}

$\bar{\rho}$ being an arbitrarily fixed number in $(\rho, \rho^*).$ The derivation of (2.4), based on the idea of Godunov [2], will be presented in §3. Here we shall check the condition (2.3). To this end, we shall find the matrices $A^\alpha, \alpha = 0, 1, 2, 3.$ explicitly. First, note from (2.4) that

\begin{equation}
(2.6) \quad v^2 = \frac{c^4}{(c^2 - u_0^2)} u^2, \quad u^2 = u_1^2 + u_2^2 + u_3^2.
\end{equation}

Substituting this into the first equation of (2.4) and putting

$$
\Phi(\rho) = \frac{K e^{\phi(\rho)}}{\rho c^2 + p(\rho)},
$$
we get

\begin{equation}
\Phi(p) = \frac{1}{r^2}((r^2 - \iota \ell)^2 - \iota^2 < \Phi(\rho^* + 0)^2.
\end{equation}

Since \( \Phi'(\rho) = -K p'(\rho) e^{\phi(\rho)}/(\rho c^2 + p)^2 < 0 \) from (2.5) and (2.6), (2.7) can be solved uniquely for \( \rho \in (\rho^*, \rho^*) \) provided

\begin{equation}
\Phi(\rho^* - 0)^2 < (1 - \frac{u_0}{c^2})^2 - \frac{u^2}{c^2} < \Phi(\rho^* + 0)^2.
\end{equation}

Thus, the map (2.1) defined with (2.4) is a diffeomorphism from

\begin{equation}
\Omega = \{ \rho < \rho < \rho^*, v^2 < c^2 \}
\end{equation}

onto

\begin{equation}
\Omega_u = \{ u_0 < c^2, (2.8) \text{ holds} \}.
\end{equation}

After a straight but tedious computation, we find the coefficients \( A^0(\rho) = (A_{ij}^0), \alpha, \beta, \gamma = 0, 1, 2, 3, \) as follows:

\begin{align}
A_{00}^0 &= A_1 \Psi(\rho), \quad A_{0i}^0 = A_{i0}^0 = A_2 \Psi(\rho)v_i,
A_{ij}^0 &= A_3 \Psi(\rho)v_i v_j + A_4 \Psi(\rho)\delta_{ij},
A_{0u}^0 &= A_2 \Psi(\rho), \quad A_{0i}^u = A_{i0}^u = A_3 \Psi(\rho)v_i v_t + A_5 \Psi(\rho)\delta_{it},
A_{ij}^u &= A_3 \Psi(\rho)v_i v_j + A_4 \Psi(\rho)(v_i \delta_{jt} + v_j \delta_{it} + v_t \delta_{ij}),
\end{align}

for \( i, j, \ell = 1, 2, 3, \) where

\[ \Psi(\rho) = \frac{1}{K}(p c^2 + p)^2 e^{-\phi(\rho)}, \]

and

\begin{align}
A_1 &= \frac{c^4 + 3p^2 v^2}{c^3 p'(c^2 - v^2)^{3/2}}, \quad A_2 = \frac{c^4 + 2p' c^2 + p^2 v^2}{c^3 p'(c^2 - v^2)^{3/2}},
A_3 &= \frac{c^2 + 3p'}{c p'(c^2 - v^2)^{3/2}}, \quad A_4 = \frac{1}{c(c^2 - v^2)^{1/2}},
A_5 &= \frac{1}{c(p c^2 + p)(c^2 - v^2)^{1/2}}.
\end{align}

These coefficients can be calculated by the chain rule and the formula

\begin{align}
\frac{\partial \rho}{\partial u_0} &= \frac{A_4}{p} \Psi(\rho), \quad \frac{\partial \rho}{\partial u_j} = \frac{A_1}{p} \Psi(\rho)v_j,
\frac{\partial \rho}{\partial v_i} &= A_6 \Psi(\rho)v_i, \quad \frac{\partial \rho}{\partial v_j} = c^2 A_6 \Psi(\rho)\delta_{ij}, \quad i, j = 1, 2, 3,
\end{align}
with

\[ A_6 = \frac{(c^2 - \nu^2)^{1/2}}{c^3(p^2 + \nu)}. \]

It is clear from (2.11) that the matrices \( A^0(u) \) are all real symmetric and smooth in \( \Omega_2 \), and hence in \( \Omega_u \). To see that \( A^0(u) \) is positive definite, let \( \Xi = (\xi_0, \xi)^T \in \mathbb{R}^4 \) be a 4-vector with \( \xi \in \mathbb{R}^3 \). We should calculate the inner product

\[ (A^0(u)\Xi | \Xi) = \Psi(\rho)\{A_1\xi_0^2 + 2A_2\xi_0\nu \xi + A_3(\nu \xi)^2 + A_4\xi^2\}, \]

\( A_j \) being those in (2.12). In the same way as in [6], we can get an estimate

(2.13) \[ (A^0\Xi | \Xi) \geq \frac{1}{2}(\kappa_0 \xi_0^2 + \kappa \xi^2), \]

with

\[ \kappa_0 = \frac{(c^2 - \nu^2)^{1/2}(c^4 - \nu^2)\Psi(\rho)}{c^3(c^4 + 2c^2\nu^2 + c^4\nu')}, \]

\[ \kappa = \frac{(c^2 - \nu^2)^{1/2}(c^4 - \nu^2)\Psi(\rho)}{c^3(c^4 + 3\nu^2\nu')}, \]

which implies that (2.3)(ii) is also satisfied in \( \Omega_u \) since (1.3) is fulfilled. Thus, (2.2) with (2.11) for the elements of the matrices \( A^0(u) \) is a symmetric hyperbolic system, which entails the existence of smooth local solutions to (2.2), thanks to the Friedrichs–Lax–Kato theory [3],[5]. Since (2.4) is a diffeomorphism, we can go back from (2.2) to the original system (1.1) to conclude Theorem 1.1.

3 Strictly convex entropy function

In this section, we shall follow Godunov [2] and explain how to find out the change of variables (2.4). First of all, we rewrite (1.1) in the form of the conservation laws,

(3.1) \[ w_t + \sum_{k=1}^{3}(f^k(w))_{x_k} = 0, \]
where \( w = (w_0, w_1, w_2, w_3)^T \) and \( f^k(w) = (w_k, f_1^k, f_2^k, f_3^k)^T \) are defined by

\[
\begin{align*}
w_0 &= \frac{\rho c^2 + p}{c^2 - v^2} - \frac{p}{c^2}, \quad w_j = \frac{\rho c^2 + p}{c^2 - v^2} v_j, \\
f_i^k &= \frac{\rho c^2 + p}{c^2 - v^2} v_i v_k + p \delta_{ik}.
\end{align*}
\]

(3.2)

A scalar function \( \eta = \eta(w) \) is called an entropy function to (3.1) if there exist scalar functions, \( q^k = q^k(w), \ k = 1, 2, 3 \), satisfying

\[
D_u \eta(w) D_w f^k(w) = D_u q^k.
\]

(3.3)

Then, the symmetrizing variable \( u \) can be given by

\[
u = (D_u \eta)^T.
\]

(3.4)

For the detail, see Godunov [2] or Kawashima-Shizuta [4].

Now, we shall solve (3.3). To this end, it is convenient to employ \( z = (\rho, v_1, v_2, v_3) \), instead of \( w \) of (3.2), as the independent variables in (3.3). This is possible since \( D_z w \) is regular;

\[
det D_z w = \frac{(\rho c^2 + p)(c^4 - v^2 p')}{c^2(c^2 - v^2)} > 0,
\]

which comes by noting

\[
\begin{align*}
\frac{\partial w_0}{\partial \rho} &= B_1, & \frac{\partial w_0}{\partial v_j} &= B_2 v_j, \\
\frac{\partial w_i}{\partial \rho} &= B_3 v_i, & \frac{\partial w_i}{\partial v_j} &= B_2 v_i v_j + B_4 \delta_{ij},
\end{align*}
\]

(3.5)

where

\[
\begin{align*}
B_1 &= \frac{c^2 + p'}{c^2 - v^2} - \frac{p}{c^2}, & B_2 &= \frac{2(\rho c^2 + p)}{(c^2 - v^2)^2}, \\
B_3 &= \frac{c^2 + p'}{c^2 - v^2}, & B_4 &= \frac{\rho c^2 + p}{c^2 - v^2}.
\end{align*}
\]

Thus the mapping \( z \to w \) is a diffeomorphism in a neighbourhood of each point of \( \Omega_z \). Moreover, using (3.5), we get

\[
(D_z w)^{-1} = (e_{\alpha \beta})_{\alpha, \beta = 0, 1, 2, 3}
\]

(3.6)
as
\[ e_{00} = c^2(c^2 + v^2)E_1, \quad e_{0j} = -2c^2E_1v_j, \]
\[ e_{i0} = -c^2(c^3 + p')E_1E_2v_i, \quad e_{ij} = 2p'E_1E_2v_iv_j + E_2\delta_{ij}, \]

with
\[ E_1 = \frac{1}{c^4 - v^2p'}, \quad E_2 = \frac{v^2 - \rho^2}{\rho c^2 + p}. \]

In view of (3.2) and (3.6), (3.3) can now be rewritten as

\[(3.7) \quad D_z\eta C^k = D_z q^k, \quad k = 1, 2, 3, \]

where
\[ C^k = (D_z w)^{-1} D_z f^k = (c^k_{\alpha\beta})_{\alpha,\beta=0,1,2,3}, \]

are given by
\[ c^k_{00} = c^2C_1v_k, \quad c^k_{i0} = -C_1C_2v_iv_k + C_2\delta_{kj}, \]
\[ c^k_{0j} = C_3\delta_{ki}, \quad c^k_{ij} = C_4v_i\delta_{kj} + v_k\delta_{ij}, \]

with
\[(3.8) \quad C_1 = \frac{c^2 - p'}{c^4 - v^2p'}, \quad C_2 = \frac{\rho'(c^2 - \rho^2)}{\rho c^2 + \rho}, \]
\[ C_3 = \frac{c^2(\rho c^2 + \rho)}{c^4 - v^2p'}, \quad C_4 = \frac{\rho'(c^2 - \rho^2)}{c^4 - v^2p'}. \]

Let us solve (3.7) for \((\eta, q^1, q^2, q^3)\). A quick count shows that (3.7) constitutes 12 equations for 4 unknowns, that is, it forms an over-determined system. We shall look for the solution of the form

\[(3.9) \quad \eta = H(\rho, y), \quad q^k = Q(\rho, y)v_k, \]

where
\[ y = v^2 = v_1^2 + v_2^2 + v_3^2. \]

This ansatz reduces (3.7) to the following system of first order linear partial differential equations;

\[(3.10) \quad H_y = Q_y, \]
\[(3.11) \quad c^2C_1H_\rho + 2C_2(-C_1y + 1)H_y = Q_\rho, \]
\[(3.12) \quad C_3H_\rho - 2C_4yH_y = Q, \]
having as in (3.8). Seemingly, we have still an over-determined system. However, making \((3.11) \times (\rho c^2 + p) - (3.12) \times (c^2 - p')\) and using (3.10), we get a single equation for \(Q\):

\[
2(c^2 - y)p'Q_y = (\rho c^2 + p)Q_{\rho} - (c^2 - p')Q. \tag{3.13}
\]

On the other hand, it follows from (3.10) that there should exist a function \(G = G(\rho)\) of \(\rho\) only such that

\[H = Q(\rho, y) + G(\rho).\]

Substitution of this into (3.11), together with (3.13), then yields

\[
\rho G_{\rho} = \frac{c^2 - y}{\rho c^2 + p}Q - \frac{c^2 - y}{c^2}Q_{\rho},
\]

or putting \(q = (c^2 - y)Q\),

\[
G_{\rho} = \frac{1}{\rho c^2 + p}q - \frac{1}{c^2}q_{\rho}. \tag{3.14}
\]

Since the left hand side of (3.14) is a function of \(\rho\) only, \(q\) must be of the form

\[
q = e^{\phi(\rho)}[g(\rho) + h(y)], \tag{3.15}
\]

where \(\phi(\rho)\) is as in (2.5) while \(g\) and \(h\) are arbitrary functions. Substituting (3.15) into (3.13) and separating the variables, we have

\[
\frac{\rho c^2 + p}{p'} \frac{dg}{d\rho} - g = 2(c^2 - y) \frac{dh}{dy} + h = \text{constant},
\]

which can be easily solved as

\[
q = e^{\phi(\rho)}[K_1(c^2 - y)^{1/2} + K_2 e^{\psi(\rho)}] = K_1(c^2 - y)^{1/2} e^{\psi(\rho)} + K_2(\rho c^2 + p), \tag{3.16}
\]

where \(K_j\)'s are integration constants and

\[
\psi(\rho) = \int_{\rho}^{\rho} \frac{p'(\rho)}{\rho c^2 + p(\rho)} d\rho = -\phi(\rho) + \log \frac{\rho c^2 + p(\rho)}{\rho c^2 + p(\bar{\rho})}. \tag{3.17}
\]
\( \dot{\rho} \) being as in (2.5). Now, (3.14) combined with (3.16) gives \( G' = -K_2 \dot{p}' / c^2 \), so that
\[
G = -\frac{K_2}{c^2} p + K_3, \tag{3.18}
\]
\( K_3 \) being also an integration constant. In view of (3.16) and (3.18), we get
\[
\eta = H = \frac{K_1}{(c^2 - v^2)^{1/2}} e^{\phi(\rho)} + K_2 \left( \frac{\rho c^2 + p}{c^2 - v^2} - \frac{p}{c^2} \right) + K_3, \tag{3.19}
\]
\[
Q = \frac{K_1}{(c^2 - v^2)^{1/2}} e^{\phi(\rho)} + \frac{K_2}{c^2 - v^2} (\rho c^2 + p). \tag{3.20}
\]

For the later purpose, we wish to choose the constants \( K_j, \ j = 1, 2, 3 \), so that (3.19) converges, as \( c \to \infty \), to the entropy function for the non-relativistic case,
\[
\eta^{(\infty)} = \frac{1}{2} \rho v^2 + \rho \int_{\rho}^{p} \frac{dp}{\rho} - p, \tag{3.21}
\]
which can be obtained exactly in the same way as (3.19). In view of (3.17), \( \Phi(\rho) \) of (2.6) equals \( e^{-\psi(\rho)} \) so that it can be expanded for large \( c \) as
\[
\Phi(\rho) = 1 - \frac{1}{c^2} \int_{\rho}^{p} \frac{dp}{\rho} + O(c^{-4}),
\]
for each fixed \( \rho \in (\rho_*, \rho^*) \). Insert this into (3.19) to deduce
\[
\eta = \frac{\rho + p/c^2}{(1 - v^2/c^2)^{1/2}} \left\{ \frac{c K_1}{K} + \frac{K_2}{(1 - v^2/c^2)^{1/2}} - \frac{K_1}{c K} \int_{\rho}^{p} \frac{dp}{\rho} - \frac{K_1}{K} O(c^{-3}) \right\} - \frac{K_2 p}{c^2} + K_3,
\]
where \( K \) is as in (2.5). Therefore, the right choice is found to be
\[
K_1 = -cK, \quad K_2 = c^2, \quad K_3 = 0,
\]
with which (3.19) becomes
\[
\eta = -\frac{cK}{(c^2 - v^2)^{1/2}} e^{\phi(\rho)} + c^2 \left( \frac{\rho c^2 + p}{c^2 - v^2} - \frac{p}{c^2} \right). \tag{3.22}
\]
The change of variables (2.4) was derived from (3.22) via the formula (3.4) or
\[ u = ((D_{x}w)^{T})^{-1}(D_{x}d)^{T}, \]
combined with (3.6). Since the matrix \( A^{0}(u) \) is positive definite in \( \Omega_{u} \) as was shown in the preceding section, the entropy function (3.22) is strictly convex there.

## 4 Non-relativistic limit

In order to study the limit \( c \to \infty \), we consider \( c \geq c_{0} \) with a fixed \( c_{0} \) sufficiently large and assume, without loss of generality, that (1.3) is satisfied for all \( c \geq c_{0} \) with the same constants \( \rho^{*} \) and \( \rho^{*} \). For the sake of simplicity, we discuss only the case \( \rho^{*} < \infty \). The case \( \rho^{*} = \infty \) can be treated similarly. Given \( \delta > 0 \) sufficiently small, define
\[
\Omega_{u}(\delta, c_{0}) = \{ \rho_{*} + \delta \leq \rho \leq \rho_{*} - \delta, \ v^{2} \leq (1 - \delta)c_{0}^{2} \}.
\]
Firstly, note that (2.4) is a diffeomorphism from the domain (4.1) onto
\[
\Omega_{u}(\delta, c_{0}, c) = \{ u_{0} < c^{2}, \ \frac{1}{c^{2}} - \frac{u}{c_{0}^{2}(1 - \delta)} \geq 0, \ \Phi(\rho^{*} - \delta)^{2} \leq (1 - \frac{u_{0}}{c^{2}})^{2} - \frac{u^{2}}{c^{2}} \leq \Phi(\rho^{*} + \delta)^{2} \},
\]
\[ \Omega_{u}(\delta, c_{0}, c) \in \Omega_{u}(\delta, c_{0}, c). \]

Firstly, note that (2.4) is a diffeomorphism from the domain (4.1) onto
\[
\Omega_{u}(\delta, c_{0}, c) = \{ u_{0} < c^{2}, \ \frac{1}{c^{2}} - \frac{u}{c_{0}^{2}(1 - \delta)} \geq 0, \ \Phi(\rho^{*} - \delta)^{2} \leq (1 - \frac{u_{0}}{c^{2}})^{2} - \frac{u^{2}}{c^{2}} \leq \Phi(\rho^{*} + \delta)^{2} \},
\]
\[ \Omega_{u}(\delta, c_{0}, c) \in \Omega_{u}(\delta, c_{0}, c). \]

This means that the Friedlichs-Kato-Lax theory applies for (2.2) uniformly for all \( c \geq c_{0} \). Go back to (1.1), which is possible due to the diffeomorphism (2.4), to conclude

**Theorem 4.1.** Let \( s \geq 3 \). For any fixed \( M_{0}, \ c_{0} > 0 \) sufficiently large and \( \delta_{0} > 0 \) sufficiently small, there exist positive constants \( M \) and \( T \) such that for any initial \( z_{0} = (\rho_{0}, v_{01}, v_{02}, v_{03}) \in H_{x}^{s} \) satisfying
\[ ||z_{0}||_{H_{x}^{s}} \leq M_{0}, \quad z_{0}(x) \in \Omega_{u}(\delta_{0}, c_{0}) \text{ for any } x \in \mathbb{R}^{3}, \]
and for any $c \geq c_0$, the Cauchy problem (1.1), (1.2) and (1.4) possesses a unique solution $z = (\rho, v_1, v_2, v_3)$ belonging to the class (1.5), (1.6) and satisfying
\[ \|z(t)\|_{H^s} \leq M, \quad z(t, x) \in \Omega_{\delta_0/2, c_0} \text{ for any } x \in \mathbb{R}^3, \]
for almost all $t \in [0, T]$.

Let us show that the solutions $z$ thus obtained converge as $c \to \infty$ to the solution of the non-relativistic Euler equation,
\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \sum_{k=1}^{3} \frac{\partial}{\partial x_k} (\rho v_k) &= 0, \\
\frac{\partial (\rho v_i)}{\partial t} + \sum_{k=1}^{3} \frac{\partial}{\partial x_k} (\rho v_i v_k + p\delta_{ik}) &= 0, \quad i = 1, 2, 3,
\end{align*}
\] with the same initial $z_0$.

The symmetrizing variables for (4.3) associated with the entropy function (3.21) are given by
\[
\begin{align*}
u_0^{(\infty)} &= -\frac{1}{2} v^2 + \int_{p}^{\rho} \frac{dp}{p}, \\
u_j^{(\infty)} &= v_j, \quad j = 1, 2, 3,
\end{align*}
\] and the resulting system is
\[
A'^{(\infty)0}(u^{(\infty)})u^{(\infty)}_i + \sum_{\ell=1}^{3} A'^{(\infty)\ell}(u^{(\infty)})u^{(\infty)}_{\ell i} = 0,
\]
with
\[
\begin{align*}
a_{00}^{(\infty)0} &= \frac{\rho}{p'}, \
a_{10}^{(\infty)0} &= a_{0i}^{(\infty)10} = \frac{\rho}{p} v_i, \\
a_{ij}^{(\infty)0} &= \frac{\rho}{p'} v_i v_j + \rho \delta_{ij},
\end{align*}
\]
for the matrix elements of $A'^{(\infty)10}$ and so on. Between the transformations (2.4) and (4.4), it holds that
\[
u(z) = u^{(\infty)}(z) + O(c^{-2}),
A^0(u(z)) = A^{(\infty)\alpha}(u^{(\infty)}(z)) + O(c^{-2}), \quad \alpha = 0, 1, 2, 3,
\]
uniformly for $c \geq c_0$ and $z \in \Omega_{\delta_0/2, c_0}$, which implies, together with the uniform properties stated before Theorem 4.1 and by the arguments in [6], the uniform convergence of the solutions $u$ of (2.2) to the solution of (4.5). Again we can go back to (1.1) and conclude
Theorem 4.2. Let \( s \geq 3 \). Then, as \( c \to \infty \), the solution \( z \) of (1.1), (1.2) and (1.4) given in Theorem 4.1 converges to the solution \( z^{(\infty)} \) to (4.3) with the same initial data, uniformly on the time interval \([0, T]\) with \( T \) specified in Theorem 4.1, strongly in \( H_{ul}^{s-\epsilon} \) for any \( \epsilon > 0 \).

References


