ON MEROMORPHICALLY MULTIVALENT FUNCTIONS

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Abstract. The purpose of this paper is to derive some properties of certain meromorphically multivalent functions in annulus.

1. Introduction

Let \( \Sigma_\alpha \) be the class of functions of the form

\[
f(z) = \frac{1}{z^n} + a_0/z^{n-1} + \cdots + a_{k_\alpha} z^{k_\alpha} + \cdots,
\]

which are analytic in the annulus \( \mathbb{D} = \{ z : |z| < 1 \} \), where \( \alpha \in \mathbb{N} = \{1, 2, 3, \ldots \} \). For \( f(z) \in \Sigma_\alpha \), we define the operator \( D^{n+p} f(z) \) by

\[
D^{n+p} f(z) = \left( z^{n+2p} f(z)/(n+p-1)! \right)^{n+p-1} / z^p
\]

\[
= 1/z^n + (n+p) a_0 / z^{n-1} + (n+p)(n+p+1)a_1 / (2! z^{n-2}) + \cdots
\]

\[
+ (n+p)(n+p+1) \cdots (n+k_\alpha 2p-1) a_{k_\alpha} z^{k_\alpha} / (k_\alpha!) + \cdots,
\]

where \( n \) is an integer and \( n > -p \).

Recently, Cho and Nunokawa [1] proved that

\[
\text{Re}\{z^{n+p}(D^{n+p} f(z))'\} < -\alpha \quad (0 \leq \alpha < p; |z| < 1)
\]

implies \( \text{Re}\{z^{n+p}(D^{n+p} f(z))'\} < -\beta \quad (|z| < 1) \)

where

\[
\beta = (p+2\alpha(n+p))/(1+2(n+p)).
\]
In the present paper, we show another properties of functions \( f(z) \in \Sigma_p \) concerning with the operator \( D^{n+p-1} f(z) \).

2. Main results

We need the following lemma due to Jack [2] (or, due to Miller and Mocanu[3]).

 Lemma. Let \( w(z) \) be non-constant analytic in \( U = \{ z: |z| < 1 \} \) with \( w(0) = 1 \). If \( |w(z)| \) attains its maximum value at a point \( z_0 \) on the circle \( |z| = |z_0| < 1 \), then we have

\[ z_0 w' (z_0) = k w(z_0) \]

where \( k \) is real and \( k \geq 1 \).

Theorem 1. If \( f(z) \in \Sigma_p \) satisfies

(2.1) \[ \text{Re} \{ z^{n+1} (D^{n+p} f(z))' \} > -\alpha \quad (z \in U) \]

for some \( \alpha (\alpha > p) \), then

(2.2) \[ \text{Re} \{ z^{n+1} (D^{n+p-1} f(z))' \} > -\beta \quad (z \in U), \]

where

\[ \beta = (p + 2 \alpha (n+p))/(1 + 2(n+p)). \]

Proof. Define the function \( w(z) \) by

(2.3) \[ z^{n+1} (D^{n+p-1} f(z))' = ((p-2\beta) w(z) -p)/(1+w(z)), \]

\( w(z) \neq -1 \), with

\[ \beta = (p + 2 \alpha (n+p))/(1 + 2(n+p)). \]

Then \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \). Note that

(2.4) \[ z(D^{n+p-1} f(z))' = (n+p)D^{n+p} f(z) - (n+2p) D^{n+p-1} f(z). \]

It follows from (2.3) that

(2.5) \[ (n+p) z^n D^{n+p} f(z) - (n+2p) z^{n-1} D^{n+p-1} f(z) \]

\[ = ((p-2\beta) w(z) - p)/(1+w(z)). \]

Taking the differentiations in both sides of (2.5), we have
Suppose there exists a point \( z_0 \in U \) such that

\[
\max_{|z| \leq 1} |\omega(z)| = |\omega(z_0)| \quad (\omega(z_0) \neq -1),
\]

then, by Lemma, we have

\[
z_0 \omega'(z_0) = k\omega(z_0) \quad (k \geq 1).
\]

Therefore, letting \( \omega(z_0) = e^{i\theta} \) \( (0 \leq \theta \leq 2\pi) \), we see that

\[
(2.7) \quad \Re\{z_0^{p-1}(D^{n-p}f(z_0))'\} + \alpha
\]

\[
= \alpha + \Re\{((p-2\beta)e^{i\theta}-p)/(1+e^{i\theta})\} + 2(p-\beta)k/(n+p) \Re\{e^{i\theta}(1+e^{i\theta})^{-2}\}
\]

\[
= \alpha - \beta + (p-\beta)k/(n+p)(1 + \cos \theta)
\]

\[
\leq \alpha - \beta + (p-\beta)/(2(n+p)) = 0
\]

for \( \alpha > p \) and \( \beta = (p + 2\alpha(n+p))/(1+2(n+p)) \).

This contradicts our condition (2.1). Therefore, \( |\omega(z)| < 1 \) for all \( z \in U \), or

\[
\Re\{z^{p-1}(D^{n-p-1}f(z))'\} > -\beta \quad (z \in U).
\]

Next, we prove

Theorem 2. Let

\[
(2.8) \quad F_c(z) = cz^{-p} \int_0^z t^{n-p-1} f(t) dt \quad (c > 0)
\]
for \( f(z) \in \Sigma_{\nu} \). If \( f(z) \) satisfies

\[
\Re \{ z^{p+1} (D^{n+1} f(z))' \} > -\alpha \quad (z \in \mathcal{U})
\]

for some \( \alpha \ (\alpha > p) \), then

\[
\Re \{ z^{p+1} (D^{n+1} F_{c}(z))' \} > -\beta \quad (z \in \mathcal{U})
\]

where \( \beta = (p+2\alpha c)/(1+2c) \).

**Proof.** We define the function \( w(z) \) by

\[
z^{p+1} (D^{n+1} F_{c}(z))' = ((p-2\beta)w(z)-p)/(1+w(z)) \quad (w(z) \neq -1).
\]

Then \( w(z) \) is analytic in \( \mathcal{U} \) and \( w(0)=0 \). Noting that

\[
z(D^{n+1} F_{c}(z))' = cD^{n+1} f(z) - (c+p) D^{n+1} F_{c}(z),
\]

therefore we have

\[
z^{p+1} (D^{n+1} f(z))' = ((p-2\beta)w(z)-p)/(1+w(z)) + 2(p-\beta) z w' (z)/(1+w(z))^2).
\]

Therefore, if we assume that there exists a point \( z_0 \in \mathcal{U} \) such that

\[
\max_{1 \leq |z| \leq 1} |w(z)| = |w(Z_0)| = 1 \quad (w(z_0) \neq -1)
\]

then Lemma gives us that

\[
\Re \{ z_0^{p+1} (D^{n+1} f(z_0))' \} + \alpha \leq \alpha - \beta + (p-\beta)/2c
\]

\[
= 0
\]
which contradicts our condition (2.9). This completes the proof of Theorem 2.

References


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