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Properties of Operator L^{λ} in the Classes $J_{\alpha}^{*}(k)$ and $E_{\alpha}(k)$

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Abstract . We investigate the relationship between $f(z)= z + \sum_{n=2}^{\infty} a_n z^n$ and $L^{\lambda} f(z)= z + \sum_{n=2}^{\infty} n^{-\lambda} a_n z^n$, λ real, when $f(z)$ is analytic and univalent in the unit disk, and when $f(z)$ is in the classes $J_{\alpha}^{*}(k)$ and $E_{\alpha}(k)$ of analytic univalent functions defined in terms of certain operators of fractional calculus .

1 . Introduction

Let S, S^{*} and K denote the classes consisting of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \quad z \in U = \{z: |z| < 1\} \tag{1}$$

that are, respectively, univalent, starlike, and convex in U. For an analytic function $f(z)$ given by (1), Komatu [2] defined the linear integral transformation $L^{\lambda} f$ by

$$L^{\lambda} f(z) = z + \sum_{n=2}^{\infty} n^{-\lambda} a_n z^n \qquad (\lambda \text{ real}, \ z \in U). \tag{2}$$

The function $L^{\lambda} f(z)$ is clearly analytic in U . Questions arise as to when $L^{\lambda} f$ will be in the same class as f . For example, what is the smallest λ for which $L^{\lambda} f \in S$ whenever f is ? In [2] , Komatu proved that if $f \in S$, then $L^{\lambda} f \in S^{*}$ at least for $\lambda \geq \lambda_0$, where $\lambda_0 \in (3,4)$ is the unique root of equation $\zeta(\lambda-2)=2$ (ζ denotes the Riemann zeta function), and conjectured that

(I) If $f \in S$, then $L^{\lambda} f \in S$ at least for $\lambda \geq 1$;

(II) If $f \in K$ (or, more generally, $f \in S^{*}$), then $L^{\lambda} f \in K$ at least for $\lambda \geq 1$.

Lewis [3] essentially showed that the conjecture (II) is true (cf.[5]). In the case $\lambda=1$, the conjecture (I) reduces to the Biernarcki conjec-ture which is false [1,P.257]. We note that the conjecture (I) is also false in the case $\lambda=2$. In fact, for the function $f(z)=z(1-z)^{-(1+i)} \in S$, we have $z(L^2f(z))' = L^1f(z) = -i((1-z)^{-i}-1)$. Hence $z(L^2f(z))' = 0$ at $z = 1-e^{-2\pi} \in U$, this shows that $L^2f(z) \notin S$.

Owa [4] and Silverman [5] investigated the relationship between $f(z)$ and $L^\lambda f(z)$, λ real, when $f(z)$ is in the subsets of S, S^* and K.

Let $J_\alpha^*(k)$ denote the class of functions

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqslant 0,\ z \in U) \tag{3}$$

which are analytic and univalent in U and satisfy the condition

$$\mathrm{Re}\left\{\frac{\Gamma(2-\alpha)\ z^\alpha D_z^\alpha f(z)}{f(z)}\right\} > k \qquad (z \in U) \tag{4}$$

for $0 \leqslant \alpha < 1$ and $0 \leqslant k < 1$, where $D_z^\alpha f(z)$ denotes the fractional derivative of $f(z)$ of order α (cf. [6]). Furthermore, let $E_\alpha(k)$ denote the class of analytic univalent functions $f(z)$ defined by (3) such that $\Gamma(2-\alpha)z^\alpha D_z^\alpha f(z) \in J_\alpha^*(k)$.

Srivastava and Owa [6] investigated Komatu's conjectures for two general classes $J_\alpha^*(k)$ and $E_\alpha(k)$. In [6], the following results, supporting conjectures (I)(II), were established.

Theorem A. If $f(z) \in J_\alpha^*(k)$, then (i) $L^\lambda f(z) \in J_\alpha^*(k)$ for $\lambda \geqslant 2$; (ii) $L^\lambda f(z) \in E_\alpha(k)$ for $\lambda \geqslant 3$; (iii) $L^\lambda f(z) \in J_\alpha^*(0)$ for $\lambda \geqslant 2$.

Theorem B. If $f(z) \in E_\alpha(k)$, then (i) $L^\lambda f(z) \in E_\alpha(k)$ for $\lambda \geqslant 2$; (ii) $L^\lambda f(z) \in J_\alpha^*(k)$ for $\lambda \geqslant \ln(4-\alpha)/\ln 2$; (iii) $L^\lambda f(z) \in J_\alpha^*(0)$ for $\lambda \geqslant 1 + \{\ln(4-\alpha-4k+2\alpha k) - \ln(2-2k+\alpha k)\}/\ln 2$; (iv) $L^\lambda f(z) \in E_\alpha(0)$

for $\lambda \geqslant 2$.

In the present paper, we shall improve the results of Theorem A and Theorem B further .

2 . Main Results

In our investigation of Komatu's conjectures for the classes $J_\alpha^*(k)$ and $E_\alpha(k)$, we need the following lemmas (cf.[6]) .

Lemma 1. The function f(z) defined by (3) is in the class $J_\alpha^*(k)$ if and only if

$$\sum_{n=2}^{\infty}\left(\frac{\Gamma(n+1)\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k\right) a_n \leqslant 1-k . \tag{5}$$

Lemma 2. The function f(z) defined by (3) is in the class $E_\alpha(k)$ if and only if

$$\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left(\frac{\Gamma(n+1)\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k\right) a_n \leqslant 1-k . \tag{6}$$

Theorem 1. (i) If $f(z)\in J_\alpha^*(k)$, then $L^\lambda f(z)\in J_\alpha^*(k)$ for $\lambda \geqslant 0$.

(ii) If $f(z)\in E_\alpha(k)$, then $L^\lambda f(z)\in E_\alpha(k)$ for $\lambda \geqslant 0$.

These results are all sharp.

Proof. (i) Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in J_\alpha^*(k)$. By using Lemma 1, we show that (5) implies

$$\sum_{n=2}^{\infty}\left(\frac{\Gamma(n+1)\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k\right)\frac{a_n}{n^\lambda} \leqslant 1-k \tag{7}$$

for $\lambda \geqslant 0$, where $0 \leqslant \alpha < 1$, $0 \leqslant k < 1$.

For any real $\lambda \geqslant 0$, it follows from (5) that

$$\sum_{n=2}^{\infty}\left(\frac{\Gamma(n+1)\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k\right)\frac{a_n}{n^\lambda}$$

$$\leqslant \sum_{n=2}^{\infty}\left(\frac{\Gamma(n+1)\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k\right) a_n$$

$$\leqslant 1 - k . \tag{8}$$

Hence $L^{\lambda} f(z) \in J_{\alpha}^{*}(k)$ for $\lambda \geqslant 0$.

To show sharpness, set

$$f_n(z) = z - \frac{(1-k)\,\Gamma(n+1-\alpha)}{\Gamma(n+1)\,\Gamma(2-\alpha) - k\,\Gamma(n+1-\alpha)}\, z^n ,$$

and observe that $f_n(z) \in J_{\alpha}^{*}(k)$, but $L^{\lambda} f_n(z) \notin J_{\alpha}^{*}(k)$ for $\lambda < 0$.

(ii) The proof of (ii) is much akin to that of (i) detailed already : indeed, instead of Lemma 1 and $f_n(z)$, it uses Lemma 2 and

$$g_n(z) = z - \frac{(1-k)\{\Gamma(n+1-\alpha)\}^2}{\Gamma(n+1)\,\Gamma(2-\alpha)\{\Gamma(n+1)\Gamma(2-\alpha) - k\Gamma(n+1-\alpha)\}}\, z^n \in E_{\alpha}(k).$$

The proof of Theorem 1 is completed.

Corollary 1 . (i) If $f(z) \in J_{\alpha}^{*}(k)$, then $L^{\lambda} f(z) \in J_{\alpha}^{*}(0)$ for $\lambda \geqslant 0$, and $L^{\lambda} f(z) \notin J_{\alpha}^{*}(0)$ for $\lambda < \lambda_0 = \{\ln(2-2k) - \ln(2-2k+k\alpha)\}/\ln 2$. (ii) If $f(z) \in E_{\alpha}(k)$, then $L^{\lambda} f(z) \in E_{\alpha}(0)$ for $\lambda \geqslant 0$, and $L^{\lambda} f(z) \notin E_{\alpha}(0)$ for $\lambda < \lambda_0 = \{\ln(2-2k) - \ln(2-2k+k\alpha)\}/\ln 2$.

Proof. Since $J_{\alpha}^{*}(k) \subset J_{\alpha}^{*}(0)$, $E_{\alpha}(k) \subset E_{\alpha}(0)$, we have from Theorem 1 that if $f(z) \in J_{\alpha}^{*}(k)$ then $L^{\lambda} f(z) \in J_{\alpha}^{*}(0)$ for $\lambda \geqslant 0$; if $f(z) \in E_{\alpha}(k)$ then $L^{\lambda} f(z) \in E_{\alpha}(0)$ for $\lambda \geqslant 0$. For the functions $f_2(z)$ and $g_2(z)$, we have $L^{\lambda} f_2(z) \notin J_{\alpha}^{*}(0)$ and $L^{\lambda} g_2(z) \notin E_{\alpha}(0)$ for $\lambda < \lambda_0$. The proof of Corollary 1 is completed .

Theorem 2. If $f(z) \in J_{\alpha}^{*}(k)$, then $L^{\lambda} f(z) \in E_{\alpha}(k)$ for $\lambda \geqslant \alpha$, and $L^{\lambda} f(z) \notin E_{\alpha}(k)$ for $\lambda < 1 - \ln(2-\alpha)/\ln 2$.

Proof . By virtue of Lemma 1 and Lemma 2 , we show that (5) implies

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left(\frac{\Gamma(n+1)\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k \right) \frac{a_n}{n^{\lambda}} \leqslant 1 - k \tag{9}$$

for $\lambda \geq \alpha$. It suffices to prove that $H(n) = \Gamma(n+1)\,\Gamma(2-\alpha)/(n^{\lambda}\Gamma(n+1-\alpha)) \leq 1$ for $\lambda \geq \alpha$ and $n \geq 2$. Let $h(n) = H(n+1)/H(n)$. Then $\lim_{n\to\infty} h(n) = 1$, $h'(n) \geq 0$ for $\lambda \geq \alpha$ and $n \geq 2$. Thus $H(n)$ is a decreasing function of n $(n \geq 2)$ for $\lambda \geq \alpha$. Since $H(2) = 1/((2-\alpha)\,2^{\lambda-1}) \leq 1$ when $\lambda \geq 1-\ln(2-\alpha)/\ln 2$, and $\max\{\alpha, 1-\ln(2-\alpha)/\ln 2\} = \alpha$, we have $H(n) \leq 1$ for $\lambda \geq \alpha$ and $n \geq 2$.

For the function $f_2(z) = z - (1-k)(2-\alpha)/(2-k(2-\alpha))\, z^2 \in J^*_\alpha(k)$, we have $L^\lambda f_2(z) \notin E_\alpha(k)$ for $\lambda < 1 - \ln(2-\alpha)/\ln 2$. This completes the proof.

Theorem 3. If $f(z) \in E_\alpha(k)$, then $L^\lambda f(z) \in J^*_\alpha(k)$ for $\lambda \geq -2\alpha/(3-\alpha)$, and $L^\lambda f(z) \notin J^*_\alpha(k)$ for $\lambda < \ln(2-\alpha)/\ln 2 - 1$.

Proof. Since $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in E_\alpha(k)$, by using Lemma 2, we have

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left(\frac{\Gamma(n+1)\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k \right) a_n \leq 1 - k \quad (10)$$

for $0 \leq \alpha < 1$, $0 \leq k < 1$. Let $H(n) = \Gamma(n+1)\,\Gamma(2-\alpha)\, n^{\lambda}/\Gamma(n+1-\alpha)$, $h(n) = H(n+1)/H(n)$. Then $\lim_{n\to\infty} h(n) = 1$, $h'(n) \leq 0$ for $n \geq 2$ and $\lambda \geq -2\alpha/(3-\alpha)$. Thus $H(n)$ is an increasing function of n $(n \geq 2)$ for $\lambda \geq -2\alpha/(3-\alpha)$. But $H(2) = 2^{\lambda+1}(2-\alpha) \geq 1$ for $\lambda \geq \{\ln(2-\alpha) - \ln 2\}/\ln 2$, and

$$\max\left\{ \frac{-2\alpha}{3-\alpha}, \frac{\ln(2-\alpha) - \ln 2}{\ln 2} \right\} = \frac{-2\alpha}{3-\alpha} \quad (0 \leq \alpha < 1), \quad (11)$$

we get $H(n) \geq 1$ for $\lambda \geq -2\alpha/(3-\alpha)$ and $n \geq 2$, i.e.,

$$\frac{\Gamma(n+1)\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \geq \frac{1}{n^{\lambda}} \quad \left(\lambda \geq \frac{-2\alpha}{3-\alpha},\ n \geq 2\right). \quad (12)$$

It follows from (10) and (12) that

$$\sum_{n=2}^{\infty} \left(\frac{\Gamma(n+1)\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - k \right) \frac{a_n}{n^{\lambda}} \leq 1 - k$$

for $\lambda \geq -2\alpha/(3-\alpha)$. Hence $L^\lambda f(z) \in J^*_\alpha(k)$.

For the function

$$g_2(z) = z - \frac{(1-k)(2-\alpha)^2}{2(2-k(2-\alpha))} z^2 \in E_\alpha(k),$$

we have $L^\lambda g_2(z) \notin J_\alpha^*(k)$ for $\lambda < (\ln(2-\alpha) - \ln 2)/\ln 2$. This completes the proof .

Corollary 2 . If $f(z) \in E_\alpha(k)$, then $L^\lambda f(z) \in J_\alpha^*(0)$ for $\lambda \geq -2\alpha/(3-\alpha)$ and $L^\lambda f(z) \notin J_\alpha^*(0)$ for $\lambda < \{\ln(2(1-k) - \alpha(1-k)) - \ln(2(1-k) + k\alpha)\}/\ln 2$.

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